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SOME RESULTS FOR CONVEX SEQUENCES OF ORDER k

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Let S_k^* be the set of all real sequences $a = \{a_n\}_{n \in \mathbf{N}}$ which are convex of order k, i.e.

$$S_k^* = \left\{ a \mid \Delta^k a_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} a_{n+l-m} \geqq 0 \right\}.$$

In the set S_k^*, a subset S_k

$$S_k = \{a \mid a \in S_k^* \wedge a_1 = 0\},$$

can be defined.

In this paper we shall prove some results for sequences from the set S_k. First we shall give an auxiliary result.

LEMMA 1. *For $0 \leqq r < k < n$ the identity*

$$\sum_{m=0}^{r} \binom{k}{m} (n - r + m)_{k-1} = \frac{(-1)^r \binom{k-1}{r}}{n + k - r - 1} (n)_k, \tag{1}$$

holds, where $(p)_s = p(p+1) \ldots (p + s - 1)$.

Proof. We shall deduce the proof by mathematical induction. For $r = 0, 1$ we can directly check that (1) holds. Suppose that (1) holds for some fixed r. Then.

$$\sum_{m=0}^{r+1} (-1)^m \binom{k}{m} (n - r + m - 1)_{k-1} = (-1)^{r+1} \binom{k}{r+1} (n)_{k-1} +$$

$$+ \frac{(-1)^r \binom{k-1}{r}}{n + k - r - 2} (n - 1)_{k-1} = (-1)^{r+1} \binom{k-1}{r+1} \frac{(n)_k}{n + k - r - 2},$$

which completes the proof.

Define the sequence $b^{(r)} = \{b_n^{(r)}\}_{n \in \mathbf{N}}$ (r is a fixed positive integer) by

$$(2) \qquad b_n^{(r)} = \begin{cases} 0 & (n = 1), \\ \dfrac{a_n}{(n-1)^{r-1}} & (n = 2, 3, \ldots). \end{cases}$$

THEOREM 1. *For all* $k \in \{2, 3, \ldots\}$ *implication holds,*

$$a \in S_k \Rightarrow b^{(2)} \in S_{k-1}.$$

Proof. According to (1) we have

$$\sum_{r=0}^{k-1} \frac{(-1)^r \binom{k-1}{r}}{n+k-r-1} (n)_k \, a_{n+k-r} \geqq 0,$$

i.e.

$$(3) \qquad \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \frac{a_{n+k-r}}{n+k-r-1} \geqq 0,$$

wherefrom, according to (2),

$$\sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} b_{n+k-r}^{(2)} \geqq 0,$$

i.e.

$$\Delta^{k-1} b_n^{(2)} \geqq 0.$$

This completees the proof.

On the basis of the preceeding conclusions the following result can be derived.

THEOREM 2. *If the sequence* $a \in S_k$ $(k \geqq 2)$ *is monotonicaly increasing, then the sequence* $b^{(k)}$ *is also increasing.*

Proof. Since the sequence a is convex of order $k (\geqq 2)$, it follows from Theorem 1 that the sequence $b^{(2)} \in S_{k-1}$. Iterating this statement, we conclude that $b^{(k-1)} \in S_2$. Applying the result from [3] (also, see [1]), we obtain the statement of Theorem 2.

Theorem 2 facilitates generalization of the Chebychev's inequality for convex sequences of order $k (\geqq 2)$.

THEOREM 3. *Let* $p = (p_1, \ldots, p_n)$ *be a sequence of positive numbers. If the sequences* $x_j = (0, x_{2j}, \ldots, x_{nj})$, $x_{ij} > 0$ $(i = 2, \ldots, n;\ j = 1, \ldots, r)$ *are convex of order* $k (\geqq 2)$, *then*

$$(4) \qquad \Big(\sum_{i=1}^n p_i\Big)^{r-1} \Big(\sum_{i=1}^n p_i \, x_{i1} \ldots x_{ir}\Big) \geqq N_{r,k} \sum_{i=1}^n p_i \, x_{i1} \ldots \sum_{i=1}^n p_i \, x_{ir},$$

where

$$N_{r,k} = \frac{\Big(\sum_{i=1}^n p_i\Big)^{r-1} \Big(\sum_{i=1}^n p_i \, (i-1)^{r(k-1)}\Big)}{\Big(\sum_{i=1}^n p_i \, (i-1)^{k-1}\Big)^r} \geqq 1.$$

Proof. If we substitute $q_i = p_i (i-1)^{k-1}$, $a_{i1} = (i-1)^{(k-1)(r-2)}$, $a_{i2} = (i-1)^{r(1-k)} x_{i1} \ldots x_{ir}$, in inequality (see [1])

$$(5) \qquad \Big(\sum_{i=1}^n q_i\Big)^{r-1} \Big(\sum_{i=1}^n q_i \, a_{i1} \ldots a_{ir}\Big) \geqq \sum_{i=1}^n q_i \, a_{i1} \ldots \sum_{i=1}^n q_i \, a_{ir}$$

for $r = 2$, we obtain

$$(6) \sum_{i=1}^n p_i \, (i-1)^{k-1} \sum_{i=1}^n p_i \, x_{i1} \ldots x_{ir} \geqq \sum_{i=1}^n p_i \, (i-1)^{r(k-1)} \sum_{i=1}^n p_i \frac{x_{i1} \ldots x_{ir}}{(i-1)^{(r-1)(k-1)}}.$$

