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SYMBOLIC COMPUTATION OF RECURRENCE COEFFICIENTS FOR POLYNOMIALS ORTHOGONAL WITH RESPECT TO THE SZEGŐ-BERNSTEIN WEIGHTS

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Dedicated to occasion of 60th birthday of Prof. Vijay Gupta

Abstract. The coefficients in the three-term recurrence relation for monic orthogonal polynomials with respect to the Szegő-Bernstein weight functions $w_\nu(x) = W(x)/(c-x)^\nu$, $\nu \ge 1$, on $(-1,1)$ are obtained in the explicit form for all Chebyshev cases, i.e., when $W(x)$ is $(1-x^2)^{\mp 1/2}$, $\sqrt{(1+x)/(1-x)}$ and $\sqrt{(1-x)/(1+x)}$. Chebyshev's method of modified moments is used, as well as the Mathematica package `OrthogonalPolinomials` developed in [*Facta Univ. Ser. Math. Inform.* **19** (2004), 17-36] and [*Math. Balkanica* **26** (2012), 169–184].

1. **Introduction.** In a joint paper with Mastroianni [10] (see also [11, §5.1.7]) we considered an integration of (2π)-periodic functions on the real line with respect to an even rational weight function, developing a transformation method for reducing such integrals to the integration over the finite interval $(-1,1)$ with respect to Szegő-Bernstein weight functions

$$w_\nu(x) = \frac{1}{(c-x)^\nu}\frac{1}{\sqrt{1-x^2}} \qquad (\nu \ge 1), \tag{1.1}$$

where $\nu \in \mathbb{N}$ and $c = \cosh b$, $b > 0$. For the sequence of monic polynomials $\{\pi_{k,\nu}\}_{k=0}^{\infty}$, orthogonal with respect to the weight function w_ν, which satisfy the three-term recurrence relation

$$\begin{aligned}\pi_{k+1,\nu}(x) &= (x-\alpha_k)\pi_{k,\nu}(x) - \beta_k \pi_{k-1,\nu}(x), \quad k = 0,1,\dots, \\ \pi_{0,\nu}(x) &= 1, \quad \pi_{-1,\nu}(x) = 0,\end{aligned} \tag{1.2}$$

the coefficients $\alpha_k = \alpha_{k,\nu}$ and $\beta_k = \beta_{k,\nu}$, $k = 0,1,\dots$, were determined in [10] in analytic form for $\nu = 1,2,3$. Otherwise, it is well known that the polynomials $\pi_{k,\nu}$ can be calculated explicitly provided $\nu < 2k$ (cf. [15, p. 31]). Also, there is a nonlinear algorithm to produce the recursion coefficients $\alpha_{k,\nu}$ and $\beta_{k,\nu}$ for polynomials $\pi_{k,\nu}$ in terms of ones for the polynomials $\pi_{k-1,\nu}$. However, such an algorithm is quite numerically unstable unless c is very close to 1 (cf. [6, p. 102]). In [4] Fischer and Golub also discussed two numerical algorithms for this purpose. In connection

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with the Kronrod extensions of Gaussian quadrature formulas, Gautschi and Notaris [9] and Notaris [13] studied weight functions on $(-1,1)$ consist of any one of the four Chebyshev weights divided by an arbitrary positive quadratic polynomial on $(-1,1)$.

The orthogonal polynomials and their three-term recurrence relation for the generalized Marchenko-Pastur measure are recently obtained in explicit form, analytically as well as symbolically using MATHEMATICA, by Gautschi and Milovanović [8]. Special cases involve Chebyshev polynomials of all four kinds. Otherwise, the Marchenko-Pastur probability measure is appears in the asymptotic theory of large random matrices. Here, we also mention a very recent survey paper by Wong and Zhao [16] on some new developments in asymptotics in the last fifty years, including the Riemann-Hilbert approach and more than a hundred references. The advantage of this method is that it can be applied to orthogonal polynomials that do not satisfy any difference or differential equations or have any integral representations.

In this paper we consider the four Chebyshev weight functions modified by the factor like in (1.1), i.e.,

$$w_\nu^T(x) = \frac{1}{(c-x)^\nu}\frac{1}{\sqrt{1-x^2}}, \tag{1.3}$$

$$w_\nu^U(x) = \frac{1}{(c-x)^\nu}\sqrt{1-x^2}, \tag{1.4}$$

$$w_\nu^V(x) = \frac{1}{(c-x)^\nu}\sqrt{\frac{1+x}{1-x}}, \tag{1.5}$$

$$w_\nu^W(x) = \frac{1}{(c-x)^\nu}\sqrt{\frac{1-x}{1+x}}, \tag{1.6}$$

where $c = \cosh b > 1$ and $\nu \geq 1$, and the upper indices correspond to the Chebyshev cases of the first (T), second (U), third (V), and fourth (W) kind. Our aim is to obtain the coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials in a symbolic form. For this purpose we use our symbolic/variable MATHEMATICA package `OrthogonalPolynomials` (see [3, 12]).

The paper is organized as follows. In Section 2 we compute the modified moments for each of cases (1.3), (1.4), (1.5) and (1.6), with respect to the monic Chebyshev polynomials of the first, second, third and fourth kind, respectively. Section 3 is devoted to application of methods of modified moments for finding coefficients in the three-term recurrence relation (1.2). Also, in this section we present the obtained results for the recurrence coefficients in all four Chebyshev's cases.

2. **Modified moments for Chebyshev's weights.**

For calculating recurrence coefficients in the three-term relation of the form (1.2) we will use the Chebyshev method of modified moments developed by Gautschi [5] (see also [7, pp. 76–78], [11, pp. 160–162]).

The modified moments $m_{k,\nu}^P$ for the weight functions (1.3), (1.4), (1.5) and (1.6) are given by

$$m_{k,\nu}^Q = \int_{-1}^{1} \widehat{Q}_k(x) w_\nu^Q(x)\,\mathrm{d}x, \quad k = 0, 1, \ldots, \tag{2.1}$$

where the upper index $Q \in \{T, U, V, W\}$, and $\widehat{Q}_k(x)$ are monic Chebyshev polynomials satisfying

$$\widehat{Q}_{k+1}(x) = (x - a_k)\widehat{Q}_k(x) - b_k\widehat{Q}_{k+1}(x), \quad k = 0, 1, \ldots,$$

$$\widehat{Q}_0(x) = 1, \quad \widehat{Q}_{-1}(x) = 0,$$

with the coefficients $a_k = 0$ in all cases, except $a_0 = 1/2$ and $-1/2$ for the monic polynomials of the third and fourth kind ($\widehat{V}_k$ and $\widehat{W}_k$), respectively. For $k \geq 1$ the coefficients $b_k = 1/4$ in all cases, except $b_1 = 1/2$ for the monic Chebyshev polynomials of the first kind $\widehat{T}_k$.

