CONSTRUCTIVE MATHEMATICAL ANALYSIS
2 (2019), No. 4, pp. 168-182
https://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Quadrature Formulas of Gaussian Type for Fast Summation of Trigonometric Series

GRADIMIR V. MILOVANOVIĆ*

ABSTRACT. A summation/integration method for fast summing trigonometric series is presented. The basic idea in this method is to transform the series to an integral with respect to some weight function on $\mathbb{R}_+$ and then to approximate such an integral by the appropriate quadrature formulas of Gaussian type. The construction of these quadrature rules, as well as the corresponding orthogonal polynomials on $\mathbb{R}_+$, are also considered. Finally, in order to illustrate the efficiency of the presented summation/integration method two numerical examples are included.

Keywords: Summation, trigonometric series, Gaussian quadrature rule, weight function, orthogonal polynomial, three-term recurrence relation, convergence, Laplace transform.

2010 Mathematics Subject Classification: 65B10, 41A55, 65D32.

1. INTRODUCTION

Let $\mathcal{P}$ be the space of all polynomials and $\mathcal{P}_n$ its subspace of polynomials of degree at most n. In a joint paper with Walter Gautschi [9], we developed the Gauss-Christoffel quadratures on $(0, +\infty)$,

$$\int_0^{+\infty} f(t)w_\nu(t)\,\mathrm{d}t = \sum_{k=1}^{N} A_{\nu,k}^{(N)} f\big(\tau_{\nu,k}^{(N)}\big) + R_{N,\nu}(f) \quad (\nu = 1, 2), \tag{1.1}$$

with respect to the Bose-Einstein and Fermi-Dirac weights, which are defined by

$$w_1(t) = \varepsilon(t) = \frac{t}{\mathrm{e}^t - 1} \quad \text{and} \quad w_2(t) = \varphi(t) = \frac{1}{\mathrm{e}^t + 1}, \tag{1.2}$$

respectively. These N-point quadrature formulas are exact on the space of all algebraic polynomials of degree at most $2N-1$, i.e., $R_{N,\nu}(\mathcal{P}_{2N-1}) = 0$, $\nu = 1, 2$.

The weight functions (1.2) and the corresponding quadratures (1.1) are widely used in solid state physics, e.g., the total energy of thermal vibration of a crystal lattice can be expressed in the form $\int_0^{+\infty} f(t)\varepsilon(t)\,\mathrm{d}t$, where $f(t)$ is related to the phonon density of states. Also, integrals with the second weight function $\varphi(t)$ are encountered in the dynamics of electrons in metals.

In the same paper [9], we showed that these quadrature formulas can be used for summation of slowly convergent series of the form

$$T = \sum_{k=1}^{+\infty} a_k \quad \text{and} \quad S = \sum_{k=1}^{+\infty} (-1)^k a_k. \tag{1.3}$$

Received: 02 September 2019; Accepted: 17 October 2019; Published Online: 22 October 2019
*Corresponding author: Gradimir V. Milovanović; gvm@mi.sanu.ac.rs
DOI: 10.33205/cma.613948

In general, the basic idea in such the so-called *summation/integration* procedures is to transform the sum to an integral with respect to some weight function $w(t)$ on $\mathbb{R}$ or $\mathbb{R}_+ = [0, +\infty)$, and then to approximate this integral by a finite quadrature sum, i.e.,

$$\int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x \approx Q_N(f) = \sum_{\nu=1}^{N} A_\nu^{(N)} f\big(x_\nu^{(N)}\big),$$

where the function f is connected with a_k in some way, and $x_\nu^{(N)}$ and $A_\nu^{(N)}$, $\nu = 1, \dots, N$, are nodes and weights of the quadrature rule $Q_N(f)$ (usually of Gaussian type), which is efficient for approximating a large class of functions with a relatively small number of quadrature nodes N.

As a transformation method of sums to integrals, we can use the Laplace transform as in [9] (see also [5, 8]) or some methods of complex contour integration as in our papers [12, 13] (see also [14, 15, 20, 17]). An account on summation/integration methods for the computation of slowly convergent power series and finite sums was given in [16].

In order to apply the quadrature rules (1.1) to the series T and S in (1.3), in the mentioned paper [9], we supposed that the general term of series is expressible in terms of the Laplace transform, or its derivative, of a known function. For example, let $a_k = F(k)$ and $F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t$ for $\mathrm{Re}\, s \geq 1$. Then

$$T = \sum_{k=1}^{+\infty} F(k) = \sum_{k=1}^{+\infty} \int_0^{+\infty} \mathrm{e}^{-kt} f(t)\,\mathrm{d}t = \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \mathrm{e}^{-kt} \right) f(t)\,\mathrm{d}t,$$

i.e.,

$$T = \int_0^{+\infty} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} f(t)\,\mathrm{d}t = \int_0^{+\infty} \frac{t}{\mathrm{e}^t - 1} \frac{f(t)}{t}\,\mathrm{d}t = \int_0^{+\infty} \varepsilon(t) \frac{f(t)}{t}\,\mathrm{d}t. \tag{1.4}$$

Similarly, for "alternating" series, we have

$$S = \sum_{k=1}^{+\infty} (-1)^k F(k) = \int_0^{+\infty} \frac{1}{\mathrm{e}^t + 1} (-f(t))\,\mathrm{d}t, \tag{1.5}$$

where the Fermi-Dirac weight function $\varphi(t)$ on $(0, +\infty)$ is appeared on the right-hand side in (1.5).

In this way, the summation of the series T and S is transformed to the integration problems with respect to the weight functions $w_1(t) = \varepsilon(t)$ and $w_2(t) = \varphi(t)$, respectively. An application of quadrature formulas (1.1) for $\nu = 1$ and $\nu = 2$ to the integrals in (1.4) and (1.5), respectively, provides an acceptable procedure for summation of slowly convergent series T and S.

In this paper we consider the corresponding summation for the convergent trigonometric series

$$C(x) = \sum_{k=1}^{+\infty} a_k \cos k\pi x \quad \text{and} \quad S(x) = \sum_{k=1}^{+\infty} a_k \sin k\pi x \quad (-1 < x < 1). \tag{1.6}$$

The corresponding series

$$A(x) = \sum_{k=1}^{+\infty} (-1)^{k-1} a_k \cos k\pi x \quad \text{and} \quad B(x) = \sum_{k=1}^{+\infty} (-1)^{k-1} a_k \sin k\pi x,$$

can be also considered, putting $x := x - 1$. Then $A(x) = -C(x-1)$ and $B(x) = -S(x-1)$.