By the new substitutions $q_i = p_i (i-1)^{k-1}$, $a_{ij} = \dfrac{x_{ij}}{(i-1)^{k-1}}$ $(j = 1, \ldots, r)$ the inequality (5) becomes

$$(7) \qquad \Big(\sum_{i=1}^n p_i \, (i-1)^{k-1}\Big)^{r-1} \sum_{i=1}^n p_i \frac{x_{i1} \ldots x_{ir}}{(i-1)^{(r-1)(k-1)}} \geqq \sum_{i=1}^n p_i \, x_{i1} \ldots \sum_{i=1}^n p_i \, x_{ir}.$$

Combining inequalities (6) and (7) we obtain (4). Let us note that the previously stated substitutions provide the conditions for which the inequality (5) holds (see Theorems 1 and 2).

Let us show that the constant $N_{r,k}$, for fixed r, increases as the parameter k increases, i.e. inequality (4) becomes sharper as k increases. Namely, if in the inequality (see [2])

$$\left(\frac{\sum_{i=1}^n p_i \, a_i^r}{\sum_{i=1}^n p_i \, b_i^r}\right)^{1/r} \geqq \left(\frac{\sum_{i=1}^n p_i \, a_i^s}{\sum_{i=1}^n p_i \, b_i^s}\right)^{1/s} \qquad (r \geqq s,\ r,\ s \neq 0,\ |r|,\ |s| < +\infty),$$

we put $s = 1$, $a_i = (i-1)^{k-1}$, $b_i = (i-1)^{k-2}$ $(i = 1, \ldots, n)$, we obtain that $N_{r,k} \geqq N_{r,k-1}$. It should be noted that the given substitutions satisfy conditions stated for Theorem 2 in [2].

From Theorem 3 the following results can be derived:

COROLLARY 1. *For* $k = 2$ *and* $r = 2$ *the inequality* (4) *can be reduced to*

$$\Big(\sum_{i=1}^n p_i \, (i-1)\Big)^2 \sum_{i=1}^n p_i \, x_{i1} x_{i2} \geqq \sum_{i=1}^n p_i \, (i-1)^2 \sum_{i=1}^n p_i \, x_{i1} \sum_{i=1}^n p_i \, x_{i2}.$$

This inequality has been proved in [3].

COROLLARY 2. *For* $k = 2$ *and* $p_i = 1 (i = 1, \ldots, n)$ *inequality* (4) *becomes*

$$\sum_{i=1}^{n} x_{i1} \ldots x_{ir} \geqq M(r) \sum_{i=1}^{n} x_{i1} \ldots \sum_{i=1}^{n} x_{ir}, \tag{8}$$

where $M(r) = \dfrac{2^r}{n^r (n-1)^r} \sum_{i=1}^{n-1} i^r$.

The inequality (8) is directly analog to the B. J. Andersson's inequality [4] (also, see [5]). Let us note that

$$M(2) = \frac{2}{3} \cdot \frac{2n-1}{n(n-1)} \quad \text{and} \quad M(3) = \frac{2}{n(n-1)}.$$

Since $\lim_{n \to +\infty} n^{r-1} M(r) = \dfrac{2^r}{r+1}$, the inequality (8) can be reduced to the B. J. Andersson's inequality.

If we apply the method used in [6] on (4) we obtain the following result:

THEOREM 4. *Let positive sequence of real numbers* $(x_{i1}; \ldots; x_{ir})$, $(y_{i1}; \ldots; y_{i,r-1})$, $(p_1, \ldots, p_r)$ *for* $i = 1, \ldots, n$ *be given. Then the inequality*

$$\sum_{i=1}^{n} p_i \frac{x_{i1} \ldots x_{ir}}{y_{i1} \ldots y_{i,r-1}} \leqq N_{r,k} \frac{\sum_{i=1}^{n} p_i x_{i1} \ldots \sum_{i=1}^{n} p_i x_{ir}}{\sum_{i=1}^{n} p_i y_{i1} \ldots \sum_{i=1}^{n} p_i y_{i,r-1}}, \tag{9}$$

holds when the sequences

$$\frac{x_{i1} \ldots x_{i,j-1}}{y_{i1} \ldots y_{i,j-1}}, \ \frac{y_{i,j-1}}{x_{i,j}} \qquad (j = 2, \ldots, r),$$

are convex of the order $k (\geqq 2)$.

The equality in (9) *holds if* $x_{ij} = C_i (i-1)^{k-1}$ *for* $j = 1, \ldots, r$ *and* $y_{ij} = C_j$ *for* $j = 1, \ldots, r$.

The inequality (9) is a generalization of the inequality proved in Theorem 2 in [6] for convex sequences of the order $k (\geqq 2)$. At the end, let us note that from the inequality (9), using the method demonstrated in [6], we can obtain generalizations for a number of well known inequalities, for the case of convex sequences of the order $k (\geqq 2)$.

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