Although the coefficient b_0 can be arbitrary, it is convenient to take it as

$$b_0 = \int_{-1}^{1} w_0^Q(x)\,\mathrm{d}x = \begin{cases} \pi, & \text{for } Q = T, \\ \pi/2, & \text{for } Q = U, \\ \pi, & \text{for } Q = V, \\ \pi, & \text{for } Q = W. \end{cases} \tag{2.2}$$

In the sequel we consider four Chebyshev's cases. With $(\lambda)_k$ we denote the well known Pochhammer symbol defined for $\lambda \in \mathbb{C}$ by [14]

$$(\lambda)_k = \begin{cases} 1, & k = 0, \\ \lambda(\lambda+1)\ldots(\lambda+k-1), & k \in \mathbb{N}. \end{cases}$$

In terms of well-known Gamma function, it is written as

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} \qquad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

2.1. **Chebyshev weight of the first kind.**

Theorem 2.1. *Let $\nu \in \mathbb{N}$, $b > 0$, $k \in \mathbb{N}_0$,*

$$C(\nu, b) = \frac{\pi}{2^{\nu-2}(\nu-1)!\sinh^{2\nu-1} b} \tag{2.3}$$

and

$$q_s(k, \nu) = (k - \nu + 1)_s (k + s + 1)_{\nu-1-s}, \quad s = 0, 1, \ldots, \nu - 1. \tag{2.4}$$

Then the modified moments (2.1) for the first Chebyshev weight are given by

$$m_{0,\nu}^T = \frac{1}{2} C(\nu, b) \sum_{s=0}^{\nu-1} (-1)^s \binom{\nu-1}{s} q_s(0, \nu) \mathrm{e}^{(\nu-1-2s)b}$$

and

$$m_{k,\nu}^T = C(\nu, b) \frac{\mathrm{e}^{-kb}}{2^k} \sum_{s=0}^{\nu-1} (-1)^s \binom{\nu-1}{s} q_s(k, \nu) \mathrm{e}^{(\nu-1-2s)b}, \quad k \geq 1.$$

Proof. We consider (2.1), when $Q = T$ and $\widehat{Q}_k(x) = \widehat{T}_k(x)$. Putting $x = \cos\theta$, we have $\widehat{T}_0(\cos\theta) = 1$ and $\widehat{T}_k(\cos\theta) = 2^{1-k}\cos(k\theta)$, as well as

$$m_{k,\nu}^T = \frac{1}{2^{k-1}} \int_0^{\pi} \frac{\cos k\theta}{(\cosh b - \cos\theta)^\nu} \mathrm{d}\theta = \frac{1}{2^k} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}k\theta}}{(\cosh b - \cos\theta)^\nu} \mathrm{d}\theta \tag{2.5}$$

for $k \geq 1$. For $k = 0$ an additional factor $1/2$ is needed.

By a change of variables $z = e^{i\theta}$, the integral (2.5) reduces to a complex integral over the unit circle $|z| = 1$,

$$m_{k,\nu}^T = \frac{(-1)^\nu}{2^{k-\nu}\,i} \oint_{|z|=1} \frac{z^{k+\nu-1}}{(z^2 - 2\cosh bz + 1)^\nu}\, dz,$$

which can be calculated by the Cauchy Residue Theorem as

$$\begin{aligned} m_{k,\nu}^T &= \frac{(-1)^\nu}{2^{k-\nu}\,i} 2\pi i \frac{1}{(\nu-1)!} \cdot \frac{d^{\nu-1}}{dz^{\nu-1}} \left[\frac{z^{k+\nu-1}}{(z-e^b)^\nu}\right]_{z=e^{-b}} \\ &= C(\nu,b)\frac{e^{-kb}}{2^k} \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} (k+j+1)_{\nu-1-j} (\nu)_j e^{(\nu-1-2j)b} \left(1-e^{-2b}\right)^{\nu-1-j}, \end{aligned}$$

where $C(\nu, b)$ given by (2.3).

Using the binomial expansion

$$\begin{aligned} \left(1-e^{-2b}\right)^{\nu-1-j} &= \sum_{s=0}^{\nu-1-j} (-1)^s \binom{\nu-1-j}{s} e^{-2sb} \\ &= \sum_{s=j}^{\nu-1} (-1)^{s+j} \binom{\nu-1-j}{s-j} e^{-2(s-j)b}, \end{aligned}$$

as well as the property in the summation proccess

$$\sum_{j=0}^{\nu-1} A_j \sum_{s=j}^{\nu-1} B_{j,s} = \sum_{s=0}^{\nu-1} \sum_{j=0}^{s} A_j B_{j,s},$$

we obtain that

$$\begin{aligned} m_{k,\nu}^T = C(\nu,b)\frac{e^{-kb}}{2^k} \sum_{s=0}^{\nu-1} &\left\{ \sum_{j=0}^{s} (-1)^j \binom{\nu-1}{j} (k+j+1)_{\nu-1-j} (\nu)_j \binom{\nu-1-j}{\nu-1-s} \right\} \\ &\times (-1)^s e^{(\nu-1-2s)b}. \end{aligned} \tag{2.6}$$

For internal sum we have

$$\begin{aligned} &\sum_{j=0}^{s} (-1)^j \binom{\nu-1}{j} (k+j+1)_{\nu-1-j} (\nu)_j \binom{\nu-1-j}{\nu-1-s} \\ &= \binom{\nu-1}{s} (k+1)_{\nu-1}\, {}_2F_1(-s, \nu; k+1; 1), \end{aligned} \tag{2.7}$$

where ${}_2F_1$ is the well-known Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{j+0}^{+\infty} \frac{(a)_j (b)_j}{(c)_j} \cdot \frac{x^j}{j!}.$$

Since (cf. [1, p. 67])

$${}_2F_1(-s, \nu; k+1; 1) = \frac{(k+1-\nu)_s}{(k+1)_s},$$

the equality (2.7) becomes

$$\sum_{j=0}^{s} (-1)^j \binom{\nu-1}{j} (k+j+1)_{\nu-1-j} (\nu)_j \binom{\nu-1-j}{\nu-1-s} = \binom{\nu-1}{s} q_s(k,\nu),$$

where $q_s(k,\nu)$ is defined by (2.4).