The series (1.6) can be treated in the complex form

$$C(x) + \mathrm{i}S(x) = \sum_{k=1}^{+\infty} a_k \mathrm{e}^{\mathrm{i}k\pi x}. \tag{1.7}$$

The paper is organized as follows. In Section 2 we present the transformation of (1.7) to the "weighted" integrals over $(0, +\infty)$. The construction of the corresponding quadrature formulas of Gaussian type for such integrals is given in Section 3. A simpler method for the sinus-series is presented in Section 4. Finally, in order to illustrate our methods, some numerical examples are given in Section 5.

2. Transformation of (1.7) to "weighted" integrals

We consider the series (1.7) whose general term a_k is expressible in terms of the Laplace transform of a known function, i.e., let $a_k = F(k)$, where

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \quad \mathrm{Re}\, s \geq 1. \tag{2.1}$$

Then we have

$$C(x) + \mathrm{i}S(x) = \sum_{k=1}^{+\infty} \mathrm{e}^{\mathrm{i}k\pi x} \int_0^{+\infty} \mathrm{e}^{-kt} f(t)\,\mathrm{d}t = \pi \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \mathrm{e}^{-k\pi(t-\mathrm{i}x)} \right) f(\pi t)\,\mathrm{d}t,$$

i.e.,

$$C(x) + \mathrm{i}S(x) = \sum_{k=1}^{+\infty} a_k \mathrm{e}^{\mathrm{i}k\pi x} = \pi \int_0^{+\infty} \frac{\mathrm{e}^{\mathrm{i}\pi x}}{\mathrm{e}^{\pi t} - \mathrm{e}^{\mathrm{i}\pi x}} f(\pi t)\,\mathrm{d}t. \tag{2.2}$$

The obtained integral on the right-hand side in (2.2) is weighted with respect to the one-parametar "complex weight function"

$$w(t; x) = \frac{\mathrm{e}^{\mathrm{i}\pi x}}{\mathrm{e}^{\pi t} - \mathrm{e}^{\mathrm{i}\pi x}}, \quad -1 < x < 1. \tag{2.3}$$

We note that

$$w(t; 0) = \frac{\varepsilon(\pi t)}{\pi t}, \quad w(t; 1/2) = -\varphi(2\pi t) + \frac{\mathrm{i}}{2 \cosh \pi t}, \quad \text{and} \quad w(t; 1) = -\varphi(\pi t),$$

where $\varepsilon(t)$ and $\varphi(t)$ are given by (1.2). As we can see, only for $x = 0$ and $x = \pm 1$, the function $w(t; x)$ is real. Also, $w(t; -x) = \overline{w(t; x)}$, so that it is enough to consider only the case when $0 < x \leq 1$. The case $x = 0$ is not interesting because it leads to a numerical series.

Lemma 2.1. *The moments of the function (2.3) are given by*

$$\mu_k(x) = \int_0^{+\infty} t^k w(t; x)\,\mathrm{d}t = \begin{cases} -\dfrac{1}{\pi} \mathrm{Log}(1 - \mathrm{e}^{\mathrm{i}\pi x}), & k = 0, \\ \dfrac{k!}{\pi^{k+1}} \mathrm{Li}_{k+1}(\mathrm{e}^{\mathrm{i}\pi x}), & k \in \mathbb{N}, \end{cases} \tag{2.4}$$

where Li_n *is the polylogarithm function defined by*

$$\mathrm{Li}_n(z) = \sum_{\nu=1}^{+\infty} \frac{z^\nu}{\nu^n}. \tag{2.5}$$

Proof. In order to calculate the moments (2.4), i.e., the integrals

$$\mu_k(x) = \int_0^{+\infty} \frac{t^k e^{i\pi x}}{e^{\pi t} - e^{i\pi x}}\, dt, \quad k \geq 0,$$

we note that, for $a = e^{i\pi x}$, we have

$$\frac{a}{e^{\pi t} - a} = \frac{a e^{-\pi t}}{1 - a e^{-\pi t}} = \sum_{\nu=1}^{+\infty} a^\nu e^{-\nu\pi t} \quad \left(|a e^{-\pi t}| = e^{-\pi t} < 1\right).$$

Then, we get

$$\mu_k(x) = \sum_{\nu=1}^{+\infty} a^\nu \int_0^{+\infty} t^k e^{-\nu\pi t}\, dt = \frac{k!}{\pi^{k+1}} \sum_{\nu=1}^{+\infty} \frac{a^\nu}{\nu^{k+1}},$$

which is the desired result, having in mind (2.5). □

Remark 2.1. An analytic extension of the function Li_n is given by

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - z}\, dt \qquad (|\arg(1-z)| < \pi).$$

The function Li_n is suitable for both symbolic and numerical calculation. It has a branch cut discontinuity in the complex z-plane running from 1 to ∞. This function is implemented in MATHEMATICA software as `PolyLog[n,z]` and it can be evaluated to arbitrary numerical precision.

Separating the real and imaginary parts in (2.3), i.e.,

$$w(t;x) = \frac{1}{2}\left\{\frac{\cos\pi x - e^{-\pi t}}{\cosh\pi t - \cos\pi x} + i\frac{\sin\pi x}{\cosh\pi t - \cos\pi x}\right\}, \tag{2.6}$$

and using (2.2) we obtain the following results:

Lemma 2.2. *We have*

$$\begin{aligned} C(x) &= \sum_{k=1}^{+\infty} a_k \cos k\pi x = \frac{\pi}{2}\int_0^{+\infty} \frac{\cos\pi x - e^{-\pi t}}{\cosh\pi t - \cos\pi x} f(\pi t)\, dt, \\ S(x) &= \sum_{k=1}^{+\infty} a_k \sin k\pi x = \frac{\pi}{2}\int_0^{+\infty} \frac{\sin\pi x}{\cosh\pi t - \cos\pi x} f(\pi t)\, dt, \end{aligned}$$

where $a_k = F(k)$ *and* $f(t) = \mathcal{L}^{-1}[F(s)]$.

