Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx), xxx–xxx.
https://doi.org/10.2298/AADM180730018M

QUADRATURE WITH MULTIPLE NODES, POWER ORTHOGONALITY, AND MOMENT-PRESERVING SPLINE APPROXIMATION, PART II

Gradimir V. Milovanović *, *Miroslav S. Pranić*, *Miodrag M. Spalević*

The paper deals with new contributions to the theory of the Gauss quadrature formulas with multiple nodes that are published after 2001, including numerical construction, error analysis and applications. The first part was published in *Numerical analysis 2000, Vol. V, Quadrature and orthogonal polynomials* (W. Gautschi, F. Marcellan, and L. Reichel, eds.) [J. Comput. Appl. Math. **127** (2001), no. 1-2, 267–286].

1. INTRODUCTION

Gauss quadrature is a popular approach to approximate the value of an integral

$$I(f) = \int_{\mathbb{R}} f(t)\, \mathrm{d}\lambda(t), \tag{1}$$

where $\mathrm{d}\lambda(t)$ is a given nonnegative measure with support on the real axis, such that

$$\langle f, g\rangle = I(fg)$$

is an inner product on $\mathcal{P}$, the space of algebraic polynomials with real coefficients. The formula

$$I(f) = \sum_{\nu=1}^{n} A_\nu f(\tau_\nu) + R(f) \tag{2}$$

* Corresponding author. Gradimir V. Milovanović
2010 Mathematics Subject Classification. 65D30, 65D32, 41A15.
Keywords and Phrases. Quadratures with multiple nodes, Gauss-Turán quadrature rules, s- and σ-orthogonal polynomials, Degree of precision, Chebyshev polynomials, Weights, Nodes, Spline function, Spline defect.

is called the Gauss quadrature if its (algebraic) degree of exactness is $2n-1$, i.e., if $R(f)=0$ for all f from $\mathcal{P}_{2n-1}$, the space of polynomials of degree at most $2n-1$. It is easy to see that a degree of exactness of the quadrature (2) cannot be larger than $2n-1$.

This approach can be generalized in several directions. For example, to approximate an arbitrary linear functional on $\mathcal{P}$, or to be exact on certain space of rational functions. Here we consider the following generalization of the Gauss quadrature: Instead of (2), we use the formula of the form

$$Q_n(f)=\sum_{\nu=1}^{n}\sum_{i=0}^{m_\nu-1} A_{i,\nu} f^{(i)}(\tau_\nu) \tag{3}$$

to approximate $I(f)$. In other words, we assume that the nodes

$$\tau_1<\tau_2<\cdots<\tau_n$$

may have multiplicities larger than one. Naturally, we ask the question what is the maximal possible degree of exactness for the formula (3). For any choice of (different) nodes τ_ν, $\nu=1,\dots,n$, and their multiplicities m_ν, such that $m_1+\cdots+m_n=\ell$, it is possible to achieve that the quadrature (3) is exact for any f from $\mathcal{P}_{\ell-1}$. As shown in [**11**, Theorem 5.1], it is necessary and sufficient to take

$$A_{i,\nu}=I(h_{i,\nu}), \tag{4}$$

where $h_{i,\nu}$ are polynomials from $\mathcal{P}_{\ell-1}$ such that

$$\begin{aligned} h_{i,\nu}^{(t)}(\tau_k)&=1 \quad \text{for } \tau_k=\tau_\nu \text{ and } t=i,\\ h_{i,\nu}^{(t)}(\tau_k)&=0 \quad \text{for } \tau_k\neq\tau_\nu \text{ or } t\neq i, \end{aligned}$$

with $k=1,2,\dots,n$, and $t=0,1,\dots,m_\nu-1$. In this case we say that the quadrature (3) is *interpolatory*, since it can be obtained by applying the integral I on the generalized (Hermite) interpolating polynomial for the integrand f at the nodes τ_ν of the multiplicities m_ν. The degree of exactness of the interpolatory quadrature (3) is $\ell-1+k$ if and only if the polynomial

$$\omega_\ell(t)=(t-\tau_1)^{m_1}(t-\tau_2)^{m_2}\dots(t-\tau_n)^{m_n} \tag{5}$$

is orthogonal to $\mathcal{P}_{k-1}$, i.e.,

$$I(t^j\omega_\ell(t))=0, \quad j=0,1,\dots,k-1.$$

Indeed, any polynomial q from $\mathcal{P}_{\ell+k-1}$ can be written as $q(t)=r(t)\omega_\ell(t)+s(t)$, with $r\in\mathcal{P}_{k-1}$ and $s\in\mathcal{P}_{\ell-1}$. The orthogonality condition $\omega_\ell\perp\mathcal{P}_{k-1}$ implies

$$I(q)=I(r\omega_\ell+s)=I(s)=Q(s)=Q(s+r\omega_\ell)=Q(q).$$

On the other hand, if $I(q) = Q(q)$ for all q from $\mathcal{P}_{\ell+k-1}$, then

$$I(t^j\omega_\ell(t)) = Q(t^j\omega_\ell(t)) = 0, \quad j = 0, 1, \dots, k-1.$$

Recalling the fact that $\langle f, g\rangle = I(fg)$ is an inner product on $\mathcal{P}$, the largest possible value of k is m, where m is number of odd numbers among m_ν, $\nu = 1, \dots, n$. For example, if only m_n is even, then ω_ℓ cannot be orthogonal to the polynomial $\prod_{i=1}^{n-1}(t - \tau_i)$.

Remark. The quadrature formula (3) can be also used to approximate integrals $\int_{\mathbb{R}} f(t)\,\mathrm{d}\lambda(t)$, where $\mathrm{d}\lambda(t)$ is not a positive measure. In such a case, the degree of exactness can be larger than $\ell + m - 1$.

Definition 1.1. We say that the quadrature formula with real weights $A_{i,\nu}$ and real nodes $\tau_1 < \cdots < \tau_n$ with preassigned multiplicities $m_1, \dots, m_n$

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\lambda(t) = \sum_{\nu=1}^{n}\sum_{i=0}^{m_\nu-1} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f) \tag{6}$$

is a Gauss quadrature with multiple nodes if

$$R(f) = 0, \quad f \in \mathcal{P}_{\ell+m-1},$$

where $\mathrm{d}\lambda$ is an arbitrary measure, $\ell = m_1 + \cdots + m_n$, and m is a number of odd multiplicities m_ν.

For obvious reasons, particular attention is paid to the case when all multiplicities are odd:

$$I(f) = \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{7}$$

$$R(f) = 0, \quad f \in \mathcal{P}_{\ell+n-1}, \quad \ell = 2(s_1 + \cdots + s_n) + n.$$

The quadrature (7) is called a Chakalov-Popoviciu quadrature, in honor of Chakalov and Popoviciu who first considered such quadratures independently of each other; see [**6**], [**7**], [**86**]. The special case $s = s_1 = \cdots = s_n$ of (7), i.e.,

$$I(f) = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f) \quad (s \geq 0), \tag{8}$$

is usually referred to as a Gauss-Turán quadrature [**97**].

If some of the nodes in (3) are preassigned, then the interpolatory quadrature (3) has the form

$$\widetilde{Q}(f) = \sum_{\nu=1}^{n_1}\sum_{i=0}^{m_\nu-1} A_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n_2}\sum_{j=0}^{r_\mu-1} B_{j,\mu} f^{(j)}(\alpha_\mu), \tag{9}$$

where $\alpha_1, \dots, \alpha_{n_2}$ are preassigned nodes and $n = n_1 + n_2$. Let

$$m = m_1 + \cdots + m_{n_1}, \qquad r = r_1 + \cdots + r_{n_2},$$

and

$$\widetilde{\omega}_m(t) = \prod_{\nu=1}^{n_1} (t - \tau_\nu)^{m_\nu}, \quad \varphi_r(t) = \prod_{\mu=1}^{n_2} (t - \alpha_\mu)^{r_\mu}. \tag{10}$$

We have $I(f) = \widetilde{Q}(f)$ for all $f \in \mathcal{P}_{m+r-1+k}$ if and only if

$$\int_{\mathbb{R}} t^j \widetilde{\omega}_m(t) \varphi_r(t) \, \mathrm{d}\lambda(t) = 0, \quad j = 0, \dots, k-1. \tag{11}$$