Finally, (2.6) reduces to

$$m_{k,\nu}^T = C(\nu,b)\frac{\mathrm{e}^{-kb}}{2^k}\sum_{s=0}^{\nu-1}(-1)^s\binom{\nu-1}{s}q_s(k,\nu)\mathrm{e}^{(\nu-1-2s)b}.$$

It holds for $k \ge 1$. As we mentioned before the expression for $m_{0,\nu}^T$ requires an additional factor $1/2$. □

Remark 1. According to Mastroianni and Milovanović [10, Lemma 3.1] the moment $m_{0,\nu}^T$ can be expressed also in terms of the Legendre polynomials

$$m_{0,\nu}^T = \frac{\pi}{\sinh^\nu b}P_{\nu-1}(\coth b). \tag{2.8}$$

In order to get connection with the case $\nu = 0$ (Chebyshev weight of the first kind) it is convenient to put $P_{-1}(x) = 1$, and then (2.8) gives $m_{0,0}^T = \pi$.

Remark 2. It would be interesting to find a similar form for $m_{k,\nu}^T$ if any for an arbitrary $k \in \mathbb{N}$. On the other side, such compact formulas for the modified moments will not be useful in our computation of the recurrence coefficients, using the MATHEMATICA package `OrthogonalPolynomials` [3], [12]. However, formulas given in Theorem 2.1 are suitable for such a purpose, because they appear as linear combinations of exponential functions, excluding the constant $C(\nu,b)$.

2.2. Chebyshev weight of the second kind.

Theorem 2.2. *Let $\nu \in \mathbb{N}$, $b > 0$, $k \in \mathbb{N}_0$, and let $C(\nu,b)$ and $q_s(k,\nu)$ be defined as in (2.3) and (2.4), respectively. Then the modified moments (2.1) for the second Chebyshev weight are given for each $k \in \mathbb{N}_0$ by*

$$m_{k,\nu}^U = C(\nu,b)\frac{\mathrm{e}^{-kb}}{2^{k+2}}\sum_{s=0}^{\nu-1}(-1)^s\binom{\nu-1}{s}\left[q_s(k,\nu) - \mathrm{e}^{-2b}q_s(k+2,\nu)\right]\mathrm{e}^{(\nu-1-2s)b}.$$

Proof. Here we consider (2.1), when $Q = U$ and $\widehat{Q}_k(x) = \widehat{U}_k(x)$.

Putting $x = \cos\theta$, we have $\widehat{U}_k(\cos\theta) = 2^{-k}\sin(k+1)\theta/\sin\theta$ for each $k \ge 0$, so that

$$m_{k,\nu}^U = \frac{1}{2^{k+1}}\int_\pi^\pi \frac{\sin(k+1)\theta\sin\theta}{(\cosh b - \cos\theta)^\nu}\mathrm{d}\theta = \frac{1}{2^{k+2}}\int_{-\pi}^\pi \frac{\mathrm{e}^{\mathrm{i}k\theta} - \mathrm{e}^{\mathrm{i}(k+2)\theta}}{(\cosh b - \cos\theta)^\nu}\mathrm{d}\theta,$$

i.e.,

$$m_{k,\nu}^U = \frac{1}{4}m_{k,\nu}^T - m_{k+2,\nu}^T.$$

Using Theorem 2.1 we obtain the desired result. □

2.3. Chebyshev weight of the third kind.

Theorem 2.3. *Let $\nu \in \mathbb{N}$, $b > 0$, $k \in \mathbb{N}_0$, and let $C(\nu,b)$ and $q_s(k,\nu)$ be defined as in (2.3) and (2.4), respectively. Then the modified moments (2.1) for the third Chebyshev weight are given for each $k \in \mathbb{N}_0$ by*

$$m_{k,\nu}^V = C(\nu,b)\frac{\mathrm{e}^{-kb}}{2^{k+1}}\sum_{s=0}^{\nu-1}(-1)^s\binom{\nu-1}{s}\left[\mathrm{e}^{-b}q_s(k+1,\nu) + q_s(k,\nu)\right]\mathrm{e}^{(\nu-1-2s)b}.$$

Proof. Here we consider (2.1), when $Q = V$ and $\widehat{Q}_k(x) = \widehat{V}_k(x)$. Since

$$\widehat{V}_k(x) = \frac{1}{2^k}\frac{\cos\left(k+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}, \quad x = \cos\theta,$$

we have

$$m_{k,\nu}^V = \frac{1}{2^k}\int_0^\pi \frac{\cos\left(k+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\cdot\frac{\sin\theta}{(\cosh b-\cos\theta)^\nu}\,\mathrm{d}\theta$$

$$= \frac{1}{2^{k+1}}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}(k+1)\theta}+\mathrm{e}^{\mathrm{i}k\theta}}{(\cosh b-\cos\theta)^\nu}\mathrm{d}\theta,$$

i.e.,

$$m_{k,\nu}^V = m_{k+1,\nu}^T + \frac{1}{2}m_{k,\nu}^T.$$

Using Theorem 2.1 we obtain the desired result. □

2.4. **Chebyshev weight of the forth kind.**

Theorem 2.4. *Let $\nu \in \mathbb{N}$, $b > 0$, $k \in \mathbb{N}_0$, and let $C(\nu, b)$ and $q_s(k,\nu)$ be defined as in (2.3) and (2.4), respectively. Then the modified moments (2.1) for the third Chebyshev weight are given for each $k \in \mathbb{N}_0$ by*

$$m_{k,\nu}^W = C(\nu,b)\frac{\mathrm{e}^{-kb}}{2^{k+1}}\sum_{s=0}^{\nu-1}(-1)^s\binom{\nu-1}{s}\left[q_s(k,\nu)-\mathrm{e}^{-b}q_s(k+1,\nu)\right]\mathrm{e}^{(\nu-1-2s)b}.$$

Proof. In this case we have

$$\widehat{Q}_k(x) = \widehat{W}_k(x) = \frac{1}{2^k}\frac{\sin\left(k+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \quad x = \cos\theta,$$

and

$$m_{k,\nu}^W = \frac{1}{2}m_{k,\nu}^T - m_{k+1,\nu}^T.$$

Using Theorem 2.1 we obtain the desired result. □

Remark 3. We mention that $\widehat{W}_k(x) = (-1)^k\widehat{V}_k(-x)$.