Remark 2.2. Similar formulas as in Lemma 2.2 are mentioned in [22, p. 725].

The real and imaginary parts of $2w(t;x) = w_R(t;x) + i w_I(t;x)$ for different values of x are presented in Figure 1.

As we can see, the imaginary part $t \mapsto w_I(t;x) = \mathrm{Im}\,(2w(t;x))$ is a positive function on $\mathbb{R}_+$ for each $0 < x < 1$, and all its moments are

$$\mu_k^I(x) = \int_0^{+\infty} t^k \frac{\sin\pi x}{\cosh\pi t - \cos\pi x}\, dt = \begin{cases} 1 - x, & k = 0, \\ \dfrac{2k!}{\pi^{k+1}} \mathrm{Im}\left\{\mathrm{Li}_{k+1}(e^{i\pi x})\right\}, & k \in \mathbb{N}, \end{cases}$$

so that the orthogonal polynomials with respect to the inner product

$$(p,q) = \int_0^{+\infty} p(t)q(t)w_I(t;x)\, dt, \tag{2.7}$$

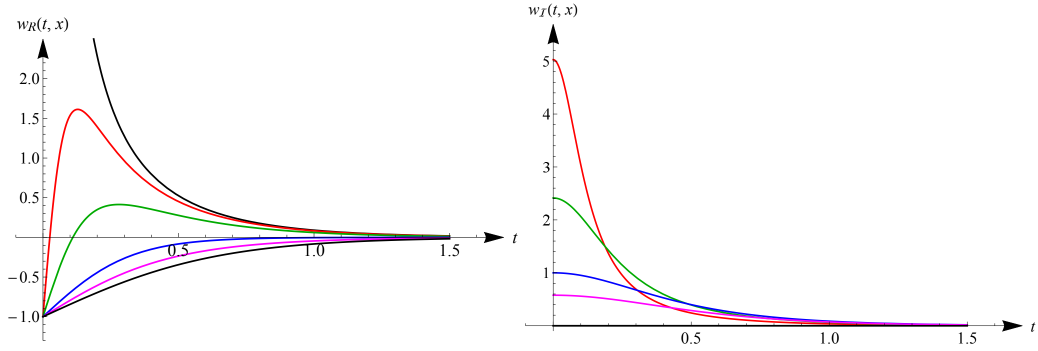

FIGURE 1. Real part (left) and imaginary part (right) of $t \mapsto 2w(t;x)$ on $[0, 1.5]$ for $x = 0$ (black line), $x = 1/8$ (red line), $x = 1/4$ (green line), $x = 1/2$ (blue line), $x = 2/3$ (magenta line), and $x = 1$ (black line)

as well as the corresponding quadrature formulas of Gaussian type exist for each $n \in \mathbb{N}$.

However, the real part $t \mapsto w_R(t;x) = \operatorname{Re}(2w(t;x))$ changes its sign at the point $t = \pi^{-1}\log(1/\cos\pi x) \in (0,+\infty)$, when $0 < x < 1/2$, while for $1/2 \le x \le 1$ this function is negative for each $t \in \mathbb{R}_+$. The moments of the function $w_R(t;x)$ are

$$\mu_k^R(x) = \int_0^{+\infty} t^k \frac{\cos\pi x - \mathrm{e}^{-\pi t}}{\cosh\pi t - \cos\pi x}\,\mathrm{d}t = \begin{cases} -\dfrac{2}{\pi}\log\left(2\sin\dfrac{\pi x}{2}\right), & k = 0, \\ \dfrac{2k!}{\pi^{k+1}}\operatorname{Re}\left\{\operatorname{Li}_{k+1}(\mathrm{e}^{\mathrm{i}\pi x})\right\}, & k \in \mathbb{N}. \end{cases}$$

Regarding these facts a system of orthogonal polynomials with respect to $t \mapsto w_R(t;x)$ on $\mathbb{R}_+$ exists for each $1/2 \le x \le 1$. However, for $0 \le x < 1/2$ the existence is not guaranteed.

3. CONSTRUCTION OF POLYNOMIALS ORTHOGONAL WITH RESPECT TO THE WEIGHTS $t \mapsto w_I(t;x)$ AND $t \mapsto w_R(t;x)$ ON $\mathbb{R}_+$ AND CORRESPONDING GAUSSIAN RULES

As we mentioned in the previous section, the (monic) polynomials $p_k^I(t;x)$, $k = 0, 1, \ldots$, orthogonal with respect to the inner product (2.7) exist uniquely, as well as the corresponding quadrature formulas of Gaussian type

$$\int_0^{+\infty} g(t) w_I(t;x)\,\mathrm{d}t = \sum_{\nu=1}^{N} A_\nu^I g(\tau_\nu^I) + R_N(g;x), \tag{3.1}$$

where $\tau_\nu^I\ (\equiv \tau_\nu^I(N,x))$ and $A_\nu^I\ (\equiv A_\nu^I(N,x))$ are their nodes and weight coefficients. The corresponding remainder term $R_N(g;x)$ vanishes for each $g \in \mathcal{P}_{2n-1}$. Some error estimates of Gaussian rules for certain classes of functions can be found in [11, Sect. 5.1.5].

The monic polynomials $p_k^I(t;x)$ satisfy the three-term recurrence relation

$$p_{k+1}^I(t;x) = (t - \alpha_k^I(x))p_k^I(t;x) - \beta_k^I(x)p_{k-1}^I(t;x), \quad k = 0, 1, \ldots, \tag{3.2}$$

with $p_0^I(t;x) = 1$ and $p_{-1}^I(t;x) = 0$.