If we define the new measure $\mathrm{d}\mu(t) = \varphi_r(t) \, \mathrm{d}\lambda(t)$, then the orthogonality conditions (11) can be written in the form

$$\int_{\mathbb{R}} t^j \widetilde{\omega}_m(t) \, \mathrm{d}\mu(t) = 0, \quad j = 0, 1, \dots, k-1. \tag{12}$$

Usually, $\mathrm{d}\mu$ is not a positive measure. If we require $k \geq n_1$, then we refer to (9) as the Gauss-Stancu quadrature. The special case of the Gauss-Stancu quadrature when $n_1 = n_2 + 1$, $r_1 = \cdots = r_{n_2} = 2s + 1$ and $\varphi_r \perp \mathcal{P}_{n_2 - 1}$ with respect to $\mathrm{d}\lambda$ is known as the Kronrod extension of the Gauss-Turan quadrature

$$\sum_{\mu=1}^{n_2} \sum_{j=0}^{2s} C_{j,\mu} f^{(j)}(\alpha_\mu).$$

This paper is written as a follow-up of [**47**] published in 2001, where a survey of the Gauss quadrature with multiple nodes was given, including many historical remarks. Thus we will not repeat them here, but rather focus on the results published after 2001. The paper is organized as follows. For completeness, we discuss in Section 2 the question of existence and uniqueness of the Gauss quadrature with multiple nodes. This is closely related to the theory of so-called σ-orthogonal polynomials. In Section 3 we present a numerical procedure for computing the nodes and weights in (7). Section 4 is devoted to the error term $R(f)$ for an analytic function f. Most of results about Gauss quadrature with multiple nodes published after 2001 deal with analytically derived error bounds (estimates). They are presented in Section 5. Some applications, particularly those to moment-preserving spline approximation, are presented in Section 6.

2. σ-ORTHOGONAL POLYNOMIALS

Existence and uniqueness of Chakalov-Popoviciu quadrature (7) for integrals with a positive measure is shown by Ghizzetti and Ossicini in [**25**]. It means that the following problem has an unique solution.

Problem I: Given a positive measure $\mathrm{d}\lambda$ such that $\langle f, g\rangle = \int_{\mathbb{R}} fg\,\mathrm{d}\lambda$ is an inner product on $\mathcal{P}$, and a sequence of nonnegative integers $\sigma = (s_1, s_2, \ldots)$, find n real numbers $\tau_1^{(n,\sigma)} < \tau_2^{(n,\sigma)} < \cdots < \tau_n^{(n,\sigma)}$ satisfying

$$\int_{\mathbb{R}} t^j \left[\prod_{i=1}^{n} \bigl(t - \tau_i^{(n,\sigma)}\bigr)\right]^{2s_i+1} \mathrm{d}\lambda(t) = 0, \quad j = 0, 1, \ldots, n-1. \tag{13}$$

The solution of Problem I defines the nth degree polynomial

$$\pi_{n,\sigma}(t) = \prod_{i=1}^{n} \bigl(t - \tau_i^{(n,\sigma)}\bigr), \quad n = 1, 2, \ldots,$$

that is known as the σ-orthogonal polynomial.

Thus, σ-orthogonal polynomials are unique when $\tau_1^{(n,\sigma)} < \tau_2^{(n,\sigma)} < \cdots < \tau_n^{(n,\sigma)}$ is imposed, with corresponding multiplicities $m_i = 2s_i + 1$, $i = 1, \ldots, n$. Otherwise, the number of distinct σ-polynomials is

$$\frac{n!}{k_1! k_2! \cdots k_q!}$$

for some q $(1 \le q \le n)$, where k_i is the number of nodes of multiplicity $m_j = i$, each node counted exactly once, $\sum_{i=1}^{q} k_i = n$. For example, in the case $n = 4$, with multiplicities $3, 3, 3, 5$, we have four different σ-polynomials, which correspond to $\sigma = (1, 1, 1, 2)$, $(1, 1, 2, 1)$, $(1, 2, 1, 1)$, and $(2, 1, 1, 1)$.

The special case $s = s_1 = s_2 = \cdots$ gives so-called s-orthogonal polynomials $\pi_{n,s}$ satisfying

$$\int_{\mathbb{R}} \pi_{j,s}(t)[\pi_{n,s}(t)]^{2s+1}\,\mathrm{d}\lambda(t) = 0, \quad j = 0, 1, \ldots, n-1. \tag{14}$$

For $s = 0$, (14) reduces to the standard orthogonality conditions.

In the definition of σ-orthogonal polynomials it is tacitly assumed that all their zeros are real and simple. In other words, polynomials with complex or multiple zeros are disqualified to be σ-orthogonal by definition. This is not the case if (14) is used as a definition of s-orthogonal polynomials. However, it cannot lead to confusion since the monic nth degree polynomial whose zeros are the solution of the proper version of Problem I is the only monic nth degree polynomial satisfying (14). As in the case of standard orthogonal polynomials, the zeros of $\pi_{n,s}$ and $\pi_{n+1,s}$ interlace. For more properties of s-orthogonal polynomials see [**47**, Section 2]. There are only a couple of measures when explicit formulas for corresponding s-orthogonal polynomials are known. All these measures are of the form $\mathrm{d}\lambda(t) = w(t)\,\mathrm{d}t$ on $[-1, 1]$:

- If $w_1(t) = (1 - t^2)^{-1/2}$, then for all s, $\pi_{n,s}(t) = \widehat{T}_n(t) = 2^{1-n}T_n(t)$, where $T_n(t) = \cos(n \arccos t)$ is the Chebyshev polynomial of the first kind; see [**1**].

- If $w_2(t) = (1-t^2)^{1/2+s}$, then $\pi_{n,s} = \widehat{U}_n(t) = 2^{-n}U_n(t)$, where $U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta$ is the Chebyshev polynomial of the second kind; see [**74**].

- If $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$, then $\pi_{n,s} = \widehat{V}_n(t) = 2^{-n}V_n(t)$, where $V_n(\cos\theta) = \cos(n+\frac{1}{2})\theta/\cos\frac{1}{2}\theta$ is the Chebyshev polynomial of the third kind; see [**74**].

- If $w_4(t) = (1+t)^{-1/2}(1-t)^{1/2+s}$, then $\pi_{n,s} = \widehat{W}_n(t) = 2^{-n}W_n(t)$, where $W_n(\cos\theta) = \sin(n+\frac{1}{2})\theta/\sin\frac{1}{2}\theta$ is the Chebyshev polynomial of the fourth kind; see [**74**].

- If $w_{n,\mu}(t) = |U_{n-1}(t)/n|^{2\mu+1}(1-t^2)^{\mu}$, with $\mu > -1$, then $\pi_{n,s}(t) = \widehat{T}_n(t) = 2^{1-n}T_n(t)$ for all s; see [**31**].

We refer to the weight functions w_1, w_2, w_3 and w_4 as the Chebyshev weight functions of the first, second, third and fourth kind. Note that w_2, w_3 and w_4 depend on s. The weight functions $w_{n,\mu}$ do not depend on s, but depend on n. This class of weight functions is known as Gori-Micchelli weight functions. As for the σ-orthogonal polynomials, the corresponding theory is not yet developed.

The previous discussion cannot be applied to the Gauss-Stancu quadrature since, in general, the measure $\mathrm{d}\mu$ in (12) is not positive, i.e., $\langle f, g\rangle = \int_{\mathbb{R}} fg\,\mathrm{d}\mu$ is not an inner product on $\mathcal{P}$. Even if the polynomial $\widetilde{\omega}_m$ satisfying (12) for $k = n_1$ exists, all its zeros are not necessarily real. This happens, for example, in the case of the standard Gauss-Kronrod quadrature with Gegenbauer weight function $(1-t^2)^{\ell-1/2}$ for $\ell > 3$; see [**84**]. However, if φ_r does not change sign on the support of $\mathrm{d}\lambda$ and all numbers m_ν are odd, then we can claim the existence and uniqueness of Gauss-Stancu quadrature. It can be done in the case of the Lobatto extension of the Chakalov-Popoviciu quadrature

$$\int_{-1}^{1} f(t)w(t)\,\mathrm{d}t = \sum_{j=0}^{r_1-1} C_j f^{(j)}(-1) + \sum_{\nu=1}^{n-2}\sum_{i=0}^{m_\nu-1} A_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{j=0}^{r_2-1} B_j f^{(j)}(1), \tag{15}$$

where we assume that $\mathrm{d}\lambda$ is an absolutely continuous measure on $[-1,1]$, and w is corresponding weight function. If we omit in (15) the first or the third sum on the right-hand side, i.e., if we preassign only one endpoint of the interval $(-1,1)$, we get the Radau extension of the Chakalov-Popoviciu quadrature.

