Double-sided Inequalities of Ostrowski's Type and Some Applications

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Abstract

We construct a new general Ostrowski type inequality for differentiable mappings whose first derivatives are bounded in terms of pre-assigned continuous functions. Applications to composite quadrature rules are also given.

1 Introduction

In 1938, A. Ostrowski [14] introduced the following interesting and useful integral inequality for differentiable mappings with bounded derivatives:

Theorem 1.1 *Let $f : [a,b] \to \mathbb{R}$ be continuous mapping on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)| < \infty$, then for all $x \in [a,b]$*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty . \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

Ostrowski's inequality is one of the most famous inequalities in the integral calculus. It measures the deviation of a function from its integral mean. Also, an estimation of approximating area under the curve of a function by a rectangle can be obtained in this case.

In 1975, Milovanović [10] (see also [12, pp. 26–29]) proposed a generalization of (1.1) for a function f of several variables as follows:

Theorem 1.2 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function defined on $\overline{D}$ and let $\left|\frac{\partial f}{\partial x_i}\right| \leq M_i$ in D, where $M_i > 0$ for each $i = 1, \ldots, m$. Then, for every $X = (x_1, \ldots, x_m) \in \overline{D}$, we have*

$$\begin{aligned} \Bigg| f(x_1, \ldots, x_m) \quad &- \quad \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(y_1, \ldots, y_m) \,\mathrm{d}y_1 \cdots \mathrm{d}y_m \Bigg| \\ &\leq \quad \sum_{i=1}^{m} \left[\frac{1}{4} + \frac{\left(x_i - \frac{a_i+b_i}{2}\right)^2}{(b_i - a_i)^2} \right] (b_i - a_i) M_i . \end{aligned}$$

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One year later, in 1976, Milovanović and Pečarić [11] presented the following generalization when $\left|f^{(n)}(x)\right| \leq M$ $(\forall x \in (a,b))$, and $n > 1$:

Theorem 1.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be $n\,(>1)$ times differentiable function such that $\left|f^{(n)}(x)\right| \leq M$ $\big(\forall x \in (a,b)\big)$. Then, for every $x \in [a,b]$*

$$\left|\frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k\right) - \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right| \leq \frac{M}{n(n+1)!} \cdot \frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a},$$

where F_k is defined by

$$F_k \equiv F_k(f;n;x;a;b) \equiv \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}.$$

For $n = 2$, Theorem 1.3 gives

$$\begin{aligned}&\left|\frac{1}{2}\left(f(x) \quad + \quad \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right) - \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right| \\ &\leq \quad \frac{M\,(b-a)^2}{4}\left[\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].\end{aligned}$$

2 Preliminaries

Associated with differentiable mappings, there has been extensive research in the literature on related results. Over the past few decades, many studies on obtaining sharp bounds of Ostrowski's tpye inequalities have been conducted. Most of the calculations within these sharp bounds depend mainly on the magnitudes of Lebesgue norms of derivatives of given functions.

In [5]–[8], Dragomir and Wang obtained the following bounds on the deviation of an absolutely continuous mapping f, defined over the interval $[a,b]$, from its integral mean

$$\left|f(x) - \frac{1}{b-a}\int_a^x f(t)\mathrm{d}t\right| \leq \begin{cases} \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\frac{\|f'\|_\infty}{b-a}, & f' \in L_\infty[a,b]; \\ \frac{1}{q+1}\left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{1/q}\frac{\|f'\|_p}{b-a}, & f' \in L_p[a,b],\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; \\ \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\frac{\|f'\|_1}{b-a}, & f' \in L_1[a,b]. \end{cases}$$

In [9], Masjed-Jamei and Dragomir provided the following analogues of the Ostrowski's inequality for a differentiable function f whose first derivative f' is bounded, bounded from below, and bounded from above in terms of two functions $\alpha, \beta \in C[a,b]$ as follows:

Theorem 2.1 *Let $f : I \to \mathbb{R}$, where I is an interval, be a function differentiable in the interior $\overset{\circ}{I}$ of I, and let $[a,b] \subset \overset{\circ}{I}$. For any $\alpha, \beta \in C[a,b]$ and $x \in [a,b]$, we have the following three cases:*

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1° *If* $\alpha(x) \le f'(x) \le \beta(x)$, *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\mathrm{d}t + \int_x^b (t-b)\,\beta(t)\mathrm{d}t\right) \le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$\le \frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\mathrm{d}t + \int_x^b (t-b)\,\alpha(t)\mathrm{d}t\right); \tag{2.1}$$