The nodes τ_ν^I in the Gaussian quadrature rule (3.1) are eigenvalues of the symmetric tridiagonal Jacobi matrix (cf. [11, pp. 325–328])

$$J_N(w_I(\,\cdot\,;x)) = \begin{bmatrix} \alpha_0^I(x) & \sqrt{\beta_1^I(x)} & & & \mathbf{O} \\ \sqrt{\beta_1^I(x)} & \alpha_1^I(x) & \sqrt{\beta_2^I(x)} & & \\ & \sqrt{\beta_2^I(x)} & \alpha_2^I(x) & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{N-1}^I(x)} \\ \mathbf{O} & & & \sqrt{\beta_{N-1}^I(x)} & \alpha_{N-1}^I(x) \end{bmatrix}, \tag{3.3}$$

and the weight coefficients A_ν^I are given by $A_\nu^I = \beta_0^I(x)v_{\nu,1}^2$, $\nu = 1,\ldots,N$, where $v_{\nu,1}$ is the first component of the eigenvector $\mathbf{v}_\nu$ $(= [v_{\nu,1}\ \ldots\ v_{\nu,n}]^{\mathrm{T}})$ corresponding to the eigenvalue τ_ν^I and normalized such that $\mathbf{v}_\nu^{\mathrm{T}}\mathbf{v}_\nu = 1$. The most popular method for solving this eigenvalue problem is the Golub-Welsch procedure, obtained by a simplification of the QR algorithm [10].

Unfortunately, the coefficients in the three-term recurrence relation (3.2) are not known. They are known explicitly only for some narrow classes of orthogonal polynomials, including a famous class of the *classical orthogonal polynomials* (Jacobi, the generalized Laguerre, and Hermite polynomials). Orthogonal polynomials for which the recursion coefficients are not known are known as *strongly non–classical polynomials*. In the eighties of the last century Walter Gautschi developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, including effective algorithms for numerically generating the recurrence coefficients for non-classical orthogonal polynomials, a detailed stability analysis of such algorithms as well as the corresponding software and several new applications of orthogonal polynomials (in particular see [4], [6], [7], as well as [18, 19, 20]).

On the other side, recent progress in symbolic computation and variable-precision arithmetic now makes it possible to generate the recurrence coefficients directly by using the original Chebyshev method of moments, but in a sufficiently high precision arithmetic. Such an approach allows us to overcome numerical instability in the map, in notation $\mathbf{K}_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, of the first $2n$ moments to $2n$ recursive coefficients,

$$\boldsymbol{\mu} = (\mu_0^I(x), \mu_1^I(x), \ldots, \mu_{2n-1}^I(x)) \ \mapsto\ \boldsymbol{\rho} = (\alpha_0^I(x)\ldots, \alpha_{n-1}^I(x), \beta_0^I(x), \ldots, \beta_{n-1}^I(x)),$$

which is a major construction problem. Respectively symbolic/variable-precision software for orthogonal polynomials is now available: Gautschi's package SOPQ in MATLAB and our MATHEMATICA package `OrthogonalPolynomials` (see [3] and [21]), which is downloadable from the web site `http://www.mi.sanu.ac.rs/~gvm/`.

The package `OrthogonalPolynomials`, beside the numerical construction of the recurrence coefficients, enables also the construction in a symbolic form for a reasonable value of n. For example, executing the following commands

```
<< orthogonalPolynomials`
 muI[x_, n_]:=Table[If[k==0,1-x,
       2k!/Pi^(k+1) Im[PolyLog[k+1,Exp[I Pi x]]]],{k,0,2n-1}];
 mom = muI[x,5];
 {alI,beI}=aChebyshevAlgorithm[mom,Algorithm->Symbolic];
```

we obtain the first five coefficients $\alpha_k^I(x)$ and $\beta_k^I(x)$, $k = 0,1,2,3,4$, whose graphics are presented in Figure 2.

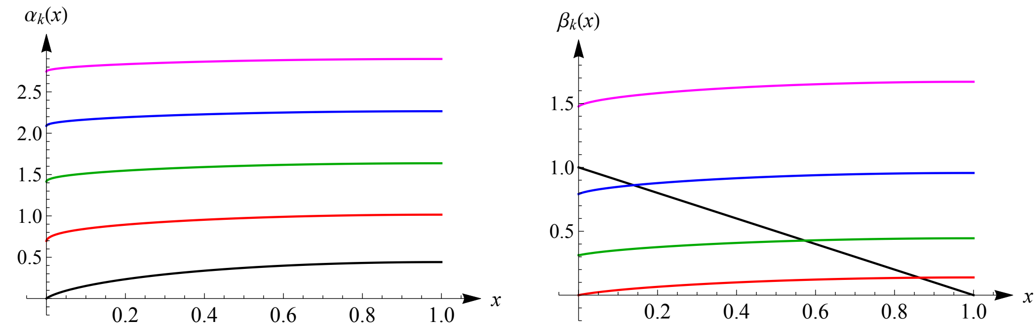

FIGURE 2. The coefficients $\alpha_k^I(x)$ (left) and $\beta_k^I(x)$ (right), for $k = 0$ (black line), $k = 1$ (red line), $k = 2$ (green line), $k = 3$ (blue line), and $k = 4$ (magenta line)

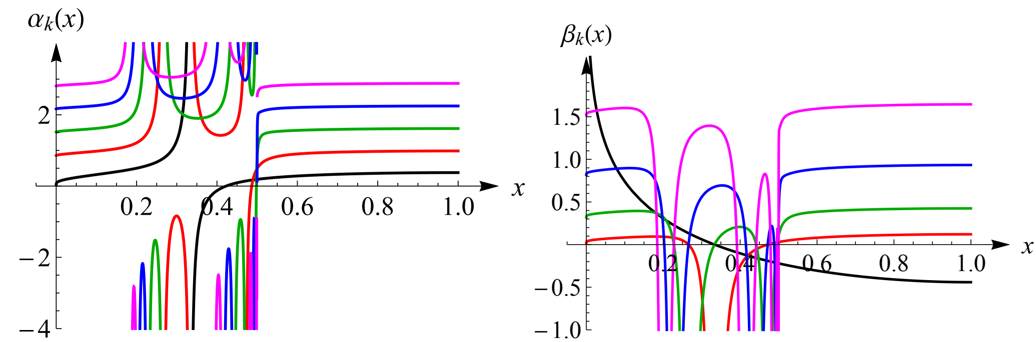

FIGURE 3. The coefficients $\alpha_k^R(x)$ (left) and $\beta_k^R(x)$ (right), for $k = 0$ (black line), $k = 1$ (red line), $k = 2$ (green line), $k = 3$ (blue line), and $k = 4$ (magenta line)

Using these coefficients we can calculate Gaussian parameters (nodes and weights) for each $N \le n = 5$ and each x. For larger values of n and a given x, it is more convenient to use the option for numerical construction in the function `aChebyshevAlgorithm`, instead of symbolic construction. A numerical example is given in Section 5.