3. NUMERICAL CONSTRUCTION

In this section we consider the numerical construction of s- and σ-orthogonal polynomials, as well as the calculation the coefficients $A_{i,\nu}$ in the quadrature with multiple nodes (6), i.e., in the Gauss-Turán and the Chakalov-Popoviciu quadrature formulas (8) and (7). These methods have enabled progress in the theory of quadratures with multiple nodes.

3.1. Construction of s- and σ-orthogonal polynomials

The first attempt in the numerical construction of s-orthogonal polynomials with respect to an even weight function on a symmetric interval $(-c, c)$ was given by Vincenti in 1986 [**100**], describing an iterative process for computing their coefficients. This process was applied to Legendre s-orthogonal polynomials and corresponding numerical results for coefficients were obtained only for low degrees of polynomials n and relatively small values of s, because the process becomes numerically unstable when n and s increase.

At the Third Conference on Numerical Methods and Approximation Theory (Niš, 18–21 August, 1987) (see [**45**]) one of us presented a stable method for numerically constructing s-orthogonal polynomials and their zeros. It uses an iterative method with quadratic convergence based on a discretized Stieltjes-Gautschi procedure and the Newton-Kantorovič method. This approach opened the door to the numerical construction of not only s and σ-orthogonal polynomials, but also to the quadrature rules of Gauss-Turán and other generalized rules with multiple nodes, mentioned in the previous sections.

The basic idea of this method for constructing s-orthogonal polynomials with respect to the measure $\mathrm{d}\lambda(t)$ on the real line $\mathbb{R}$ is a reinterpretation of the "*orthogonality conditions*," given by (14) or (13), i.e.,

$$\int_{\mathbb{R}} t^j [\pi_{n,s}(t)]^{2s+1}\,\mathrm{d}\lambda(t) = 0, \quad j = 0, 1, \ldots, n-1,$$

in the following form

$$\int_{\mathbb{R}} t^j \pi_{n,s}(t)\,\mathrm{d}\mu_{n,s}(t) = 0, \quad j = 0, 1, \ldots, n-1, \tag{16}$$

where $\mathrm{d}\mu_{n,s}(t) = [\pi_{n,s}(t)]^{2s}\,\mathrm{d}\lambda(t)$ is a new positive measure, implicitly defined for given n and s. It is clear that for such a measure $\mathrm{d}\mu_{n,s}(t)$ (n and s are fixed), there exists a unique sequence of (monic) polynomials $\{\pi_k^{(n,s)}\}_{k\geq 0}$ ($\deg \pi_k^{(n,s)} = k$), such that

$$\int_{\mathbb{R}} t^j \pi_k^{(n,s)}(t)\,\mathrm{d}\mu_{n,s}(t) = 0, \quad j = 0, 1, \ldots, k-1. \tag{17}$$

Thus, $\{\pi_k^{(n,s)}\}_{k\geq 0}$ is a sequence of standard monic polynomials orthogonal with respect to the measure $\mathrm{d}\mu_{n,s}(t)$ on $\mathbb{R}$. Due to the unicity of this sequence, by comparing (17) with (16), we conclude that its nth member must be the s-orthogonal polynomial, i.e.,

$$\pi_{n,s}(t) = \pi_n^{(n,s)}(t).$$

For a given $s \in \mathbb{N}$ and each $n = 0, 1, 2, \ldots$, this construction is presented in Table 1, whose diagonal (boxed) elements are, in fact, the s-orthogonal polynomials, i.e., $\pi_n^{(n,s)} = \pi_{n,s}$, $n = 0, 1, 2, \ldots$.

Table 1: Construction of s-orthogonal polynomials

n	$\mathrm{d}\mu_{n,s}(t)$	Orthogonal polynomials				
0	$[\pi_{0,s}(t)]^{2s}\,\mathrm{d}\lambda(t)$	$\boxed{\pi_0^{(0,s)}}$				
1	$[\pi_{1,s}(t)]^{2s}\,\mathrm{d}\lambda(t)$	$\pi_0^{(1,s)}$	$\boxed{\pi_1^{(1,s)}}$			
2	$[\pi_{2,s}(t)]^{2s}\,\mathrm{d}\lambda(t)$	$\pi_0^{(2,s)}$	$\pi_1^{(2,s)}$	$\boxed{\pi_2^{(2,s)}}$		
3	$[\pi_{3,s}(t)]^{2s}\,\mathrm{d}\lambda(t)$	$\pi_0^{(3,s)}$	$\pi_1^{(3,s)}$	$\pi_2^{(3,s)}$	$\boxed{\pi_3^{(3,s)}}$	
4	$[\pi_{4,s}(t)]^{2s}\,\mathrm{d}\lambda(t)$	$\pi_0^{(4,s)}$	$\pi_1^{(4,s)}$	$\pi_2^{(4,s)}$	$\pi_3^{(4,s)}$	$\boxed{\pi_4^{(4,s)}}$
$\vdots$						

As we can see, for a fixed n, the polynomials π_k $(\equiv \pi_k^{(n,s)})$, $k = 0, 1, \ldots$, from one row in Table 1, are implicitly defined, because the measure $\mathrm{d}\mu_{n,s}(t)$ depends on $\pi_{n,s}(t)$. However, they satisfy the usual three-term recurrence relation with coefficients α_k $(\equiv \alpha_k(n,s))$ and β_k $(\equiv \beta_k(n,s))$, $k \geq 0$, i.e.,

$$
\begin{aligned}
\pi_{k+1}(t) &= (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0, 1, \ldots, \\
\pi_{-1}(t) &= 0, \quad \pi_0(t) = 1,
\end{aligned} \tag{18}
$$

where, because of orthogonality,

$$
\alpha_k = \frac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}, \quad \beta_k = \frac{(\pi_k, \pi_k)}{(\pi_{k-1}, \pi_{k-1})}, \tag{19}
$$

where

$$
(p, q) = (p, q)_{\,\mathrm{d}\mu_{n,s}} = \int_{\mathbb{R}} p(t)q(t)\,\mathrm{d}\mu_{n,s}(t), \tag{20}
$$

and, by convention, $\beta_0 = (1,1) = \int_{\mathbb{R}} \mathrm{d}\mu_{n,s}(t)$.

Knowing the coefficients α_k and β_k, for $k = 0, 1, \ldots, n-1$, gives us access to the first $n+1$ orthogonal polynomials $\pi_0, \pi_1, \ldots, \pi_n$. Here, we are interested only in the last one, i.e., $\pi_n \equiv \pi_{n,s}$. Unfortunately, in this case the discretized Stieltjes-Gautschi procedure cannot be applied directly, because the measure $\mathrm{d}\mu_{n,s}(t)$ involves the unknown s-orthogonal polynomial $\pi_{n,s}$. However, these recurrence coefficients for $k \leq n-1$ are solutions of the following system of nonlinear equations

$$
\begin{aligned}
f_{2k+1} &\equiv \int_{\mathbb{R}} (\alpha_k - t)\pi_k^2(t)\pi_n^{2s}(t)\,\mathrm{d}\lambda(t) = 0, && k = 0, 1, \ldots, n-1, \\
f_{2k} &\equiv \int_{\mathbb{R}} \left[\beta_k\pi_{k-1}^2(t) - \pi_k^2(t)\right]\pi_n^{2s}(t)\,\mathrm{d}\lambda(t) = 0, && k = 1, 2, \ldots, n-1,
\end{aligned} \tag{21}
$$

obtained from Darboux's formulae (19).

The previous reinterpretation of the "*orthogonality conditions*" (14) in terms of the ordinary orthogonality, as well as the method for solving the system of $2n$ nonlinear equations (21), using the Newton-Kantorovič method, were given in [**45**]. All these integrals in (21), as well as the ones in the elements of the Jacobian, can be computed exactly, except for rounding errors, by using a Gauss-Christoffel quadrature formula with respect to the measure $\mathrm{d}\lambda(t)$, taking $N = (s+1)n$ nodes. Notice that the partial derivatives $a_{k,\nu} = \partial\pi_k/\partial\alpha_\nu$ and $b_{k,\nu} = \partial\pi_k/\partial\beta_\nu$ satisfy the the same recurrence relation as π_k, but with other (delayed) initial values.