3. **Application of Chebyshev method of modified moments.** Using the modified moments given in the previous theorems, we are able to calculate coefficients in the corresponding recurrence relations of the form (1.2) for monic polynomials orthogonal on $(-1,1)$ with respect to the weight functions (1.3), (1.4), (1.5) and (1.6). In order to get a better stability in numerical construction of orthogonal polynomials, Walter Gautschi [5, §2.4] proposed the *method of modified moments* as a improvement of the original Chebyshev method of moments [2]. These methods are implemented in recent symbolic/variable-precision software for orthogonal polynomials (Gautschi's package in MATLAB and our MATHEMATICA package).

In this section we use our MATHEMATICA package `OrthogonalPolynomials` [3], [12], which is downloadable from the web site http://www.mi.sanu.ac.rs/~gvm/. When we have the (modified) moments in (linearized) analytic form, an application of this package in symbolic mode is very simple. We illustrate it in the case (1.3).

For obtaining the first n recurrence coefficients α_k and β_k, $k = 0, 1, \ldots, n-1$, in (1.2), using MATHEMATICA package `OrthogonalPolynomials`, in symbolic form for a given $\nu\,(\geq 1)$ we need the following commands:

```
<< orthogonalPolynomials`
aRecurT[n_,v_,b_]:=
  Module[{aa,bb,q,alpha,beta,s,k,n,mom,Cvb},
   Cvb=Pi/(2^(v-2)(v-1)! Sinh[b]^(2v-1))
   q[s_,k_]:=Pochhammer[k-v+1,s] Pochhammer[k+s+1,v-1-s];
   aa = Table[0, {k,0,2 n-1}];
   bb = Table[If[k == 0,Pi,If[k==1,1/2,1/4]], {k,0,2 n-1}];
   mom=Table[If[k == 0,1/2,1] Exp[-k b]/2^k
   Sum[(-1)^s Binomial[v-1,s]q[s,k]Exp[(v-1-2s)b],
        {s,0,v-1}], {k,0,2n-1}];
   {alpha, beta}=aChebyshevAlgorithmModified[mom,aa,bb,
         Algorithm -> Symbolic]//Simplify;
   beta[[1]]=Cvb beta[[1]];
   Return[{alpha, beta}]];
```

Remark 4. Without loss of generality, the common factor $C(\nu, b)$ (`Cvb`) can be eliminate (i.e., we can divide all moments by this factor, so that the first moment be 1). The obtained results should be corrected only in β_0 (`beta[[1]]`), because $\beta_0 = m^T_{0,\nu}$. This approach is applied in the previous `Module`.

`aChebyshevAlgorithmModified`-function returns the sequences of three-term recurrence coefficients (`alpha` and `beta`) of the length `n` in symbolic form, for a given sequence of modified moments `mod`. The sequences `aa` and `bb` represent the recurrence coefficients, in this case, of the monic Chebyshev polynomials of the first kind.

For other cases (1.4), (1.5) and (1.6), we need only to change the sequences `aa`, `bb` and `mod`.

In the sequel we present the obtained results for recurrence coefficients in each of the Chebyshev cases, as well as graphics of the corresponding recurrence coefficients when b runs over $(0, 4)$ (i.e., when $c \in (0, 27.31)$.

3.1. **Chebyshev weight of the first kind.** We present the obtained results for the recurrence coefficients $\alpha_k \equiv \alpha^T_{k,\nu}$ and $\beta_k \equiv \beta^T_{k,\nu}$, $k = 0, 1, \ldots$, and $\nu = 1, \ldots, 5$.

<u>Case $\nu = 1$</u>

$$\alpha_0 = \mathrm{e}^{-b}, \quad \alpha_1 = -\frac{1}{2}\mathrm{e}^{-b}, \quad \alpha_k = 0 \quad (k \ge 2);$$

$$\beta_0 = \frac{\pi}{\sinh b}, \quad \beta_1 = \frac{1}{2}\big(1 - \mathrm{e}^{-2b}\big), \quad \beta_k = \frac{1}{4} \quad (k \ge 2);$$

<u>Case $\nu = 2$</u>

$$\alpha_0 = \frac{2\mathrm{e}^{b}}{\mathrm{e}^{2b}+1}, \quad \alpha_1 = \frac{1-\mathrm{e}^{2b}}{\mathrm{e}^{3b}+\mathrm{e}^{b}}, \quad \alpha_k = 0 \quad (k \ge 2);$$

$$\beta_0 = \frac{\pi\cosh b}{\sinh^3 b}, \quad \beta_1 = \frac{\mathrm{e}^{-2b}\left(\mathrm{e}^{2b}-1\right)^3}{2\left(\mathrm{e}^{2b}+1\right)^2}, \quad \beta_2 = \frac{1}{4}\big(1+\mathrm{e}^{-2b}\big), \quad \beta_k = \frac{1}{4} \quad (k \ge 3);$$

<u>Case $\nu = 3$</u>

$$\alpha_0 = \frac{3\mathrm{e}^{b}\left(\mathrm{e}^{2b}+1\right)}{\mathrm{e}^{4b}+4\mathrm{e}^{2b}+1}, \quad \alpha_1 = \frac{\mathrm{e}^{-3b}+7\mathrm{e}^{-b}+7\mathrm{e}^{b}-3\mathrm{e}^{3b}}{2\big(\mathrm{e}^{4b}+4\mathrm{e}^{2b}+1\big)}, \quad \alpha_2 = -\frac{1}{2}\mathrm{e}^{-3b},$$

$$\alpha_k = 0 \quad (k \ge 3);$$

$$\beta_0 = \frac{\pi\left(\mathrm{e}^{-2b}+4+\mathrm{e}^{2b}\right)}{4\sinh^5 b}, \quad \beta_1 = \frac{\left(\mathrm{e}^{2b}-1\right)^4}{2\left(\mathrm{e}^{4b}+4\mathrm{e}^{2b}+1\right)^2},$$

$$\beta_2 = \frac{1}{4}\left(1+3\mathrm{e}^{-2b}-3\mathrm{e}^{-4b}-\mathrm{e}^{-6b}\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 3);$$