In the same way, we can obtain the graphics of the coefficients $\alpha_k^R(x)$ and $\beta_k^R(x)$, $k = 0, 1, 2, 3, 4$, for the polynomials $p_k^R(t; x)$, $k = 0, 1, \ldots$, orthogonal with respect to the function $t \mapsto w_R(t; x)$ on $\mathbb{R}_+$ (see Figure 3). As we mention before, these polynomials exist uniquely for $1/2 \le x \le 1$, but for $0 \le x < 1/2$ their existence is not guaranteed.

4. Some classes of polynomials orthogonal on the semiaxis and corresponding Gaussian quadrature rules

There are orthogonal polynomials related to Bernoulli numbers, discovered as early as Stieltjes [23] and later extended by Touchard [24] and Carlitz [2] (for details see Chihara [1, pp. 191–193]. Carlitz defined polynomials

$$\Omega_k^{(\lambda)}(t) = \frac{(-1)^k (\lambda+1)_k k!}{2^k \left(\frac{1}{2}\right)_k} F_k^{\lambda}(1 - \lambda + 2t),$$

where, as usual $(\lambda)_k$ is the well known Pochhammer symbol (or the raised factorial, since $(1)_k = k!$), defined by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+k-1),$$

and $F_k^\lambda(t) = {}_3F_2\big[-k, k+1, \frac{1}{2}(1+\lambda+t); 1, \lambda+1; 1\big]$ is the so-called Pasternak polynomial. These polynomials are orthogonal (but not positive-definite) on a line in the complex plane $L = (c - \mathrm{i}\infty, c + \mathrm{i}\infty)$, $-1 < c < 0$, with respect to the complex weight function $z \mapsto 1/(\sin(\pi z)\sin\pi(z-\lambda))$. However, taking $(\lambda - 1 + \mathrm{i}t)/2$ instead of t (see [1, p. 192]), we get the positive-definite monic polynomials

$$G_k^{(\lambda)}(t) = (-\mathrm{i})^k \Omega_k^{(\lambda)}\left(\frac{\lambda - 1 + \mathrm{i}t}{2}\right), \quad -1 < \lambda < 1, \tag{4.1}$$

orthogonal with respect to the weight function

$$t \mapsto w^G(t;\lambda) = \frac{1}{\cosh \pi t + \cos \pi\lambda} \quad \text{on} \quad \mathbb{R}. \tag{4.2}$$

These polynomials satisfy the three-term recurrence relation

$$G_{k+1}^{(\lambda)}(t) = tG_k^{(\lambda)}(t) - B_k(\lambda)G_{k-1}^{(\lambda)}(t), \quad G_0^{(\lambda)}(t) = 1,\ G_{-1}^{(\lambda)}(t) = 0,$$

where the recurrence coefficients are given by

$$B_0(\lambda) = \int_{-\infty}^{+\infty} w^G(t;\lambda)\,\mathrm{d}t = \frac{2\lambda}{\sin\pi\lambda}, \quad B_k(\lambda) = \frac{k^2(k^2-\lambda^2)}{4k^2-1}, \quad k = 1, 2, \ldots . \tag{4.3}$$

Remark 4.1. When $\lambda \to 0$ these polynomials $G_k^{(\lambda)}$ reduce to orthogonal polynomials with respect to the logistic weight $t \mapsto 1/(\cosh \pi t + 1) = 2\mathrm{e}^{-\pi t}/(1+\mathrm{e}^{-\pi t})^2$ (see [16, p. 49]).

As we can see, there is a connection between the weights $w_I(t;x)$ and $w^G(t;\lambda)$ on $\mathbb{R}_+$. Namely,

$$w_I(t;x) = \sin(\pi x)\, w^G(t; 1-x), \quad 0 \le t < +\infty.$$

Using this fact and some results from [11, pp. 102–103] we can get the recurrence coefficients in an explicit form for polynomials $M_k(t;x)$ orthogonal with respect to the weight function

$$t \mapsto w^M(t;x) = \frac{\sin \pi x}{\sqrt{t}\big(\cosh \pi\sqrt{t} - \cos \pi x\big)} \quad \text{on} \quad \mathbb{R}_+ \quad (0 < x < 1). \tag{4.4}$$

Here, $\sin(\pi x)$ is a constant factor and it can be omitted.

Theorem 4.1. *The polynomials $\{M_k(t;x)\}_{k=0}^{+\infty}$ orthogonal with respect to the weight function $w^M(t;x)$, given by (4.4), satisfy the three-term recurrence relation*

$$M_{k+1}(t;x) = (t - \alpha_k^M(x))M_k(t;x) - \beta_k^M(x)M_{k-1}(t;x), \quad k = 0, 1, \ldots, \tag{4.5}$$

with $M_0(t;x)=1$ and $M_{-1}(t;x)=0$. The recurrence coefficients are

$$
\text{(4.6)}\qquad \begin{cases}
\alpha_0^M(x)=\dfrac{1}{3}x(2-x),\\[2mm]
\alpha_k^M(x)=\dfrac{32(k+1)k^3-8k^2(x-2)x-4k(x-1)^2+(x-2)x}{(4k-1)(4k+3)}, & k\in\mathbb{N};\\[2mm]
\beta_0^M(x)=2(1-x),\\[2mm]
\beta_k^M(x)=\dfrac{4k^2(2k-1)^2\left(4k^2-(1-x)^2\right)\left((2k-1)^2-(1-x)^2\right)}{(4k-3)(4k-1)^2(4k+1)}, & k\in\mathbb{N}.
\end{cases}
$$

In terms of polynomials (4.1), these polynomials can be expressed in the form

$$
\text{(4.7)}\qquad M_k(t;x)=G_{2k}^{(1-x)}\left(\sqrt{t}\,\right),\quad k=0,1,2,\ldots\,.
$$

Proof. According to (4.1), (4.2) and [11, Theorem 2.2.11] we conclude that $G_{2k}^{(1-x)}\left(\sqrt{t}\,\right)$ are monic polynomials orthogonal with respect to the weight function $t\mapsto w^G(\sqrt{t};1-x)/\sqrt{t}$ on $\mathbb{R}_+$, so that (4.7) holds.