Thus, for all calculations we use only the fundamental three-term recurrence relation (18) and the Gauss-Christoffel quadrature rule with respect to the measure $\mathrm{d}\lambda(t)$.

As usual, the main problem are starting values for $\alpha_k^{[0]} = \alpha_k^{[0]}(n,s)$ and $\beta_k^{[0]} = \beta_k^{[0]}(n,s)$ in the Newton-Kantorovič iterative process. In [**45**] (see also [**19**] and [**47**]) we proposed to take the values obtained for $n-1$, i.e., $\alpha_k^{[0]} = \alpha_k(n-1,s)$ and $\beta_k^{[0]} = \beta_k(n-1,s)$, $k \le n-2$, and the corresponding extrapolated values for $\alpha_{n-1}^{[0]}$ and $\beta_{n-1}^{[0]}$. In the case $n = 1$ we have only one nonlinear equation $\int_{\mathbb{R}}(t-\alpha_0)^{2s+1}\,\mathrm{d}\lambda(t) = 0$.

In the case of the Laguerre and Hermite weight functions, Gautschi [**16**] recently presented procedures and corresponding Matlab software for generating Gauss-Turán quadrature rules to arbitrarily high accuracy. The initial approximations have been taken sufficiently accurate to be capable of producing n-point quadrature formulae for n as large as 42 in the case of the Laguerre weight function, and 90 in the case of the Hermite weight function.

In the general case, a more convenient procedure for solving (21) is to apply a method of continuation in relation to the corresponding problem with $s = 0$ (or even better in relation to the problem with some smaller s, for example $s-1$).

Following the previous idea, Gori, Lo Cascio and Milovanović [**33**] developed the corresponding iterative method for the construction of σ-orthogonal polynomials. In this case, the corresponding reinterpretation of the *"orthogonality conditions"* (13) leads to the conditions

$$\int_{\mathbb{R}} t^j \pi_k^{(n,\sigma)}(t)\,\mathrm{d}\mu^{(n,\sigma)}(t) = 0, \quad j = 0,1,\ldots,k-1, \tag{22}$$

where

$$\mathrm{d}\mu^{(n,\sigma)}(t) = \prod_{\nu=1}^{n}(t-\tau_\nu^{(n,\sigma)})^{2s_\nu}\,\mathrm{d}\lambda(t), \tag{23}$$

and we conclude that $\{\pi_k^{(n,\sigma)}\}$ is a sequence of (standard) orthogonal polynomials with respect to the measure $\mathrm{d}\mu^{(n,\sigma)}(t)$, so that $\pi_n^{(n,\sigma)}(\,\cdot\,)$ is the desired σ-orthogonal polynomial $\pi_{n,\sigma}(\,\cdot\,)$. Unfortunately, the same procedure as in the case of s-orthogonal polynomials cannot be applied. Namely, the determination of the Jacobian requires the partial derivatives of the zeros $\tau_k^{(n,\sigma)}$ with respect to α_ν and

β_ν, which is not possible in an analytic form. Therefore in [**33**], it was replaced by a discrete analogue of the Newton-Kantorovič method (a version of the secant method). The convergence of such a method is superlinear and strongly depends on the choice of the starting points.

An alternative approach with quadratic convergence was given in 2004 by Milovanović, Spalević and Cvetković [**64**] (see also [**88**]). It can be used in constructions for measures with the bounded and unbounded supports.

For a given sequence $\sigma = (s_1, s_2, \ldots)$, the orthogonality conditions (22) can be rewritten as the following system of nonlinear equations

$$F_j(\tau_1, \ldots, \tau_n) \equiv \int_{\mathbb{R}} p_{j-1}(t)\pi_{n,\sigma}(t)\, \mathrm{d}\mu^{(n,\sigma)}(t) = 0, \quad j = 1, \ldots, n, \tag{24}$$

where the measure $\mathrm{d}\mu^{(n,\sigma)}(t)$ is given by (23), and, instead of monomials $\{1, t, \ldots\}$, we take the sequence of orthonormal polynomials $\{p_j\}$, with respect to the measure $\mathrm{d}\lambda(t)$ on $\mathbb{R}$, which satisfy the three-term recurrence relation

$$\sqrt{\beta_{j+1}}\, p_{j+1}(t) + \alpha_j p_j(t) + \sqrt{\beta_j}\, p_{j-1}(t) = t p_j(t), \quad j = 0, 1, \ldots,$$

with $p_{-1}(t) = 0$ and $p_0(t) = 1/\sqrt{\beta_0}$, where $\beta_0 = \int_{\mathbb{R}} \mathrm{d}\lambda(t)$.

The nonlinear system of equations (24) can be written in matrix form as

$$\mathbf{F}(\mathbf{t}) = \begin{bmatrix}\mathbf{F}_1(\mathbf{t}) & \ldots & \mathbf{F}_n(\mathbf{t})\end{bmatrix}^T = \mathbf{0},$$

where $\mathbf{t} = [\tau_1 \ \ldots \ \tau_n]^T$. If

$$W = W(\mathbf{t}) = \begin{bmatrix} w_{j,k} \end{bmatrix}_{n\times n} = \left[\frac{\partial F_j}{\partial \tau_k}\right]_{n\times n}$$

is the corresponding Jacobian of $\mathbf{F}(\mathbf{t})$, then for finding the zeros of the σ-orthogonal polynomial $\pi_{n,\sigma}(t)$ we can apply the Newton-Kantorovič method [**44**, p. 239]

$$\mathbf{t}^{(m+1)} = \mathbf{t}^{(m)} - W^{-1}\big(\mathbf{t}^{(m)}\big)\mathbf{F}\big(\mathbf{t}^{(m)}\big), \quad m = 0, 1, \ldots, \tag{25}$$

where $\mathbf{t}^{(m)} = \big[\tau_1^{(m)} \ \ldots \ \tau_n^{(m)}\big]^T$.

The elements of the Jacobian, in this case, are

$$w_{j,k} = \frac{\partial F_j}{\partial \tau_k} = -(2s_k+1)\int_{\mathbb{R}} \frac{p_{j-1}(t)}{t-\tau_k}\left(\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\right)\mathrm{d}\lambda(t), \quad j,k = 1, \ldots, n,$$

and they can be calculated by the recurrence relation [**64**]

$$\sqrt{\beta_{j+1}}\, w_{j+2,k} = (\tau_k - \alpha_j)w_{j+1,k} - \sqrt{\beta_j}\, w_{j,k} - (2s_k+1)F_{j+1}, \quad j = 0, 1, \ldots, n-2,$$

where $w_{0.k} = 0$ and

$$w_{1,k} = -\frac{2s_k+1}{\sqrt{\beta_0}}\int_{\mathbb{R}} (t-\tau_k)^{2s_k}\left(\prod_{\nu=1,\nu\neq k}^{n}(t-\tau_\nu)^{2s_\nu+1}\right)\mathrm{d}\lambda(t). \tag{26}$$

Thus, for calculating the elements of the Jacobian by the previous recurrence relation we only need to know F_j and $w_{1,j}$ for each $j = 1, \ldots, n$. All of the integrals in (24) and (26) can be calculated exactly, except for rounding errors, for example, by using a Gauss-Christoffel quadrature rule with respect to the measure $\mathrm{d}\lambda(t)$,

$$\int_{\mathbb{R}} g(t)\,\mathrm{d}\lambda(t) = \sum_{\nu=1}^{N} A_\nu^{(N)} g\big(\tau_\nu^{(N)}\big) + R_N(g),$$

taking $N = \sum_{\nu=1}^{n} s_\nu + n$ nodes. Such a formula is exact for all polynomials of degree at most $2N - 1 = 2\sum_{\nu=1}^{n} s_\nu + 2n - 1$, which is enough in this case. Numerical experiments with respect to several different measures are considered in [**64**].

3.2. Calculation of coefficients

A stable method for determining the coefficients $A_{i,\nu}$ in the Gauss–Turán quadrature formula (8) was given by Gautschi and Milovanović [**19**]. Some alternative methods were proposed earlier by Stroud and Stancu [**96**] (see also [**95**]), Golub and Kautsky [**27**], and Milovanović and Spalević [**50**]. A generalization of the method from [**19**] to the general case of Chakalov-Popoviciu quadrature formulas (7) has been derived in [**51**]. These methods have enabled further progress in the theory of quadratures with multiple nodes.