Case $\nu = 4$

$$\alpha_0 = \frac{4\mathrm{e}^b\left(\mathrm{e}^{4b}+3\mathrm{e}^{2b}+1\right)}{\mathrm{e}^{6b}+9\mathrm{e}^{4b}+9\mathrm{e}^{2b}+1}, \quad \alpha_1 = -\frac{2\mathrm{e}^b\left(1-4\mathrm{e}^{2b}-14\mathrm{e}^{4b}-4\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)}{\left(\mathrm{e}^{2b}+1\right)\left(\mathrm{e}^{4b}+1\right)\left(\mathrm{e}^{4b}+8\mathrm{e}^{2b}+1\right)},$$

$$\alpha_2 = \frac{2(1-\mathrm{e}^{2b})}{\mathrm{e}^{5b}+\mathrm{e}^b}, \quad \alpha_k = 0 \quad (k \geq 3);$$

$$\beta_0 = \frac{\pi\left(\mathrm{e}^{-3b}+9\mathrm{e}^{-b}+9\mathrm{e}^{b}+\mathrm{e}^{3b}\right)}{8\sinh^7 b}, \quad \beta_1 = \frac{\left(\mathrm{e}^{2b}-1\right)^4\left(\mathrm{e}^{4b}+1\right)}{2\left(\mathrm{e}^{6b}+9\mathrm{e}^{4b}+9\mathrm{e}^{2b}+1\right)^2},$$

$$\beta_2 = \frac{\mathrm{e}^{-4b}\left(\mathrm{e}^{2b}-1\right)^3\left(\mathrm{e}^{2b}+1\right)\left(\mathrm{e}^{4b}+8\mathrm{e}^{2b}+1\right)}{4\left(\mathrm{e}^{4b}+1\right)^2},$$

$$\beta_3 = \frac{1}{4}\left(1+\mathrm{e}^{-4b}\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 4);$$

Case $\nu = 5$

$$\alpha_0 = \frac{5\mathrm{e}^b\left(\mathrm{e}^{2b}+1\right)\left(1+5\mathrm{e}^{2b}+\mathrm{e}^{4b}\right)}{1+16\mathrm{e}^{2b}+36\mathrm{e}^{4b}+16\mathrm{e}^{6b}+\mathrm{e}^{8b}},$$

$$\alpha_1 = -\frac{5\mathrm{e}^b\left(\mathrm{e}^{2b}+1\right)\left(1-6\mathrm{e}^{2b}-45\mathrm{e}^{4b}-40\mathrm{e}^{6b}-45\mathrm{e}^{8b}-6\mathrm{e}^{10b}+\mathrm{e}^{12b}\right)}{2\left(1+\mathrm{e}^{2b}+6\mathrm{e}^{4b}+\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)\left(1+16\mathrm{e}^{2b}+36\mathrm{e}^{4b}+16\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)},$$

$$\alpha_2 = \frac{\mathrm{e}^{-5b}\left(1+\mathrm{e}^{2b}+11\mathrm{e}^{4b}+\mathrm{e}^{6b}+16\mathrm{e}^{8b}-10\mathrm{e}^{10b}\right)}{2\left(1+\mathrm{e}^{2b}+6\mathrm{e}^{4b}+\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)},$$

$$\alpha_3 = -\frac{1}{2}\mathrm{e}^{-5b}, \quad \alpha_k = 0 \quad (k \geq 4);$$

$$\beta_0 = \frac{\pi\left(\mathrm{e}^{-4b}+16\mathrm{e}^{-2b}+36+16\mathrm{e}^{2b}+\mathrm{e}^{4b}\right)}{16\sinh^9 b},$$

$$\beta_1 = \frac{\left(\mathrm{e}^{2b}-1\right)^4\left(1+\mathrm{e}^{2b}+6\mathrm{e}^{4b}+\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)}{2\left(1+16\mathrm{e}^{2b}+36\mathrm{e}^{4b}+16\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)^2},$$

$$\beta_2 = \frac{\left(\mathrm{e}^{2b}-1\right)^4\left(1+16\mathrm{e}^{2b}+36\mathrm{e}^{4b}+16\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)}{4\left(1+\mathrm{e}^{2b}+6\mathrm{e}^{4b}+\mathrm{e}^{6b}+\mathrm{e}^{8b}\right)^2},$$

$$\beta_3 = \frac{1}{4}\left(1+5\mathrm{e}^{-4b}-5\mathrm{e}^{-6b}-\mathrm{e}^{-10b}\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 4).$$

The recurrence coefficients α_k, $k = 0, 1, 2, \ldots$, and β_k, $k = 1, 2, \ldots$, for $\nu = 4$, $\nu = 5$ and $\nu = 10$ as functions of b are presented in Figures 1, 2, and 3, respectively.

3.2. **Chebyshev weight of the second kind.** Here we give the recurrence coefficients $\alpha_k \equiv \alpha^U_{k,\nu}$ and $\beta_k \equiv \beta^U_{k,\nu}$, $k = 0, 1, \ldots$, and $\nu = 1, \ldots, 5$.

Case $\nu = 1$

$$\alpha_0 = \frac{1}{2}\mathrm{e}^{-b}, \quad \alpha_k = 0 \quad (k \geq 1); \quad \beta_0 = \frac{\pi}{2}\frac{1-\mathrm{e}^{-2b}}{\sinh b} = \mathrm{e}^{-b}\pi, \quad \beta_k = \frac{1}{4} \quad (k \geq 1);$$

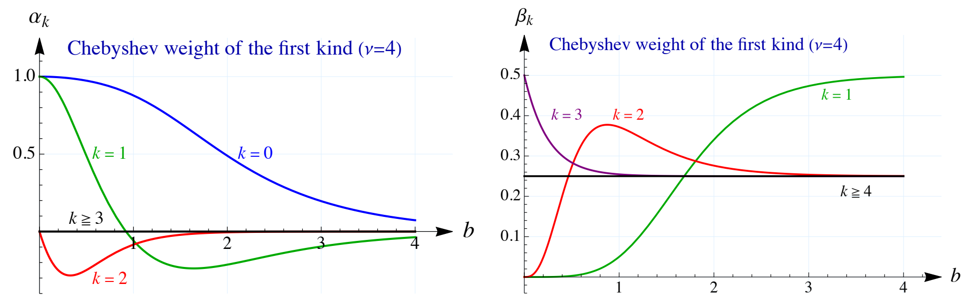

FIGURE 1. Three-term recurrence coefficients α_k (left) and β_k (right) for $\nu = 4$