2° *If* $\alpha(x) \le f'(x)$, *then*

$$\frac{1}{b-a}\Bigg[\int_a^x (t-a)\alpha(t)\mathrm{d}t + \int_x^b (t-b)\alpha(t)\mathrm{d}t - \max\{x-a, b-x\}\left(f(b) - f(a) - \int_a^b \alpha(t)\mathrm{d}t\right)\Bigg]$$
$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$\le \frac{1}{b-a}\Bigg[\int_a^x (t-a)\alpha(t)\mathrm{d}t + \int_x^b (t-b)\alpha(t)\mathrm{d}t + \max\{x-a, b-x\}\left(f(b) - f(a) - \int_a^b \alpha(t)\mathrm{d}t\right)\Bigg], \tag{2.2}$$

3° *If* $f'(x) \le \beta(x)$, *then*

$$\frac{1}{b-a}\Bigg[\int_a^x (t-a)\beta(t)\mathrm{d}t + \int_x^b (t-b)\beta(t)\mathrm{d}t - \max\{x-a, b-x\}\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\Bigg]$$
$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$\le \frac{1}{b-a}\Bigg[\int_a^x (t-a)\beta(t)\mathrm{d}t + \int_x^b (t-b)\beta(t)\mathrm{d}t + \max\{x-a, b-x\}\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\Bigg], \tag{2.3}$$

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The listed inequalities in Theorem 2.1 are significant as they improve all previous results in which the Lebesgue norms of f' come into play when handling the bounds calculations. In this case, the required computations in bounds are just in terms of pre-assigned functions. For other related general results, the reader may be refer to [3], [15], [16], [18], [1], and [2].

In this paper, motivated by [9], new integral inequalities of Ostrowski type are obtained. Namely, under certain conditions on f', we give the lower and upper bounds for the difference

$$E(f;h) = \frac{h}{2}[f(a) + f(b)] + (1-h) f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \tag{2.4}$$

where $h \in [0,1]$ and $x \in [a+h\frac{b-a}{2}, b-h\frac{b-a}{2}]$. Our results provides range of estimates including those given by [9] and [5]–[8]. Utilizing general Peano kernel, we recapture the three inequalities (2.1)–(2.3) obtained by [9]. Some special cases of our result and applications to numerical quadrature rules are also given.

3 Main Results

In order to formulate our main results, we need a kernel $K(t;\cdot):[a,b]\to\mathbb{R}$ defined by

$$K(t;x) = \begin{cases} t - \left(a + h\frac{b-a}{2}\right), & t \in [a,x], \\ t - \left(b - h\frac{b-a}{2}\right), & t \in (x,b], \end{cases} \tag{3.1}$$

for all $h \in [0,1]$ and $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$. Also, for two functions $\alpha, \beta \in C[a,b]$, such that $\alpha(t) \le \beta(t)$ for each $t \in [a,b]$, we define the functions $A(t;\cdot):[a,b]\to\mathbb{R}$ and $B(t;\cdot):[a,b]\to\mathbb{R}$ by

$$A(t;x) = \frac{1}{2}\Big\{\big[1 - \operatorname{sgn} K(t;x)\big]\beta(t) + \big[1 + \operatorname{sgn} K(t;x)\big]\alpha(t)\Big\} \tag{3.2}$$

and

$$B(t;x) = \frac{1}{2}\Big\{\big[1 - \operatorname{sgn} K(t;x)\big]\alpha(t) + \big[1 + \operatorname{sgn} K(t;x)\big]\beta(t)\Big\}, \tag{3.3}$$

respectively. We note that

$$\operatorname{sgn} K(t;x) = \begin{cases} -1, & t \in \left[a, a + h\frac{b-a}{2}\right), \\ 1, & t \in \left(a + h\frac{b-a}{2}, x\right], \\ -1, & t \in \left(x, b - h\frac{b-a}{2}\right), \\ 1, & t \in \left(b - h\frac{b-a}{2}, b\right], \end{cases} \tag{3.4}$$

and equal to zero at $t = a + h\frac{b-a}{2}$ and $t = b - h\frac{b-a}{2}$.