Now, using [11, Theorem 2.2.12] we obtain the coefficients in the three-term recurrence relation (4.8). As usual, we put (cf. [11, p. 97])

$$
\beta_0^M(x)=\int_0^{+\infty}w^M(t;x)\,\mathrm{d}t=2(1-x).
$$

Thus, we have $\alpha_0^M(x)=B_1(1-x)=x(2-x)/3$, as well as

$$
\alpha_k^M(x)=B_{2k}(1-x)+B_{2k+1}(1-x)\quad\text{and}\quad \beta_k^M(x)=B_{2k-1}(1-x)B_{2k}(1-x),
$$

where the coefficients B_k are given in (4.3). These formulas give the desired results. $\square$

Remark 4.2. A few first polynomials $M_k(t;x)$ are

$$
\begin{aligned}
M_0(t;x)&=1,\\
M_1(t;x)&=t+\frac{1}{3}x(x-2),\\
M_2(t;x)&=t^2+\frac{2}{7}\left(3x^2-6x-10\right)t+\frac{3}{35}x\left(x^2-4\right)(x-4),\\
M_3(t;x)&=t^3+\frac{5}{11}\left(3x^2-6x-28\right)t^2+\frac{1}{11}\left(5x^4-20x^3-80x^2+200x+224\right)t\\
&\quad+\frac{5}{231}x\left(x^2-4\right)\left(x^2-16\right)(x-6),\\
M_4(t;x)&=t^4+\frac{28}{15}\left(x^2-2x-18\right)t^3+\frac{14}{39}\left(3x^4-12x^3-100x^2+224x+648\right)t^2\\
&\quad+\frac{4}{2145}\left(105x^6-630x^5-4830x^4+23520x^3+54824x^2-158368x-146112\right)t\\
&\quad+\frac{7}{1287}x\left(x^2-4\right)\left(x^2-16\right)\left(x^2-36\right)(x-8),
\end{aligned}
$$

etc.

As an additional result, which will not be of interest in our summation of trigonometric series, we can prove the following statement:

Theorem 4.2. *The polynomials* $\{N_k(t;x)\}_{k=0}^{+\infty}$ *orthogonal with respect to the weight function*

$$t \mapsto w^N(t;x) = \frac{\sin \pi x \sqrt{t}}{\cosh \pi\sqrt{t} - \cos \pi x} \quad on \quad \mathbb{R}_+ \quad (0 < x < 1)$$

satisfy the three-term recurrence relation

$$N_{k+1}(t;x) = (t - \alpha_k^N(x))N_k(t;x) - \beta_k^N(x)N_{k-1}(t;x), \quad k = 0,1,\ldots, \tag{4.8}$$

with $N_0(t;x) = 1$ *and* $N_{-1}(t;x) = 0$*. The recurrence coefficients are*

$$\alpha_0^N(x) = \frac{1}{5}\left(-3x^2 + 6x + 4\right),$$

$$\alpha_k^N(x) = \frac{32(k+3)k^3 - 8k^2\left(x^2 - 2x - 12\right) - 12k(x-3)(x+1) - 3x^2 + 6x + 4}{(4k+1)(4k+5)},$$

$$\beta_k^N(x) = \frac{4k^2(2k+1)^2\left(4k^2 - (1-x)^2\right)\left((2k+1)^2 - (1-x)^2\right)}{(4k-1)(4k+1)^2(4k+3)}$$

for each $k \in \mathbb{N}$*. In terms of polynomials* (4.1)*, the polynomials* $N_k(t;x)$ *can be expressed in the form* $N_k(t;x) = G_{2k+1}^{(1-x)}\left(\sqrt{t}\right)/\sqrt{t}$ *for each* $k \in \mathbb{N}_0$*.*

The coefficient $\beta_0^N(x)$ may be arbitrary, because it multiplies $N_{-1}(t;x) = 0$, but usually, it is appropriate to take

$$\beta_0^N(x) = \int_0^{+\infty} w^N(t;x)\,\mathrm{d}t = \frac{8}{\pi^3}\,\mathrm{Im}\left\{\mathrm{Li}_3(\mathrm{e}^{\mathrm{i}\pi x})\right\}.$$

In the sequel we consider the Gaussian quadrature formula with respect to the weight function $w^M(t;x)$ on $\mathbb{R}_+$,

$$\int_0^{+\infty} g(t)w^M(t;x)\,\mathrm{d}t = \sum_{\nu=1}^{N} A_\nu^{(N)}(x)g\left(\tau_\nu^{(N)}(x)\right) + R_N(g;x), \tag{4.9}$$

where $R_N(g;x)$ is the corresponding remainder term ($g \in \mathcal{P}_{2n-1}$). As we mentioned in the previous section, the parameters of the quadrature formula (4.9), the nodes $\tau_\nu^{(N)}(x)$ and the weight coefficients $A_\nu^{(N)}(x)$, can be calculated very easy from the symmetric tridiagonal Jacobi matrix $J_N(w^M(\,\cdot\,;x))$ by the Golub-Welsch procedure. It is also implemented in the package `OrthogonalPolynomials` by the function `aGaussianNodesWeights`. Taking the recursion coefficients $\alpha_k^M(x)$ and $\beta_k^M(x)$, $k = 0,1,\ldots,n-1$, defined before in (4.6), we can calculate nodes and weights in (4.9) for a given x and any $N \le n$.

For calculating values of the series $S(x)$, presented in the form

$$S(x) = \sum_{k=1}^{+\infty} a_k \sin k\pi x = \frac{\pi}{4}\int_0^{+\infty} \frac{\sin \pi x}{\sqrt{t}\left(\cosh \pi\sqrt{t} - \cos \pi x\right)} f\left(\pi\sqrt{t}\right)\mathrm{d}t, \tag{4.10}$$

we use the quadrature rule (4.9). Thus, we approximate $S(x)$ by the quadrature sum $Q_N(f;x)$, where

$$Q_N(f;x) = \frac{\pi}{4}\sum_{\nu=1}^{N} A_\nu^{(N)}(x) f\left(\xi_\nu^{(N)}(x)\right) \tag{4.11}$$

and $\xi_\nu^{(N)}(x) = \pi\sqrt{\tau_\nu^{(N)}(x)}$, $\nu = 1,\ldots,N$. The corresponding (relative) error is given by

$$E_N(x) = \left|\frac{Q_N(f;x) - S(x)}{S(x)}\right|. \tag{4.12}$$

5. Numerical examples

Through two examples we illustrate the efficiency of our methods. All computations were performed in Mathematica, Ver. 12, on MacBook Pro (15-inch, 2017), OS X 10.14.6.