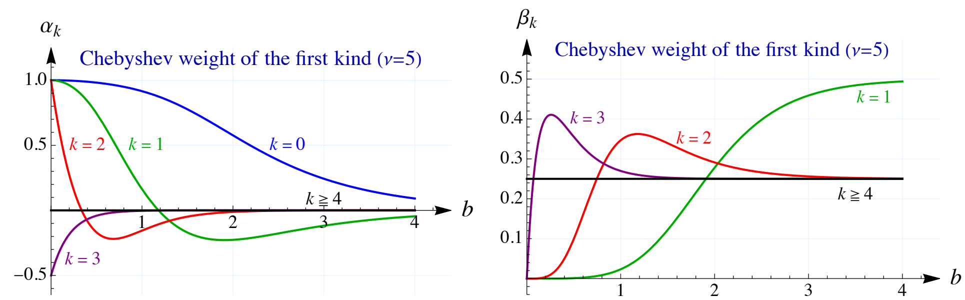

FIGURE 2. Three-term recurrence coefficients α_k (left) and β_k (right) for $\nu = 5$

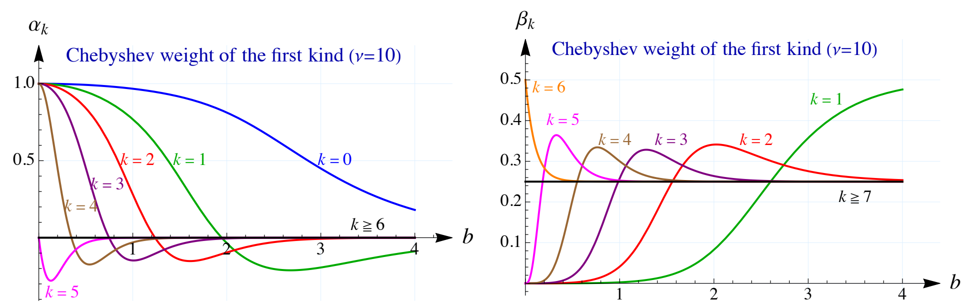

FIGURE 3. Three-term recurrence coefficients α_k (left) and β_k (right) for $\nu = 10$

Case $\nu = 2$

$$\alpha_0 = \mathrm{e}^{-b}, \quad \alpha_k = 0 \quad (k \geq 1);$$

$$\beta_0 = \frac{2\pi}{\mathrm{e}^{2b} - 1}, \quad \beta_1 = \frac{1}{4}\left(1 - \mathrm{e}^{-2b}\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 2);$$

Case $\nu = 3$

$$\alpha_0 = \frac{1}{2}\mathrm{e}^{-3b}\left(3\mathrm{e}^{2b} - 1\right), \quad \alpha_1 = \frac{1}{2}\mathrm{e}^{-3b}, \quad \alpha_k = 0 \quad (k \geq 2);$$

$$\beta_0 = \frac{4\pi e^{3b}}{\left(e^{2b}-1\right)^3}, \quad \beta_1 = \frac{1}{4}\left(1-e^{-2b}\right)^3, \quad \beta_k = \frac{1}{4} \quad (k \geq 2).$$

Case $\nu = 4$

$$\alpha_0 = \frac{2e^b}{e^{2b}+1}, \quad \alpha_1 = \frac{2}{e^{3b}+e^b}, \quad \alpha_k = 0 \quad (k \geq 2);$$

$$\beta_0 = \frac{8\pi e^{4b}\left(e^{2b}+1\right)}{\left(e^{2b}-1\right)^5}, \quad \beta_1 = \frac{e^{-4b}\left(e^{2b}-1\right)^4}{4\left(e^{2b}+1\right)^2}, \quad \beta_2 = \frac{1}{4}\left(1-e^{-4b}\right),$$

$$\beta_k = \frac{1}{4} \quad (k \geq 3);$$

Case $\nu = 5$

$$\alpha_0 = \frac{5e^b\left(e^{2b}+1\right)}{2\left(e^{4b}+3e^{2b}+1\right)}, \quad \alpha_1 = \frac{10e^b+4e^{-b}-3e^{-3b}-e^{-5b}}{2\left(e^{4b}+3e^{2b}+1\right)},$$

$$\alpha_2 = \frac{1}{2}e^{-5b}, \quad \alpha_k = 0 \quad (k \geq 3);$$

$$\beta_0 = \frac{16\pi e^{5b}\left(e^{4b}+3e^{2b}+1\right)}{\left(e^{2b}-1\right)^7}, \quad \beta_1 = \frac{\left(e^{2b}-1\right)^4}{4\left(e^{4b}+3e^{2b}+1\right)^2},$$

$$\beta_2 = \frac{1}{4}e^{-10b}\left(e^{2b}-1\right)^3\left(e^{4b}+3e^{2b}+1\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 3).$$

The recurrence coefficients α_k, $k = 0, 1, 2, \ldots$, and β_k, $k = 1, 2, \ldots$, for $\nu = 5$ and $\nu = 10$ are presented in Figures 4 and 5, respectively.

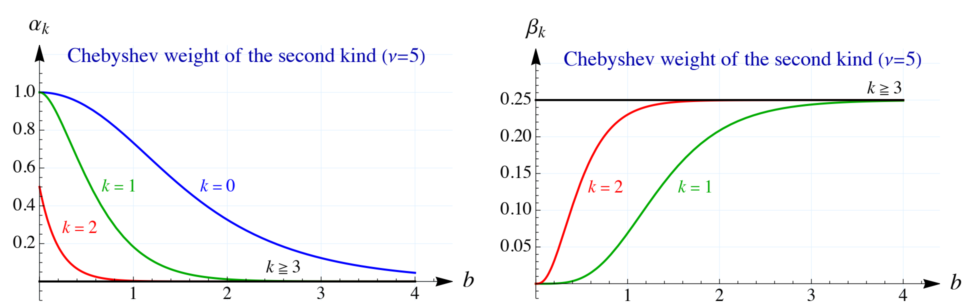

FIGURE 4. Three-term recurrence coefficients α_k (left) and β_k (right) for $\nu = 5$

3.3. **Chebyshev weight of the third kind.** Here we present the recurrence coefficients $\alpha_k \equiv \alpha^V_{k,\nu}$ and $\beta_k \equiv \beta^V_{k,\nu}$, $k = 0, 1, \ldots$, and $\nu = 1, \ldots, 5$.