Obviously, (2.4) provides range of estimates including those introduced by [9] and [5]–[8]. For

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instance, when $h = 0$, $h = 1/2$, and $h = 1$, (2.4) can be, respectively, reduced to

$$E(f;0) = f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad (x \in [a,b]), \tag{3.5}$$

$$E(f;1/2) = \frac{1}{4}\left[f(a)+f(b)+2f(x)\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]\right), \tag{3.6}$$

$$E(f;1) = \frac{1}{2}[f\,(a) + f\,(b)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{3.7}$$

Theorem 3.1 *Let $f : I \to \mathbb{R}$, where I is an interval, be a function differentiable in the interior $\mathring{I}$ of I, and let $[a,b] \subset \mathring{I}$. Also, let $E(f;h)$, $K(t;x)$, $A(t;x)$, and $B(t;x)$ be given by (2.4), (3.1), (3.2), and (3.3), respectively.*

For any $\alpha, \beta \in C[a,b]$, $h \in [0,1]$, and $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$, we have the following three cases:

1° If $\alpha(x) \le f'(x) \le \beta(x)$, then

$$\frac{1}{b-a}\int_a^b K(t;x)A(t;x)\mathrm{d}t \le E(f;h) \le \frac{1}{b-a}\int_a^b K(t;x)B(t;x)\mathrm{d}t; \tag{3.8}$$

2° If $\alpha(x) \le f'(x)$, then

$$\begin{aligned}\frac{1}{b-a}\left\{\int_a^b K(t;x)\alpha(t)\mathrm{d}t - L(x,h)\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right)\right\} &\le E(f;h)\\ \le \frac{1}{b-a}\left\{\int_a^b K(t;x)\alpha(t)\mathrm{d}t + L(x,h)\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right)\right\}&,\end{aligned} \tag{3.9}$$

where

$$L(x,h) = \max_{t\in[a,b]}\left|K(t;x)\right| = \max\left\{x - a - h\frac{b-a}{2}, b - x - h\frac{b-a}{2}, h\frac{b-a}{2}\right\}; \tag{3.10}$$

3° If $f'(x) \le \beta(x)$, then

$$\begin{aligned}\frac{1}{b-a}\left\{\int_a^b K(t;x)\beta(t)\mathrm{d}t - L(x,h)\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\right\} &\le E(f;h)\\ \le \frac{1}{b-a}\left\{\int_a^b K(t;x)\beta(t)\mathrm{d}t + L(x,h)\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\right\}&,\end{aligned} \tag{3.11}$$

where $L(x,h)$ is defined by (3.10).

Proof. By considering the kernel $K(t;x)$ in (3.1), we have

$$\begin{aligned}\int_a^b K(t;x)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)\mathrm{d}t &= \int_a^b K(t;x)f'(t)\mathrm{d}t-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t\\ &= (b-a)E(f;h)-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t, \qquad (3.12)\end{aligned}$$

because of

$$\begin{aligned}\int_a^b K(t;x)f'(t)\mathrm{d}t &= \int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]f'(x)\mathrm{d}t+\int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]f'(x)\mathrm{d}t\\ &= \int_a^b tf'(t)\mathrm{d}t-\left(a+h\frac{b-a}{2}\right)(f(x)-f(a))-\left(b-h\frac{b-a}{2}\right)(f(b)-f(x))\\ &= (b-a)\left[\frac{h}{2}[f(a)+f(b)]+(1-h)f(x)\right]-\int_a^b f(t)\mathrm{d}t\\ &= (b-a)E(f;h).\end{aligned}$$

Now, for the first inequality (3.8), the given assumption $\alpha(x)\le f'(x)\le\beta(x)$ yields

$$\left|f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\le\frac{\beta(t)-\alpha(t)}{2}. \qquad (3.13)$$

Therefore, from (3.12) and (3.13), we get

$$\begin{aligned}\left|(b-a)E(f;h)-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t\right| &= \left|\int_a^b K(t;x)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)\mathrm{d}t\right|\\ &= \frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t,\end{aligned}$$

i.e.,

$$\begin{aligned}-\frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t+\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t &\le (b-a)E(f;h)\\ \le \frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t+\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t.\end{aligned}$$

Since $\big|K(t;x)\big| = K(t;x)\,\mathrm{sgn}\,K(t;x)$, and $A(t;x)$ and $B(t;x)$ are defined by (3.2) and (3.3), respectively, the previous inequalities reduce to (3.8).