Example 5.1. We consider the following series

$$C(x) = \sum_{k=1}^{+\infty} \frac{k}{4k^2-1}\cos k\pi x \quad \text{and} \quad S(x) = \sum_{k=1}^{+\infty} \frac{k}{4k^2-1}\sin k\pi x.$$

The sum of the first series is given by (cf. [22, p. 731])

$$C(x) = -\frac{1}{4} - \frac{1}{4}\cos\frac{\pi x}{2}\log\left|\tan\frac{\pi x}{4}\right|,$$

while for the sinus series, one can find that $S(x) = (\pi/8)\cos(\pi x/2)$.

Since

$$f(t) = \mathcal{L}^{-1}\left[\frac{s}{4s^2-1}\right] = \frac{1}{4}\cosh\frac{t}{2},$$

using (2.2) and the corresponding Gaussian rules with respect to the weights $w_R(t;x)$ and $w_I(t;x)$, we have

$$C(x) = \frac{\pi}{8}\sum_{\nu=1}^{N} A_\nu^R \cosh\Big(\frac{\pi\tau_\nu^R}{2}\Big) + R_N^R(x) \quad \text{and} \quad S(x) = \frac{\pi}{8}\sum_{\nu=1}^{N} A_\nu^I \cosh\Big(\frac{\pi\tau_\nu^I}{2}\Big) + R_N^I(x),$$

respectively.

TABLE 1. Relative errors $E_N^I(x)$ and $E_N^R(x)$, when $N = 5, 10, 15, 20$ and $x = 0.1(0.1)0.7$

rel. err.	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$
$E_5^I(x)$	1.10(−5)	2.03(−5)	2.84(−5)	3.54(−5)	4.13(−5)	4.62(−5)	5.00(−5)
$E_{10}^I(x)$	2.40(−10)	4.51(−10)	6.39(−10)	8.06(−10)	1.10(−15)	1.07(−9)	1.16(−9)
$E_{15}^I(x)$	4.78(−15)	9.07(−15)	1.29(−14)	1.64(−14)	1.10(−15)	2.19(−14)	2.39(−14)
$E_{20}^I(x)$	9.14(−20)	1.74(−19)	2.50(−19)	3.17(−19)	1.88(−15)	4.25(−19)	4.64(−19)
$E_5^R(x)$	2.88(−5)	6.78(−5)	9.34(−5)	2.67(−4)	1.09(−8)	2.57(−5)	3.58(−5)
$E_{10}^R(x)$	6.40(−10)	1.15(−9)	9.49(−10)	2.97(−9)	5.16(−17)	7.74(−10)	8.14(−10)
$E_{15}^R(x)$	1.22(−14)	2.25(−14)	4.91(−14)	8.44(−14)	2.23(−25)	1.16(−14)	1.66(−14)
$E_{20}^R(x)$	5.86(−19)	4.23(−19)	8.73(−19)	1.33(−18)	9.12(−34)	2.23(−19)	3.21(−19)

The relative errors in these quadrature sums are given by

$$E_N^R(x) = \left|\frac{R_N^R(x)}{C(x)}\right| = \left|\frac{1}{C(x)}\left(\frac{\pi}{8}\sum_{\nu=1}^{N} A_\nu^R \cosh\Big(\frac{\pi\tau_\nu^R}{2}\Big) - C(x)\right)\right|$$

and

$$E_N^I(x) = \left|\frac{R_N^I(x)}{S(x)}\right| = \left|\frac{1}{S(x)}\left(\frac{\pi}{8}\sum_{\nu=1}^{N} A_\nu^I \cosh\Big(\frac{\pi\tau_\nu^I}{2}\Big) - S(x)\right)\right|.$$

For getting recurrence parameters in the three-term recurrence relations for the polynomials $p_k^R(t;x)$ and $p_k^I(t;x)$, $k = 0, 1, \ldots$, we apply the procedure described Section 3. In order to save space the relative errors are given only at the points $x = j/10$, $j = 1, \ldots, 7$, for $N = 5, 10, 15$ and 20 nodes in the Gaussian rules (see Table 1). Numbers in parentheses indicate the decimal exponents, e.g., $1.10(-5)$ means 1.10×10^{-5}. For $N = 5$ we use the symbolic construction of recurrence coefficients, while for $N > 5$ we use numerical construction with the `WorkingPrecision -> 50`, with repetitions for each x.

As we mention before, the existence of the orthogonal polynomials $p_k^R(t;x)$, as well as the corresponding Gaussian formulas, are not guaranteed for $0 \leq x < 1/2$. However, the obtained numerical results of $E_N^R(x)$, $N = 5(5)20$, for selected values of $x \in \{0.1, 0.2, 0.3, 0.4\}$ show the existence of such quadrature rules, as well as a fast convergence.

Example 5.2. Now we consider the series

$$S(x) = \sum_{k=1}^{+\infty} \frac{\sin k\pi x}{(1+k^2)^{1/2}}, \qquad 0 < x < 1.$$

With $S_n(x)$ we denote the n-th partial sum, and by $e_n(x)$ its relative error, i.e.,

$$e_n(x) = \left| \frac{S_n(x) - S(x)}{S(x)} \right|.$$

The partial sums $S_n(x)$ are displayed in Figure 4 for $n = 5, 10$, and 50. As we can observe their convergence is very slow.