Here, we shortly present the basic idea of the method for the general case when $s_\nu \in \mathbb{N}_0$, $\nu = 1, \ldots, n$. Let $\tau_\nu = \tau_\nu^{(n,\sigma)}$, $\nu = 1, \ldots, n$, be corresponding zeros of the σ-orthogonal polynomial $\pi_{n,\sigma}(t)$.

For each $1 \le j \le n$ we first define

$$\Omega_j(t) = \prod_{i \ne j} (t - \tau_i)^{2s_i+1},$$

as well as the polynomials

$$f_{k,j}(t) = (t - \tau_j)^k \Omega_j(t) = (t - \tau_j)^k \prod_{i \ne j} (t - \tau_i)^{2s_i+1} \quad (0 \le k \le 2s_j).$$

Since the Chakalov-Popoviciu quadrature formula (7) is exact for all polynomials of degree at most $\ell + n - 1 = 2(s_1 + \cdots + s_n) + 2n - 1$ and

$$\deg f_{k,j}(t) = k + \sum_{i \ne j} (2s_i + 1) \le 2(s_1 + \cdots + s_n) + n - 1,$$

we conclude that the integration by (7) is exact for each of the polynomials $f_{k,j}(t)$, $0 \le k \le 2s_j$, i.e.,

$$\int_{\mathbb{R}} f_{k,j}(t)\,\mathrm{d}\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f_{k,j}^{(i)}(\tau_\nu). \tag{27}$$

On the other hand $f_{k,j}^{(i)}(\tau_\nu) = 0$, $0 \le i \le 2s_\nu$, for each $\nu \ne j$. Therefore, for each $j = 1, \dots, n$, (27) becomes

$$\sum_{i=0}^{2s_j} A_{i,j} f_{k,j}^{(i)}(\tau_j) = \mu_{k,j}, \quad k = 0, 1, \dots, 2s_j,$$

or in the matrix form

$$\begin{bmatrix} f_{0,j}(\tau_j) & f'_{0,j}(\tau_j) & \dots & f_{0,\nu}^{(2s_\nu)}(\tau_\nu) \\ & f'_{1,j}(\tau_j) & \dots & f_{1,j}^{(2s_j)}(\tau_j) \\ & & \ddots & \\ & & & f_{2s_j,j}^{(2s_j)}(\tau_j) \end{bmatrix} \begin{bmatrix} A_{0,j} \\ A_{1,j} \\ \vdots \\ A_{2s_j,j} \end{bmatrix} = \begin{bmatrix} \mu_{0,j} \\ \mu_{1,j} \\ \vdots \\ \mu_{2s_j,j} \end{bmatrix}, \tag{28}$$

where we put

$$\mu_{k,j} = \int_{\mathbb{R}} f_{k,j}(t)\,\mathrm{d}\lambda(t) = \int_{\mathbb{R}} (t-\tau_j)^k \prod_{i \ne \nu} (t-\tau_i)^{2s_i+1}\,\mathrm{d}\lambda(t).$$

Thus, for each zero τ_j of the σ-orthogonal polynomial $\pi_{n,\sigma}(t)$, $j = 1, \dots, n$, the corresponding upper triangular system of equations (28) gives the weight coefficients (generalized Cotes numbers) $A_{i,j}$, $i = 0, 1, \dots, 2s_j$, in the Chakalov-Popoviciu quadrature formula (7). The moments $\mu_{k,j}$ can be computed exactly, except for rounding errors, by using the same Gauss–Christoffel formula as in the construction of σ-orthogonal polynomials in the previous subsection, with $N = \sum_{\nu=1}^{n} s_\nu + n$ nodes. Recurrence formulas for the generalized Cotes numbers have been derived in [**51**], after some normalizations in systems of equations (28).

A software implementation for constructing s- and σ-orthogonal polynomials and the corresponding quadrature formulas with multiple nodes is given in the Mathematica package `OrthogonalPolynomials` ([**8**], [**48**]), which is freely downloadable from the Web Site: `http://www.mi.sanu.ac.rs/~gvm/`.

4. The REMAINDER TERM

An elementary and old technique of deriving derivative-free representation of the remainder term in quadrature rules is based on Cauchy's integral formula. This technique has already been used in 1878 by Hermite [**35**], and in 1881 by Heine [**34**], to derive an error term for polynomial interpolation; these, when integrated, yield a remainder term for interpolatory quadrature rules.

Assume that the support of the measure $\mathrm{d}\lambda$ in (1) is an interval, say $[-1, 1]$, Γ is a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and D is its interior. If the integrand f is an *analytic function* on D, then the remainder term $R(f)$ in (6) admits the contour integral representation

$$R(f) = R(f; \lambda) = \frac{1}{2\pi i} \oint_{\Gamma} K(z; \lambda) f(z)\,\mathrm{d}z. \tag{29}$$

The *kernel* is given by

$$K(z;\lambda) = K_n(z;\lambda) = \frac{q(z;\lambda)}{\omega_\ell(z)}\ , \qquad z \notin [-1,1], \tag{30}$$

where

$$q(z;\lambda) = \int_{-1}^{1} \frac{\omega_\ell(t)}{z-t}\ \mathrm{d}\lambda(t),$$

and ω_ℓ is given by (5).

Note that in the case of Gauss-Stancu quadrature (9)

$$\omega_\ell = \tilde{\omega}_m\, \varphi_r,$$

with $\tilde{\omega}_m$ and φ_r given by (10).

The integral representation (29) leads to a general error estimate, by using Hölder's inequality,

$$\begin{aligned} |R(f;\lambda)| \quad &= \quad \frac{1}{2\pi}\left| \oint_\Gamma K(z;\lambda) f(z)\ \mathrm{d}z \right| \\ &\le \quad \frac{1}{2\pi}\left(\oint_\Gamma |K(z;\lambda)|^r |\,\mathrm{d}z|\right)^{1/r}\left(\oint_\Gamma |f(z)|^{r'} |\,\mathrm{d}z|\right)^{1/r'}, \end{aligned} \tag{31}$$

i.e.,

$$|R(f;\lambda)| \le \frac{1}{2\pi}\ \|K\|_r\ \|f\|_{r'}, \tag{32}$$

where $1 \le r \le +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r := \begin{cases} \left(\displaystyle\oint_\Gamma |f(z)|^r |\,\mathrm{d}z|\right)^{1/r}, & 1 \le r < +\infty, \\ \displaystyle\max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

Important features of the estimate (32), besides being derivative-free, are its sharpness, its natural conduciveness to a comparison of different quadrature processes, and the neat separation it expresses between the influence of the quadrature rule (given by $\|K_n\|_r$) and the function to which it is applied (given by $\|f\|_{r'}$). The following choices of r are natural. The case $r = +\infty$ ($r' = 1$) gives

$$|R(f;\lambda)| \le \frac{1}{2\pi}\left(\max_{z\in\Gamma} |K(z;\lambda)|\right) \|f\|_1\ , \tag{33}$$

whereas for $r = 1$ ($r' = +\infty$) we have

$$|R(f;\lambda)| \le \frac{1}{2\pi}\ \left(\oint_\Gamma |K(z;\lambda)||\,\mathrm{d}z|\right) \|f\|_\infty\ . \tag{34}$$

The bounds in (33) and (34) depend on the contour Γ, while the contour Γ is arbitrary and $R(f;\lambda)$ does not depend on Γ. Therefore, there is a possibility for optimizing these bounds over suitable families of contours. Two choices of the contours Γ have been most frequently used: concentric circles

$$C_r = \big\{z \in \mathbb{C} : |z| = r\big\}, \quad r > 1,$$

and confocal ellipses

$$\mathcal{E}_\varrho = \left\{ z \in \mathbb{C} : z = \frac{1}{2}\left(u + u^{-1}\right),\ 0 \le \theta \le 2\pi \right\}, \quad u = \varrho \mathrm{e}^{\mathrm{i}\theta},\ \varrho > 1, \tag{35}$$

having foci at ± 1 and the sum of semi-axes equal to ϱ. We limit ourselves to elliptic contours which we consider to be more flexible than circular ones. Indeed, they can be chosen to snuggle tightly around the interval $[-1,1]$ by selecting ρ sufficiently close to 1, thereby avoiding possible singularities or excessive growth of f. The circular contours are used in [**56**] and [**60**].