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For the second case, when $\alpha(x) \leq f'(x)$, we have

$$\begin{aligned}\int_a^b K(t;x)\left(f'(t)-\alpha(t)\right)\mathrm{d}t &= \int_a^b K(t;x)f'(t) - \int_a^b K(t;x)\alpha(t)\mathrm{d}t \\ &= (b-a)E(f;h) - \int_a^b K(t;x)\alpha(t)\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}\left|(b-a)E(f;h) - \int_a^b K(t;x)\alpha(t)\mathrm{d}t\right| &\leq \left|\int_a^b K(x,t)\left(f'(t)-\alpha(t)\right)\mathrm{d}t\right| \\ &\leq \int_a^x \big|K(t;x)\big|\big(f'(t)-\alpha(t)\big)\mathrm{d}t \\ &\leq \left(\max_{t\in[a,b]}|K(t;x)|\right)\int_a^b \big(f'(t)-\alpha(t)\big)\mathrm{d}t \\ &= L(x,h)\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right), \qquad (3.14)\end{aligned}$$

where $L(x,h)$ is defined by (3.10). Then, (3.14) gives (3.9).

Finally, for the third case, when $f'(x) \leq \beta(x)$, we have

$$\begin{aligned}\int_a^b K(t;x)\left(f'(t)-\beta(t)\right)\mathrm{d}t &= \int_a^b K(t;x)f'(t) - \int_a^b K(t;x)\beta(t)\mathrm{d}t \\ &= (b-a)E(f;h) - \int_a^b K(t;x)\beta(t)\mathrm{d}t,\end{aligned}$$

from which, as before, we obtain

$$\begin{aligned}\left|(b-a)E(f;h) - \int_a^b K(t;x)\beta(t)\mathrm{d}t\right| &\leq \left|\int_a^b K(x,t)\left(f'(t)-\beta(t)\right)\mathrm{d}t\right| \\ &\leq \int_a^x \big|K(t;x)\big|\big(\beta(t)-f'(t)\big)\mathrm{d}t \\ &\leq \left(\max_{t\in[a,b]}|K(t;x)|\right)\int_a^b \big(\beta(t)-f'(t)\big)\mathrm{d}t \\ &= L(x,h)\Big(\int_a^b \beta(t)\mathrm{d}t - f(b)+f(a)\Big),\end{aligned}$$

i.e., (3.11).

The proof of this theorem is completed. $\square$

Remark 3.1 According to (3.10) and $\max\{u, v\} = \frac{1}{2}(u + v + |u - v|)$, we can see that

$$L(x, h) = \frac{b-a}{2}(1-h) + \left|x - \frac{a+b}{2}\right| \qquad \text{if} \quad h \leq \frac{1}{2}.$$

This expression holds also for $h > \frac{1}{2}$, but only when $|x| > 2h - 1$. However, for $|x| \leq 2h - 1$, the

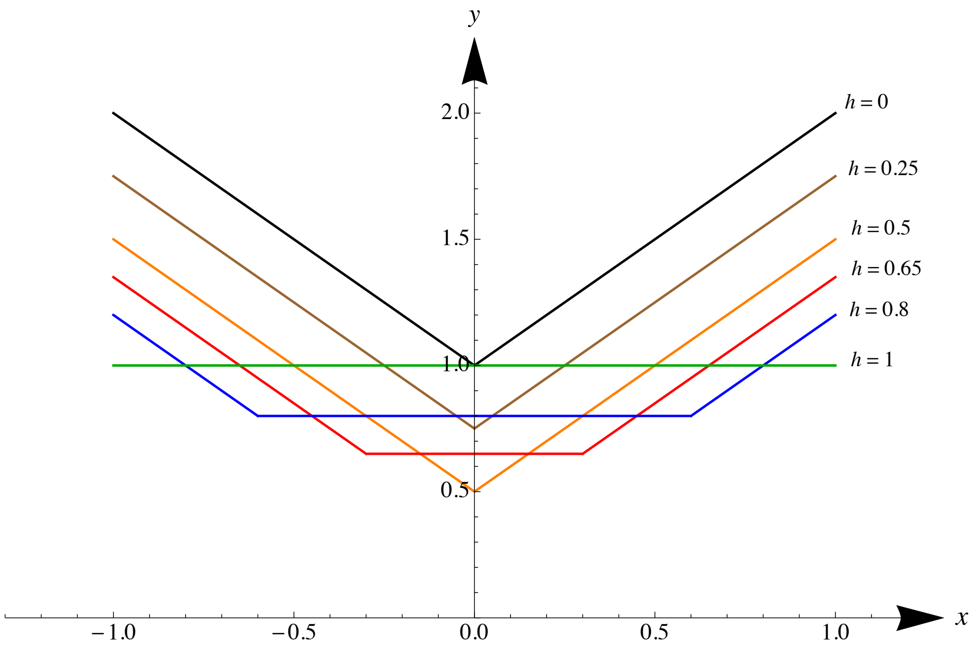

Figure 1: The function $x \mapsto L(x, h)$ for $h = 0$, 0.25, 0.5, 0.65, 0.8, and 1.