Case $\nu = 1$

$$\alpha_0 = \frac{1}{2}\left(1+e^{-b}\right), \quad \alpha_k = 0 \quad (k \geq 1);$$

$$\beta_0 = \frac{2\pi}{e^b-1}, \quad \beta_1 = \frac{1}{4}\left(1-e^{-b}\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 2);$$

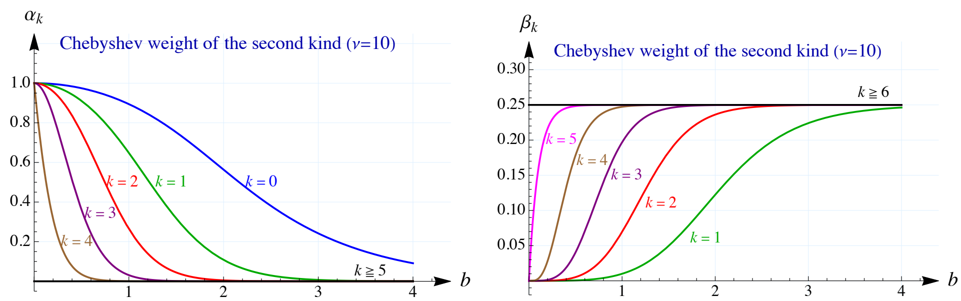

FIGURE 5. Three-term recurrence coefficients α_k (left) and β_k (right) for $\nu = 10$

Case $\nu = 2$

$$\alpha_0 = \frac{1}{2}\left(1 + 2e^{-b} - e^{-2b}\right), \quad \alpha_1 = \frac{1}{2}e^{-2b}, \quad \alpha_k = 0 \quad (k \geq 2);$$

$$\beta_0 = \frac{\pi}{2\sinh^3 b}e^{-b}\left(1+e^b\right)^2, \quad \beta_1 = \frac{1}{4}e^{-4b}\left(e^b - 1\right)^3\left(e^b+1\right),$$

$$\beta_k = \frac{1}{4} \quad (k \geq 2);$$

Case $\nu = 3$

$$\alpha_0 = \frac{1}{2}e^{-3b}\left(3e^{2b} - 1\right), \quad \alpha_1 = \frac{1}{2}e^{-3b}, \quad \alpha_k = 0 \quad (k \geq 2);$$

$$\beta_0 = \frac{4\pi e^{3b}}{\left(e^{2b}-1\right)^3}, \quad \beta_1 = \frac{1}{4}\left(1 - e^{-2b}\right)^3, \quad \beta_k = \frac{1}{4} \quad (k \geq 2);$$

Case $\nu = 4$

$$\alpha_0 = \frac{2e^b}{e^{2b}+1}, \quad \alpha_1 = \frac{2}{e^{3b}+e^b}, \quad \alpha_k = 0 \quad (k \geq 2);$$

$$\beta_0 = \frac{8\pi e^{4b}\left(e^{2b}+1\right)}{\left(e^{2b}-1\right)^5}, \quad \beta_1 = \frac{e^{-4b}\left(e^{2b}-1\right)^4}{4\left(e^{2b}+1\right)^2}, \quad \beta_2 = \frac{1}{4}\left(1 - e^{-4b}\right),$$

$$\beta_k = \frac{1}{4} \quad (k \geq 3);$$

Case $\nu = 5$

$$\alpha_0 = \frac{5e^b\left(e^{2b}+1\right)}{2\left(e^{4b}+3e^{2b}+1\right)}, \quad \alpha_1 = \frac{10e^b + 4e^{-b} - 3e^{-3b} - e^{-5b}}{2\left(e^{4b}+3e^{2b}+1\right)},$$

$$\alpha_2 = \frac{1}{2}e^{-5b}, \quad \alpha_k = 0 \quad (k \geq 3);$$

$$\beta_0 = \frac{16\pi e^{5b}\left(e^{4b}+3e^{2b}+1\right)}{\left(e^{2b}-1\right)^7}, \quad \beta_1 = \frac{\left(e^{2b}-1\right)^4}{4\left(e^{4b}+3e^{2b}+1\right)^2},$$

$$\beta_2 = \frac{1}{4}e^{-10b}\left(e^{2b}-1\right)^3\left(e^{4b}+3e^{2b}+1\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 3).$$

In Figure 6 we display α_k, $k = 0, 1, 2, \ldots$, and β_k, $k = 1, 2, \ldots$, for $\nu = 10$.

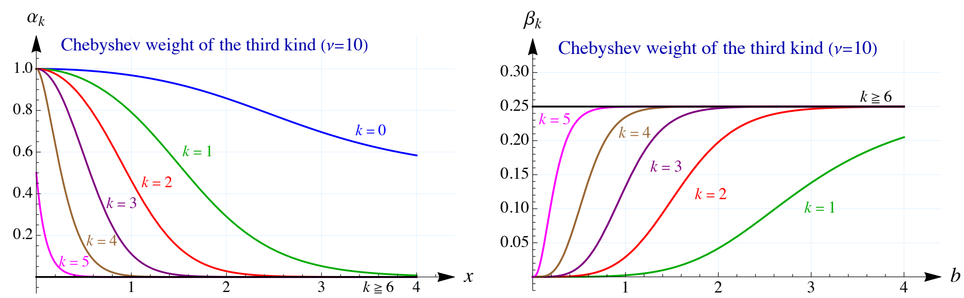

FIGURE 6. Three-term recurrence coefficients α_k (left) and β_k (right) for $\nu = 10$

3.4. **Chebyshev weight of the fourth kind.** Finally, we present here the recurrence coefficients $\alpha_k \equiv \alpha^W_{k,\nu}$ and $\beta_k \equiv \beta^W_{k,\nu}$, $k = 0, 1, \ldots$, and $\nu = 1, \ldots, 5$.