5. ANALYTICALLY DERIVED ERROR BOUNDS

We focus on the Gauss quadrature with multiple nodes in which the corresponding nodal polynomial ω_ℓ is known. Then, by (30), it is possible to derive an explicit formula for the kernel $K_n(z;\lambda)$, and investigate the values of $\max_{z\in\Gamma}|K_n(z;\lambda)|$ and $\oint_\Gamma |K_n(z;\lambda)||\,\mathrm{d}z|$; see (33) and (34).

5.1. Gauss-Turán quadrature

In Section 2 we have already mentioned five cases when the nodes in the Gauss-Turán quadrature (8) are explicitly known.

(I) *The Chebyshev weight function of the first kind* w_1. This case was considered in [**68**], [**53**] and [**55**], where it is shown that

$$\max_{z\in\mathcal{E}_\rho} |K(z;w_1)| = \frac{2^{1-s}\pi}{\varrho^n} \cdot \frac{\displaystyle\sum_{k=0}^{s} \binom{2s+1}{k} \varrho^{-2n(s-k)}}{(a_2-1)^{1/2}(a_{2n}+1)^{s+1/2}}, \quad n > n_0(w_1),$$

and

$$\frac{1}{2\pi}\oint_{\mathcal{E}_\rho} |K_{n,s}(z;w_1)|\,|\,\mathrm{d}z| \le \frac{\sqrt{\pi}}{2^{s-1/2}\varrho^{(s+1)n}} \left(\sum_{k=0}^{s} A_k\, J_k(a_{2n})\right)^{1/2},$$

where $n_0(w_1)$ is the largest zero of the function

$$\begin{aligned} F_{n,s}(\rho) &:= (a_{2n}+1)^{2s+1} - \big[(a_2-1)n^2+2\big] \\ &\quad \times \left[\sum_{k=0}^{s} 4^k \binom{2s+1}{2k+1}(a_{2n}+1)^{2s-2k} - \frac{1}{2}\sum_{k=1}^{s} 4^k \binom{2s+1}{2k}(a_{2n}+1)^{2s-2k+1}\right], \end{aligned}$$

$$a_j = a_j(\rho) = \frac{1}{2}\left(\rho^j + \rho^{-j}\right), \quad A_k = \frac{2}{\varrho^{2n(s-k)}} \sum_{\nu=0}^{s-k} \binom{2s+1}{\nu}\binom{2s+1}{\nu+k} \varrho^{4n\nu},$$

and

$$\begin{aligned} J_k(a) &= \int_0^\pi \frac{\cos k\theta}{(a+\cos\theta)^{2s+1}}\, \mathrm{d}\theta \\ &= \frac{(-1)^k \pi 2^{2s+1} x^{s-(k-1)/2}}{(x-1)^{4s+1}} \sum_{\nu=0}^{2s} \binom{2s+\nu}{\nu}\binom{2s+k}{\nu+k}(x-1)^{2s-\nu}, \end{aligned}$$

where $a = (x+1)/(2\sqrt{x})$, $x > 1$.

(*II*) *The Chebyshev weight function of the second kind* w_2. The following results can be found in [**68**], [**53**] and [**55**]:

$$\max_{z\in\mathcal{E}_\rho} |K_{n,s}(z; w_2)| = \frac{\pi}{4^s \varrho^{n+1}} \cdot \frac{(a_2+1)^{s+1/2}}{(a_{2n+2}+(-1)^n)^s} \cdot \frac{\displaystyle\sum_{k=0}^{s} (-1)^{kn}\binom{2s+1}{s+k+1}\varrho^{-2(n+1)k}}{(a_{2n+2}+(-1)^n)^{1/2}},$$

for $n > n_0(w_2)$, and

$$\frac{1}{2\pi}\oint_{\mathcal{E}_\varrho} |K_{n,s}(z; w_2)||\,\mathrm{d}z| \le \frac{\sqrt{\pi}}{2^{2s+1/2}\varrho^{(s+1)(n+1)}}\left(M_{2s+2}(\varrho^2)\sum_{k=0}^{s} A_k\, J_k(a_{2n+2})\right)^{1/2},$$

where

$$M_n(\varrho) = (2\varrho)^{-n}\sum_{\nu=0}^{n}\binom{n}{\nu}^2 \varrho^{2\nu}$$

and $n_0(w_2)$ is either zero (if n is odd) or the unique root of the equation

$$\frac{\rho+\rho^{-1}}{\rho^{n+1}+\rho^{-(n+1)}} = \frac{1}{n+1}.$$

(*III*) *The Chebyshev weight function of the third kind* w_3. In [**68**], [**53**] and [**55**] it is shown that

$$\max_{z\in\mathcal{E}_\rho} |K_{n,s}(z; w_3)| = \frac{2^{1-s}\pi}{\varrho^{n+1/2}} \cdot \frac{\displaystyle (a_1+1)^{s+1}\sum_{k=0}^{s}\binom{2s+1}{s+k+1}\varrho^{-(2n+1)k}}{(a_2-1)^{1/2}(a_{2n+1}+1)^{s+1/2}},$$

for $n > n_0(w_1) - 1/2$, and

$$\frac{1}{2\pi}\oint_{\mathcal{E}_\varrho} |K_{n,s}(z; w_3)||\,\mathrm{d}z| \le \frac{\sqrt{\pi}}{2^{s-1/2}\varrho^{(s+1)(n+1/2)}}\left(M_{2s+2}(\varrho)\sum_{k=0}^{s} A_k\, J_k(a_{2n+1})\right)^{1/2}.$$

(IV) the Chebyshev weight function of the fourth kind w_4. Note that $W_n(-t) = (-1)^n V_n(t)$ and the results for the weight function w_3 can be easily adjusted for this case.

(V) The Gori-Micchelli weight function $w_{n,\ell}$. The Gauss-Turán quadrature with the Gori-Micchelli weight function with $\mu = \ell - 1/2$, $\ell \in \mathbb{N}$, has been considered in [**72**]. It is shown that $|K(z; w_{n,\ell})|$, $z \in \mathcal{E}_\varrho$, attains its maximum on the real axis for sufficiently large ρ since

$$|K(z; w_{n,\ell})| \sim \frac{\pi\, F_{s,k}(0)}{n^{2k+2}\, 4^k\, \varrho^{2n(s+1)+1}} \left(1 + \frac{2\cos 2\theta}{\varrho^2}\right)^{1/2}, \quad \varrho \to \infty\,,$$

with

$$\begin{aligned} F_{s,k}(\lambda) &= \sum_{j+p=\lambda} (-1)^p \binom{2k+1}{k-p} \binom{2s+1}{s-j} \\ &+ \sum_{|p-j|=\lambda+1} (-1)^p \operatorname{sign}(p-j) \binom{2k+1}{k-p} \binom{2s+1}{s-j}, \end{aligned}$$

$j = 0, 1, \ldots, s;\ p = 0, 1, \ldots, k$. In addition,

$$\begin{aligned} \frac{1}{2\pi} \oint_{\mathcal{E}_\varrho} |K(z; w_{n,\ell})|\, |\mathrm{d}z| &\le \frac{\pi^{1/2}}{2^{s+2\ell-1}\, n^{2\ell}\, \varrho^{n(s+\ell+1)}} \times \\ &\times \left\{ \sum_{j=0}^{s+\ell-1} \bar{A}_j \left[a_{2n} J_j(a_{2n}) - \frac{1}{2}(J_{j+1}(a_{2n}) + J_{|j-1|}(a_{2n})) \right] \right\}^{1/2}, \end{aligned}$$

where

$$\bar{A}_0 = \frac{1}{\rho^{2n(s+k)}} \sum_{\nu=0}^{s+k} \left[F_{s,k}(s+k-\nu)\right]^2 \rho^{4n\nu},$$

$$\bar{A}_j = \frac{2}{\rho^{2n(s+k-j)}} \sum_{\nu=0}^{s+k-j} F_{s,k}(s+k-\nu)\, F_{s,k}(s+k-\nu-j)\, \rho^{4n\nu}, \quad j = 1, \ldots, s\,.$$

5.2. Gauss-Turán quadrature of Lobatto (Radau) type

Here we mention two cases of the quadrature (15) with

$$\widehat{w}(t) = (1-t)^{k-1/2}(1+t)^{m-1/2}$$

on $(-1, 1)$ considered in [**54**].