function $x \mapsto L(x, h)$ is a constant, i.e.,

$$L(x, h) = \frac{b-a}{2}h.$$

This function on $[a, b] = [-1, 1]$ for different value of h is presented in Figure 1.

Now, we consider cases with constant functions α and β, i.e., when $\alpha(x) = \alpha_0$ and $\beta(x) = \beta_0$ on $[a, b]$.

According to (3.2), (3.3), and (3.4), we get

$$\big(A(t; x), B(t; x)\big) = \begin{cases} (\beta_0, \alpha_0), & t \in \left[a, a + h\frac{b-a}{2}\right), \\ (\alpha_0, \beta_0), & t \in \left(a + h\frac{b-a}{2}, x\right], \\ (\beta_0, \alpha_0), & t \in \left(x, b - h\frac{b-a}{2}\right), \\ (\alpha_0, \beta_0), & t \in \left(b - h\frac{b-a}{2}, b\right], \end{cases}$$

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so that the corresponding bounds in (3.8) become

$$\begin{aligned}\underline{B}^{(1)} &= \frac{1}{b-a}\int_a^b K(t;x)A(t;x)\mathrm{d}t \\ &= -\frac{1}{2(b-a)}\big[b^2\beta_0 - a^2\alpha_0 - 2(b\beta_0 - a\alpha_0)x + (\beta_0-\alpha_0)x^2\big] \\ &\quad +\frac{1}{2}\big[a\alpha_0 + b\beta_0 - (\alpha_0+\beta_0)x\big]h - \frac{1}{4}(b-a)(\beta_0-\alpha_0)h^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\overline{B}^{(1)} &= \frac{1}{b-a}\int_a^b K(t;x)B(t;x)\mathrm{d}t \\ &= \frac{1}{2(b-a)}\big[a^2\beta_0 - b^2\alpha_0 + 2(b\alpha_0 - a\beta_0)x + (\beta_0-\alpha_0)x^2\big] \\ &\quad +\frac{1}{2}\big[b\alpha_0 + a\beta_0 - (\alpha_0+\beta_0)x\big]h + \frac{1}{4}(b-a)(\beta_0-\alpha_0)h^2. \end{aligned} \tag{3.16}$$

Also,

$$\begin{aligned}\frac{1}{b-a}\int_a^b K(t;x)\mathrm{d}t &= \frac{1}{b-a}\left\{\int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]\mathrm{d}t + \int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]\mathrm{d}t\right\} \\ &= \frac{1}{2}(1-h)(2x-a-b),\end{aligned}$$

so that we can find the corresponding lower and upper bounds in the inequalities (3.9) and (3.11):

$$\underline{B}^{(2)} = \frac{\alpha_0}{2}(1-h)(2x-a-b) - L(x,h)\left(\frac{f(b)-f(a)}{b-a} - \alpha_0\right), \tag{3.17}$$

$$\overline{B}^{(2)} = \frac{\alpha_0}{2}(1-h)(2x-a-b) + L(x,h)\left(\frac{f(b)-f(a)}{b-a} - \alpha_0\right), \tag{3.18}$$

$$\underline{B}^{(3)} = \frac{\beta_0}{2}(1-h)(2x-a-b) - L(x,h)\left(\beta_0 - \frac{f(b)-f(a)}{b-a}\right), \tag{3.19}$$

$$\overline{B}^{(3)} = \frac{\beta_0}{2}(1-h)(2x-a-b) + L(x,h)\left(\beta_0 - \frac{f(b)-f(a)}{b-a}\right), \tag{3.20}$$

where $L(x,h)$ is defined by (3.10).