Case $\nu = 1$

$$\alpha_0 = \frac{1}{2}\big(-1 + \mathrm{e}^{-b}\big), \quad \alpha_k = 0 \quad (k \geq 1);$$

$$\beta_0 = \frac{2\pi}{1 + \mathrm{e}^b}, \quad \beta_1 = \frac{1}{4}\big(1 + \mathrm{e}^{-b}\big), \quad \beta_k = \frac{1}{4} \quad (k \geq 2);$$

Case $\nu = 2$

$$\alpha_0 = \frac{1}{2}\big(\mathrm{e}^{-2b} + 2\mathrm{e}^{-b} - 1\big), \quad \alpha_1 = -\frac{1}{2}\mathrm{e}^{-2b}, \quad \alpha_k = 0 \quad (k \geq 2);$$

$$\beta_0 = \frac{\pi}{2\sinh^2 b}\mathrm{e}^{-b}\big(\mathrm{e}^b - 1\big)^2, \quad \beta_1 = \frac{1}{4}\mathrm{e}^{-4b}\big(\mathrm{e}^{2b} - 1\big)\big(1 + \mathrm{e}^b\big)^2,$$

$$\beta_k = \frac{1}{4} \quad (k \geq 2);$$

Case $\nu = 3$

$$\alpha_0 = -\frac{\mathrm{e}^{2b} - 4\mathrm{e}^b + 1}{2\big(1 - \mathrm{e}^b + \mathrm{e}^{2b}\big)}, \quad \alpha_1 = -\frac{3}{2}\frac{1 - \mathrm{e}^{-b}}{1 - \mathrm{e}^b + \mathrm{e}^{2b}}, \quad \alpha_k = 0 \quad (k \geq 2);$$

$$\beta_0 = \frac{\pi}{4\sinh^5 b}\mathrm{e}^{-2b}\big(\mathrm{e}^b - 1\big)^2\big(1 - \mathrm{e}^b + \mathrm{e}^{2b}\big), \quad \beta_1 = \frac{\mathrm{e}^{-3b}\left(\mathrm{e}^b - 1\right)^3\left(\mathrm{e}^b + 1\right)^4}{4\left(\mathrm{e}^{2b} - \mathrm{e}^b + 1\right)^2},$$

$$\beta_2 = \frac{1}{4}\big(1 + \mathrm{e}^{-3b}\big), \quad \beta_k = \frac{1}{4} \quad (k \geq 2);$$

Case $\nu = 4$

$$\alpha_0 = -\frac{\mathrm{e}^{4b} - 6\mathrm{e}^{3b} + 6\mathrm{e}^{2b} - 6\mathrm{e}^b + 1}{2\big(\mathrm{e}^{4b} - 2\mathrm{e}^{3b} + 4\mathrm{e}^{2b} - 2\mathrm{e}^b + 1\big)},$$

$$\alpha_1 = \frac{\mathrm{e}^{-4b} - 2\mathrm{e}^{-3b} + 4\mathrm{e}^{-2b} + 2\mathrm{e}^{-b} - 7 + 12\mathrm{e}^b - 6\mathrm{e}^{2b}}{2\big(\mathrm{e}^{4b} - 2\mathrm{e}^{3b} + 4\mathrm{e}^{2b} - 2\mathrm{e}^b + 1\big)},$$

$$\alpha_2 = -\frac{1}{2}\mathrm{e}^{-4b}, \quad \alpha_k = 0 \quad (k \geq 3);$$

$$\beta_0 = \frac{\pi}{8\sinh^7 b}\mathrm{e}^{-3b}\big(\mathrm{e}^b - 1\big)^2\big(\mathrm{e}^{4b} - 2\mathrm{e}^{3b} + 4\mathrm{e}^{2b} - 2\mathrm{e}^b + 1\big),$$

$$\beta_1 = \frac{\left(e^{2b}-1\right)^4}{4\left(e^{4b}-2e^{3b}+4e^{2b}-2e^{b}+1\right)^2},$$

$$\beta_2 = \frac{1}{4}\left(1-e^{-8b}-4e^{-5b}+4e^{-3b}\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 3);$$

<u>Case $\nu = 5$</u>

$$\alpha_0 = -\frac{1-8e^{b}+14e^{2b}-24e^{3b}+14e^{4b}-8e^{5b}+e^{6b}}{2\left(1-3e^{b}+9e^{2b}-9e^{3b}+9e^{4b}-3e^{5b}+e^{6b}\right)},$$

$$\alpha_1 = -\frac{5e^{2b}\left(2-8e^{b}+15e^{2b}-20e^{3b}+15e^{4b}-8e^{5b}+2e^{6b}\right)}{2\left(1-e^{b}+e^{2b}-e^{3b}+e^{4b}\right)\left(1-3e^{b}+9e^{2b}-9e^{3b}+9e^{4b}-3e^{5b}+e^{6b}\right)},$$

$$\alpha_2 = -\frac{5e^{-b}\left(e^{b}-1\right)}{2\left(1-e^{b}+e^{2b}-e^{3b}+e^{4b}\right)}, \quad \alpha_k = 0 \quad (k \geq 3);$$

$$\beta_0 = \frac{\pi}{16\sinh^9 b}e^{-4b}\left(e^{b}-1\right)^2\left(1-3e^{b}+9e^{2b}-9e^{3b}+9e^{4b}-3e^{5b}+e^{6b}\right),$$

$$\beta_1 = \frac{\left(e^{b}-1\right)^4\left(e^{b}+1\right)^4\left(1-e^{b}+e^{2b}-e^{3b}+e^{4b}\right)}{4\left(1-3e^{b}+9e^{2b}-9e^{3b}+9e^{4b}-3e^{5b}+e^{6b}\right)^2},$$

$$\beta_2 = \frac{e^{-5b}\left(e^{b}-1\right)^3\left(e^{b}+1\right)^4\left(1-3e^{b}+9e^{2b}-9e^{3b}+9e^{4b}-3e^{5b}+e^{6b}\right)}{4\left(1-e^{b}+e^{2b}-e^{3b}+e^{4b}\right)^2},$$

$$\beta_3 = \frac{1}{4}\left(e^{-5b}+1\right), \quad \beta_k = \frac{1}{4} \quad (k \geq 4).$$

Recurrence coefficients α_k and β_k for $\nu = 10$ are displayed in Figure 7.

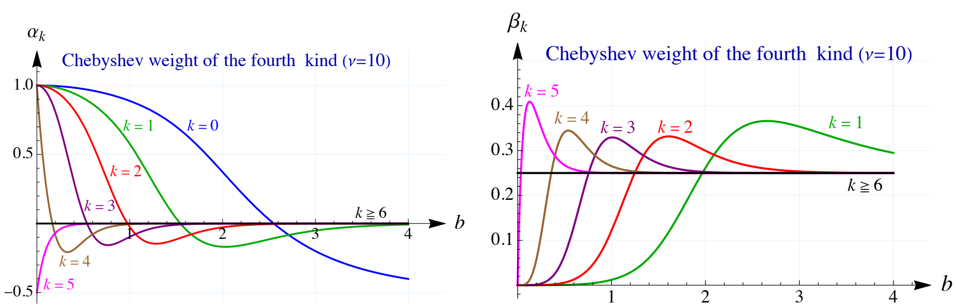

FIGURE 7. Three-term recurrence coefficients α_k (left) and β_k (right) for $\nu = 10$

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