(I) If we set $r_1 = s + 1 - m$, $r_2 = s + 1 - k$, $m_\nu = 2s + 1$ in (15), then the nodes $\tau_1, \ldots, \tau_{n-2}$ are the zeros of $U_{n-2}(t)$, and

$$|K_n(z; \widehat{w})| \le \left(\frac{2}{a_2 - 1}\right)^{s+1} (a_1 + 1)^{k+m} \max_{z \in \mathcal{E}_\varrho} \bigl|K_{n-2}(z; w_2)\bigr|,$$

where $K_{n-2}(z;w_2)$ is the kernel for the Gauss-Turán quadrature with $n-2$ nodes of multiplicities $2s+1$, with respect to the Chebyshev weight function of the second type.

(*II*) If we set $r_1 = s+1-m$, $r_2 = 0$, $m_\nu = 2s+1$ in (15), then the nodes $\tau_1, \ldots, \tau_{n-2}$ are the zeros of $V_{n-2}(t)$, and

$$K_n(z;\widehat{w}) \leq \frac{1}{(a_1-1)^{s+1-m}} \max_{z\in\mathcal{E}_\varrho} \bigl|K_{n-2}(z;w_3)\bigr|,$$

where $K_{n-2}(z;w_3)$ is the kernel for the Gauss-Turán quadrature with $n-2$ nodes of multiplicities $2s+1$, with respect to w_3.

5.3. Kronrod extension of the Gauss-Turán quadrature

Consider the special case of the Gauss-Stancu quadrature

$$\int_{-1}^{1} f(t)w(t)\,\mathrm{d}t = \sum_{\mu=1}^{n}\sum_{j=0}^{2s} B_{j,\mu} f^{(j)}(\alpha_\mu) + \sum_{\nu=1}^{n+1}\sum_{i=0}^{m_\nu-1} A_{i,\nu} f^{(i)}(\tau_\nu), \tag{36}$$

where $\prod_{\mu=1}^{n}(t-\alpha_\mu)$ is an s-orthogonal polynomial with respect to the weight function w.

(*I*) *The Chebyshev weight function of the first kind* w_1. If all the nodes τ_ν in (36) are simple, i.e., $m_\nu = 1$ for $\nu = 1\ldots, n+1$, then $\tau_1, \ldots, \tau_{n+1}$ are the zeros of the polynomial $(t^2-1)U_{n-1}(t)$. The upper bound for $\frac{1}{2\pi}\oint_{\mathcal{E}_\rho} |K_n(z;w_1)|\,|\mathrm{d}z|$ is given in [**58**, Section 3.1].

(*II*) *The Chebyshev weight function of the second kind* w_2. If we set $m_\nu = 1$ for all ν in (36), then the nodes $\tau_1, \ldots, \tau_{n+1}$ are the zeros of the polynomial $T_{n+1}(t)$. The upper bound for $\frac{1}{2\pi}\oint_{\mathcal{E}_\rho} |K_{n,s}(z;w_1)|\,|\mathrm{d}z|$ is given in [**58**, Section 3.2].

(*III*) *The Chebyshev weight function of the third kind* w_3. If we set $m_\nu = 1$ for all ν in (36), then the nodes $\tau_1, \ldots, \tau_{n+1}$ are the zeros of the polynomial $(1-t)W_n(t)$. The upper bound for $\frac{1}{2\pi}\oint_{\mathcal{E}_\rho} |K_{n,s}(z;w_1)|\,|\mathrm{d}z|$ is given in [**58**, Section 3.3].

(*IV*) *The Gori-Micchelli weight function* $w_{n\ell}$. If we set $m_1 = m_{n+1} = s-\ell+1$ and $m_2 = \cdots = m_n = 2(s-\ell)+1$ in (36), then the free nodes $\tau_1, \ldots, \tau_{n+1}$ are the zeros of the polynomial $(t^2-1)U_{n_2-1}(t)$. The upper bound for $\max_{z\in\mathcal{E}_\rho} |K(z;w_{n,\ell})|$ is given in [**67**, Theorems 1 and 2], and the upper bound for $\frac{1}{2\pi}\oint_{\mathcal{E}_\rho} |K(z;w_{n,\ell})|\,|\mathrm{d}z|$ is given in [**65**, Theorem 4.1].

6. APPLICATIONS

Here we discuss how the Gauss quadrature with multiple nodes can be used to solve some classical mathematical problems, such as the approximation of functions by defective spline and computation of Fourier coefficients.

6.1. Moment preserving spline approximation

A spline function of degree $m \geq 2$ on the (finite or infinite) interval $[a,b]$ with n distinct knots τ_ν $(\nu = 1,\dots,n)$ in the interior of $[a,b]$ having multiplicities m_ν $(\leq m)$, can be written in the form

$$S_{n,m}(t) = p_m(t) + \sum_{\nu=1}^{n} \sum_{i=0}^{m_\nu-1} \alpha_{i,\nu}(\tau_\nu - t)_+^{m-i}, \tag{37}$$

where $\alpha_{i,\nu}$ are real numbers and $p_m(t)$ is a polynomial of degree $\leq m$. We discuss a problem of approximating a function f by the spline (37) on the interval $[0,1]$ and on the interval $[0,+\infty)$.

(I) *Spline-approximation on* $[0,1]$. We discuss the two following problems. Let $N = m_1 + \cdots + m_n$.

Problem AI. Determine $S_{n,m}$ in (37) such that

$$\int_0^1 t^j S_{n,m}(t)\ \mathrm{d}t = \int_0^1 t^j f(t)\ \mathrm{d}t, \quad j = 0,1,\dots,N+n+m. \tag{38}$$

Problem AII. Determine $S_{n,m}$ in (37) such that

$$S_{n,m}^{(k)}(1) = p_m^{(k)}(1) = f^{(k)}(1), \quad k = 0,1,\dots,m, \tag{39}$$

and such that (38) holds for $j = 0,1,\dots,N+n-1$.

The following theorems give solutions for Problems AI and AII.

Theorem 6.1. *Let* $f \in C^{m+1}[0,1]$. *There exists a unique spline function* (37) *on* $[0,1]$ *satisfying* (38) *if and only if the measure*

$$\mathrm{d}\psi(t) = \frac{(-1)^{m+1}}{m!} f^{(m+1)}(t)\,\mathrm{d}t \tag{40}$$

admits the quadrature of Lobatto type

$$\int_0^1 g(t)\,\mathrm{d}\psi(t) = \sum_{k=0}^{m} \big[\alpha_k g^{(k)}(0) + \beta_k g^{(k)}(1)\big] + \sum_{\nu=1}^{n} \sum_{i=0}^{m_\nu-1} A_{i,\nu}^L g^{(i)}\big(\tau_\nu^{(n)}\big) + R(g), \tag{41}$$

satisfying

$$R(g) = 0 \quad \text{for} \quad g \in \mathcal{P}_{N+n+2m+1}, \tag{42}$$

with distinct zeros $\tau_\nu^{(n)}$, $\nu = 1,\dots,n$, *all contained in the open interval* $(0,1)$. *The spline function* $S_{n,m}$ *is given by*

$$\tau_\nu = \tau_\nu^{(n)}, \quad a_{m-k,\nu} = \frac{m!}{(m-k)!} A_{k,\nu}^L; \quad \nu = 1,2,\dots,n,\ k = 0,1,\dots,m_\nu - 1,$$

and

$$p_m^{(j)}(1) = f^{(j)}(1) + (-1)^j m!\beta_{m-j}, \quad j = 0, 1, \ldots, m.$$

Theorem 6.2. *Let $f \in C^{m+1}[0,1]$. The spline function $S_{n,m}$ satisfying conditions of the problem AI exists and is unique if and only if the measure $\mathrm{d}\psi(t)$ given by (40) admits the quadrature of Radau type*

$$\int_0^1 g(t)\ \mathrm{d}\psi(t) = \sum_{k=0}^{m} \alpha_k^* g^{(k)}(0) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}^R g^{(i)}(\tau_\nu^{(n)*}) + R(g), \tag{43}$$

satisfying

$$R(g) = 0 \quad \text{for} \quad g \in \mathcal{P}_{N+n+m},$$

with distinct nodes $\tau_\nu^{(n)*}$, $\nu = 1, \ldots, n$, all contained in the open interval $(0,1)$. The spline $S_{n,m}$ is given by

$$\tau_\nu^* = \tau_\nu^{(n)*}, \quad a_{m-k,\nu}^* = \frac{m!}{(m-k)!} A_{k,\nu}^R; \quad \nu = 1, 2, \ldots, n,\ k = 0, 1, \ldots, 2s_\nu,$$

and

$$p_m^*(t) = \sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!}(t-1)^k.$$

The special case $m_1 = \cdots = m_n = 1$ of Problems AI and AII was considered in [**13**], and generalized in [**12**] to the case $m_1 = \cdots = m_n = 2s$, $s \in \mathbb{N}$.