Thus, for constant functions α and β on $[a,b]$, we get the following result:

Corollary 3.1 *Under the assumptions of Theorem* 3.1 *with* $\alpha(x)=\alpha_0$ *and* $\beta(x)=\beta_0$, *we have:*

1° *If* $\alpha_0 \le f'(x) \le \beta_0$, *then* $\underline{B}^{(1)} \le E(f;h) \le \overline{B}^{(1)}$;

2° *If* $\alpha_0 \le f'(x)$, *then* $\underline{B}^{(2)} \le E(f;h) \le \overline{B}^{(2)}$;

3° *If* $f'(x) \le \beta_0$, *then* $\underline{B}^{(3)} \le E(f;h) \le \overline{B}^{(3)}$,

where the bounds are given in (3.15)–(3.19).

4 Some Applications in Numerical Integration

Inequalities of Ostrowski's type have attracted considerable interest over the years. Many authors have worked on this subject and proved many extensions and generalizations, including applications in numerical integration (cf. [4]). These inequalities can be considered as error estimates of certain elementary quadrature rules in some classes of functions.

Beside the bounds of (3.5)–(3.7), in this section we consider also ones for $h = 1/3,\ 1/4,\ 2/3$, and $3/4$, i.e.,

$$\begin{aligned} E(f;1/3) &= \frac{1}{6}\left[f(a)+f(b)+4f(x)\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]\right), \qquad (4.1)\\ E(f;1/4) &= \frac{1}{8}[f(a)+f(b)+6f(x)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[\frac{7a+b}{8}, \frac{a+7b}{8}\right]\right),\\ E(f;2/3) &= \frac{1}{3}[f(a)+f(b)+f(x)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right]\right),\\ E(f;3/4) &= \frac{1}{8}[3f(a)+3f(b)+2f(x)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[\frac{5a+3b}{8}, \frac{3a+5b}{8}\right]\right), \end{aligned}$$

respectively.

For $x = (a+b)/2$, $E(f;1/3)$, given before by (4.1), represents the error in the well-known Simpson formula (cf. [13, pp. 343–350]).

In order to get the corresponding estimates of (2.4), i.e.,

$$E(f;h) = \frac{h}{2}\big[f(a)+f(b)\big] + (1-h)f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]\right),$$

for different values of h, we use here Corollary 3.1 (Case 1°).

Case $h=0$. Here, the value of x can be arbitrary in $[a,b]$. Then, $\underline{B}^{(1)}$ and $\overline{B}^{(1)}$ reduce to

$$\underline{B}^{(1)} = -\frac{1}{2(b-a)}\big[b^2\beta_0 - a^2\alpha_0 - 2(b\beta_0 - a\alpha_0)x + (\beta_0-\alpha_0)x^2\big]$$

and

$$\overline{B}^{(1)} = \frac{1}{2(b-a)}\big[a^2\beta_0 - b^2\alpha_0 + 2(b\alpha_0 - a\beta_0)x + (\beta_0-\alpha_0)x^2\big],$$

so that, under the condition $\alpha_0 \le f'(x) \le \beta_0$, for each $x \in [a,b]$, we have

$$\underline{B}^{(1)} \le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \overline{B}^{(1)}. \qquad (4.2)$$

For the symmetric bounds of the first derivative f' ($|f'(x)| \le \beta_0$), i.e., if $\alpha_0 = -\beta_0$, (4.2) reduces to

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\beta_0,$$

which is, in fact, the original Ostrowski inequality (1.1).

Otherwise, (4.2) for $x = (a+b)/2$ gives the error estimate for the midpoint rule,

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \frac{1}{8}(b-a)(\beta_0 - \alpha_0),$$

while for $x = b$ it gives the error estimate for the so-called endpoint rule

$$\frac{1}{2}(b-a)\alpha_0 \le f(b) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \frac{1}{2}(b-a)\beta_0.$$

Case $h = 1$. Here x must be $(b-a)/2$! Taking $h = 1$ in (3.15) and (3.15), for the trapezoidal rule (3.7), we obtain the same bound as for the midpoint rule,

$$\left| \frac{1}{2}\big[f(a) + f(b)\big] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \frac{1}{8}(b-a)(\beta_0 - \alpha_0).$$

Case $0 < h < 1$. Now we take $x = (a+b)/2$ in (3.15) and (3.15). Since, in that case,

$$-\underline{B}^{(1)} = \overline{B}^{(1)} = \frac{1}{8}(b-a)\big(1 - 2h + 2h^2\big)(\beta_0 - \alpha_0),$$

we get

$$\left| \frac{h}{2}\big[f(a) + f(b)\big] + (1-h) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \frac{b-a}{4}\Big(\frac{1}{2} - h + h^2\Big)(\beta_0 - \alpha_0), \tag{4.3}$$

provided that $\alpha_0 \le f'(x) \le \beta_0$ for $x \in [a,b]$.