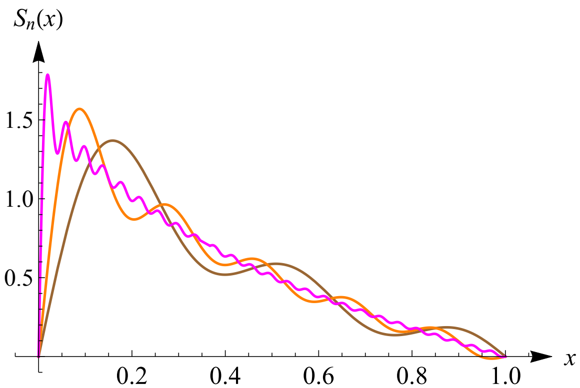

FIGURE 4. Partial sums $S_n(x)$ for $n = 5$ (brown line), $n = 10$ (orange line), and $n = 50$ (magenta line)

Now we apply our method for summing trigonometric series. Using Lemma 2.2, we can identify $F(s) = (1+s^2)^{-1/2}$ and $f(t) = J_0(t)$, where J_0 is the Bessel function. Then, according to (4.10) and (4.11), we have

$$\begin{aligned} S(x) = \sum_{k=1}^{+\infty} \frac{\sin k\pi x}{(1+k^2)^{1/2}} &= \frac{\pi}{4} \int_0^{+\infty} \frac{\sin \pi x}{\sqrt{t}\,(\cosh \pi\sqrt{t} - \cos \pi x)} J_0\big(\pi\sqrt{t}\,\big)\, \mathrm{d}t \\ &\approx \frac{\pi}{4} \sum_{\nu=1}^{N} A_\nu^{(N)}(x) J_0\big(\xi_\nu^{(N)}(x)\big). \end{aligned}$$

This quadrature process converges fast, because

$$t \mapsto J_0(\pi\sqrt{t}) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{\pi}{2}\right)^{2m} t^m$$

is an entire function.

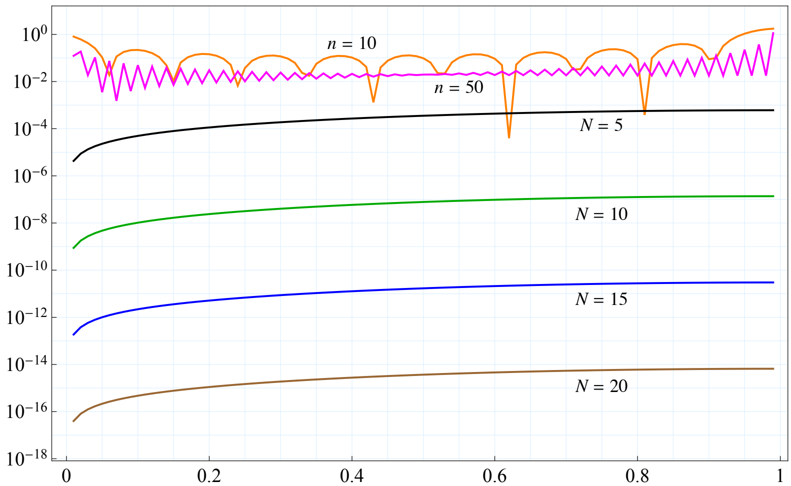

FIGURE 5. The relative errors $e_n(x)$ in the partial sums $S_n(x)$ for $n = 10$ and $n = 50$ and the relative errors $E_N(x)$ in the quadrature sums $Q_N(f;x)$ for $N = 5(5)20$

The relative errors $E_N(x)$ of quadrature approximation $Q_N(f;x)$ are given by (4.12), and presented in Figure 5 in log-scale for $N = 5, 10, 15$, and 20. Indeed, the convergence of $Q_N(f;x)$ is fast. As we can see, the five-point Gaussian formula gives three to five exact decimal digits (depending of x) of the sum $S(x)$, and the quadrature formula with $n = 20$ nodes gives an accuracy to more than 14 decimal digits.

TABLE 2. Relative errors $e_n(x)$ and $E_N(x)$, when $n = 100$ and 500 and $N = 5, 10, 20, 50$, for some selected values of x in $(0,1)$

x	$e_{100}(x)$	$e_{500}(x)$	$E_5(x)$	$E_{10}(x)$	$E_{20}(x)$	$E_{50}(x)$
0.1	2.49(−2)	4.99(−3)	4.96(−5)	1.03(−8)	4.72(−16)	4.78(−38)
0.2	1.51(−2)	3.03(−3)	1.13(−4)	2.39(−8)	1.10(−15)	1.13(−37)
0.3	1.12(−2)	2.41(−3)	1.88(−4)	4.03(−8)	1.88(−15)	1.93(−37)
0.4	1.06(−2)	2.12(−3)	2.70(−4)	5.87(−8)	2.76(−15)	2.84(−37)
0.5	9.87(−3)	1.97(−3)	3.54(−4)	7.80(−8)	3.68(−15)	3.81(−37)
0.6	9.44(−3)	1.89(−3)	4.35(−4)	9.66(−8)	4.59(−15)	4.76(−37)
0.7	9.18(−3)	1.84(−3)	5.06(−4)	1.13(−7)	5.39(−15)	5.61(−37)
0.8	9.04(−3)	1.81(−3)	5.61(−4)	1.26(−7)	6.03(−15)	6.28(−37)
0.9	8.96(−3)	1.79(−3)	5.96(−4)	1.34(−7)	6.44(−15)	6.71(−37)

Also, the relative errors $e_n(x)$ in the partial sums $S_n(x)$ for $n = 10$ and $n = 50$ are shown in the same figure. Numerical values of the errors $e_n(x)$ in the partial sums $S_n(x)$

with $n = 100$ and $n = 500$ terms are given in the second and third column of Table 2 for equidistant values of x $(= 0.1, 0.2, \dots, 0.9)$. We note that the number of exact digits in partial sums does not exceed three.

The numerical values of the corresponding relative errors $E_N(x)$ in the quadrature approximations $Q_N(f; x)$, with $N = 5, 10, 20, 50$ nodes, at the same values of x are given in the other columns of the same table. We note that the quadrature approximation $Q_{50}(f; x)$ has about 37 exact decimal digits! As an exact value of $S(x)$ we use one obtained by the Gaussian quadrature formula with $N = 100$ nodes.

6. Conclusion

In conclusion we can say that the method presented in Section 3 is general for the both series $C(x)$ and $S(x)$, but for the sinus-series $S(x)$ the method presented in 4 is much simpler in applications, because in that case we have the recurrence relation (4.8), with recurrence coefficients in the explicit form (4.6).

7. Acknowledgment

The author was supported in part by the Serbian Academy of Sciences and Arts, Project No. $\Phi - 96$.

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Serbian Academy of Sciences and Arts
11000 Belgrade, Serbia
and
University of Niš
Faculty of Sciences and Mathematics
P.O. Box 224, 18000 Niš, Serbia
ORCID: 0000-0002-3255-8127
E-mail address: gvm@mi.sanu.ac.rs