For $h = 1/2,\ 1/3,\ 1/4,\ 2/3$, and $3/4$, the inequality (4.3) reduces to

$$\begin{aligned}
\left| \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| &\le \frac{b-a}{16}(\beta_0 - \alpha_0),\\
\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| &\le \frac{5(b-a)}{72}(\beta_0 - \alpha_0),\\
\left| \frac{1}{8}\left[f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| &\le \frac{5(b-a)}{64}(\beta_0 - \alpha_0),\\
\left| \frac{1}{3}\left[f(a) + f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| &\le \frac{13(b-a)}{144}(\beta_0 - \alpha_0),
\end{aligned}$$

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and

$$\left|\frac{1}{8}\left[3f(a)+2f\left(\frac{a+b}{2}\right)+3f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leq\frac{5(b-a)}{64}(\beta_0-\alpha_0),$$

respectively.

5 Conclusion

Inspired and motivated by the work of Masjed-Jamei and Dragomir [9], new integral inequalities of Ostrowski type are obtained with bounds are just in terms of pre-assigned functions. Our results provides a generalization of error bounds that is independent of Lebesgue norms including those given by [9] and [5]–[8]. We utilize general Peano kernel to recapture the three inequalities (3.8), (3.9), and (3.11), obtained in [9]. Some special cases and applications to numerical quadrature rules are also proposed.

References

[1] W.G. Alshanti and A. Qayyum, *A note on new Ostrowski type inequalities using a generalized kernel*, Bull. Math. Anal. Appl. **9**(1) (2017), 74–91.

[2] W.G. Alshanti, A. Qayyum and M.A. Majid, *Ostrowski type inequalities by using general quadratic kernel*, J. Ineq. Spec. Funct. **8**(4) (2017), 111–135.

[3] P. Cerone, *A new Ostrowski type inequality involving integral means over end intervals*, Tamkang J. Math. **33** (2002), 109–118.

[4] S. S. Dragomir and Th. M. Rassias (Eds.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[5] S. S. Dragomir and S. Wang, *A new inequality Ostrowski's type in L_1 norm and applications to some special means and some numerical quadrature rules*. Tamkang J. Math. **28** (1997), 239–244.

[6] S. S. Dragomir and S. Wang, *An inequality Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl. **33** (1997), 15–22.

[7] S. S. Dragomir and S. Wang, *A new inequality Ostrowski's type in L_p norm*, Indian J. Math. **40** (1998), 299–304.

[8] S. S. Dragomir and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and to some numerical quadrature rules*, Appl. Math. Lett. **11** (1998), 105–109.

[9] M. Masjed-Jamei and S. S. Dragomir, *An analogue of the Ostrowski inequality and applications*, Filomat **28** (2014), 373–381.

[10] G. V. Milovanović, *On some integral inequalities.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. No. 498–541 (1975), 119–124.

[11] G. V. Milovanović and J. E. Pečarić, *On generalization of the inequality of A. Ostrowski and some related applications.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. No. 544–576 (1976), 155–158.

[12] G. V. Milovanović, *On some functional inequalities.* Univ. Beograd. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. No. 599 (1977), 1–59.

[13] G. V. Milovanović, *Numerical Analysis, Part I*, University of Niš, Niš, 1979 (Serbian).

[14] A. Ostrowski, *Über die Absolutabweichung einer differentienbaren Funktion von ihrem Integralimittelwert.* Comment. Math. Helv. **10** (1938), 226–227.

[15] N. Ujević, *New bounds for the first inequality of Ostrowski-Grüss type and applications*, Comput. Math. Appl. **46** (2003), 421–427.

[16] N. Ujević, *A generalization of Ostrowski's inequality and applications in numerical integration*, Appl. Math. Lett. **17** (2004), 133–137.

[17] A. Qayyum, M. Shoaib and I. Faye, *Some new generalized results on Ostrowski type integral inequalities with application*, J. Comput. Anal. Appl. **19** (2015), 693–712.

[18] L. Zeng, *More on Ostrowski type inequalities for some s-convex functions in the second sense*, Demonstratio Mathematica. **49**(4) (2016), 398–412.

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