Interval $[0, +\infty)$. Similarly as in the previous case, we discuss two problems of approximating a function $f(t)$, $0 \le t < +\infty$, by the spline function (37).

Problem BI. Determine $S_{n,m}$ in (37) with $p_m(t) = 0$ such that $S_{n,m}^{(k)}(0) = f^{(k)}(0)$ $(k = 0, 1, \ldots, N+n-1;\ m \ge N+n-1)$.

Problem BII. Determine $S_{n,m}$ in (37 with $p_m(t) = 0$ such that $S_{n,m}^{(k)}(0) = f^{(k)}(0)$ $(k = 0, 1, \ldots, l;\ l \le m)$ and

$$\int_0^{+\infty} t^j S_{n,m}(t)\,\mathrm{d}t = \int_0^{+\infty} t^j f(t)\,\mathrm{d}t \qquad (j = 0, 1, \ldots, N+n-l-2).$$

The next theorem gives the solution of Problem BII.

Theorem 6.3. *Let $f \in C^{m+1}[0, +\infty)$ and $\int_0^{+\infty} t^{N+n-l+m}\left|f^{(m+1)}(t)\right|\mathrm{d}t < +\infty$. Then a spline function $S_{n,m}$ with positive knots τ_ν that satisfies the conditions of Problem BII exists and is unique if and only if the measure*

$$\mathrm{d}\lambda(t) = \frac{(-1)^{m+1}}{m!}\, t^{m-l} f^{(m+1)}(t)\,\mathrm{d}t$$

admits the quadrature

$$\int_0^{+\infty} g(t)\,\mathrm{d}\lambda(t) = \sum_{\nu=1}^{n} \sum_{k=0}^{m_\nu - 1} A_{k,\nu}^{(n)} g^{(k)}(\tau_\nu^{(n)}) + R(g), \tag{44}$$

with n distinct positive nodes $\tau_\nu^{(n)}$, satisfying $R(g) = 0$ for all $g \in \mathcal{P}_{N+n-1}$. The knots in (37) are given by $\tau_\nu = \tau_\nu^{(n)}$, and the coefficients $\alpha_{\nu,i}$ by the following triangular system:

$$A_{\nu,k}^{(n)} = \sum_{i=k}^{m_\nu - i} \frac{(m-i)!}{m!} \binom{i}{k} \left[D^{i-k} t^{m-l}\right]_{t=\tau_\nu} \alpha_{\nu,i} \qquad (k = 0, 1, \ldots, m_\nu - 1).$$

If we set $l = N + n - 1$, Theorem 6.3 gives also the solution of Problem BI. The case $m_1 = m_2 = \cdots = m_n = 1$, $l = -1$, was considered by Gautschi and Milovanović in [**18**].

The error of the spline approximation can be expressed as the remainder term in (44), (43) or (41) for a particular function $\sigma_t(x) = x^{-(m-l)}(x-t)_+^m$ (see [**38**]). A generalization of these results can be found in [**90**].

6.2. Computation of Fourier coefficients

Let $\{P_k\}_{k=0}^{\infty}$ be a system of orthonormal polynomials on $[a, b]$ with respect to a weight function w. The coefficients $a_k(f)$ in the Fourier series expansion

$$f(x) = \sum_{k=0}^{\infty} a_k(f) P_k(x), \quad a_k(f) = \int_a^b w(t) P_k(t) f(t)\,\mathrm{d}t, \tag{45}$$

can be approximated by Gauss quadrature with multiple nodes

$$\int_a^b f(t) P_k(t) w(t)\,\mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \quad a < x_1 < \cdots < x_n < b. \tag{46}$$

The question of existence of the Gauss quadrature (46) can be answered by the following theorem from [**3**].

Theorem 6.4. *For any given sets of multiplicities $\bar{\mu} := (\mu_1, \ldots, \mu_k)$ and $\bar{\nu} := (\nu_1, \ldots, \nu_n)$, and nodes $y_1 < \cdots < y_k$, $x_1 < \cdots < x_n$, there exists a quadrature formula of the form*

$$\int_a^b f(t) w(t) \prod_{m=1}^{k} (t - y_m)^{\mu_m}\,\mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \tag{47}$$

having degree of exactness N if and only if there exists a quadrature formula of the form

$$\int_a^b f(t) w(t)\,\mathrm{d}t \approx \sum_{m=1}^{k} \sum_{\lambda=0}^{\mu_m - 1} b_{m\lambda} f^{(\lambda)}(y_m) + \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \tag{48}$$

which has degree of exactness $N + \mu_1 + \cdots + \mu_k$. In the case $y_m = x_j$ for some m and j, the corresponding terms in both sums combine in one term of the form

$$\sum_{\lambda=0}^{\mu_m+\nu_j-1} d_{m\lambda} f^{(\lambda)}(y_m).$$

For example, the existence of the Gauss-Turán quadrature (8) with respect to the Chebyshev weight function w_1 guarantees the existence of the following Gauss quadrature

$$\int_{-1}^{1} f(t) T_n(t) \frac{1}{\sqrt{1-t^2}} \,\mathrm{d}t \approx \sum_{i=1}^{n} \sum_{h=0}^{2s-1} \hat{A}_{hi} f^{(h)}(\tau_i).$$

In a similar way, several types of the Gauss quadrature for approximating coefficients in Fourier series in terms of the Chebyshev polynomials T_n, U_n, V_n and W_n are derived in [**62**]. Other approaches within the same topic can be found in [**41**], [**43**], [**2**]. Error bounds of the form (34) and (33) for some of these quadrature are considered in [**79**], [**80**], [**73**].

The quadrature (47) can be easily computed if we know the quadrature (48):

$$c_{ji} = \sum_{s=i}^{\nu_j-1} a_{js} \binom{s}{i} \left[\prod_{m=1}^{k} (t - y_m)^{\mu_m} \right]^{(s-i)} \Bigg|_{t=x_j}.$$

Acknowledgments. The first author was supported in part by the Serbian Academy of Sciences and Arts (No. Φ-96) and by the Serbian Ministry of Education, Science and Technological Development (No. #OI 174015). The second author was supported in part by the Ministry of Science and Technology of R. Srpska. The third author was supported in part by the Serbian Ministry of Education, Science and Technological Development (No. #OI 174002).

REFERENCES

1. S. Bernstein: *Sur les polynomes orthogonaux relatifs à un segment fini.* J. Math. Pures Appl. **9** (1930) 127–177.
2. B. Bojanov: *On a quadrature formula of Micchelli and Rivlin.* J. Comput. Appl. Math. **70** (1996) 349–356.
3. B. Bojanov, G. Petrova: *Quadrature formulae for Fourier coefficients.* J. Comput. Appl. Math. **231** (2009) 378–391.
4. D. Calvetti, G. H. Golub, W. B. Gragg, L. Reichel: *Computation of Gauss-Kronrod rules.* Math. Comp. **69** (2000) 1035–1052.
5. L. Chakalov: *Über eine allgemeine Quadraturformel.* C.R. Acad. Bulgar. Sci. **1** (1948) 9–12.

6. L. Chakalov: *General quadrature formulae of Gaussian type.* Bulgar. Akad. Nauk Izv. Mat. Inst. **1** (1954) 67–84 (Bulgarian). [English transl. East J. Approx. 1 (1995) 261–276]

7. L. Chakalov: *Formules générales de quadrature mécanique du type de Gauss.* Colloq. Math. **5** (1957) 69–73.

8. A. S. Cvetković, G. V. Milovanović: *The Mathematica package "OrthogonalPolynomials".* Facta Univ. Ser. Math. Inform. **19** (2004) 17–36.

9. A. S. Cvetković, M. M. Spalević: *Estimating the error of Gauss-Turán quadrature formulas using their extensions.* Electron. Trans. Numer. Anal. **41** (2014) 1–12.

10. B. de la Calle Ysern, F. Peherstorfer: *Ultraspherical Stieltjes polynomials and Gauss–Kronrod quadrature behave nicely for $\lambda < 0$.* SIAM J. Numer. Anal. **45** (2007) 770–786.

11. A. Draux: *Polynômes Orthogonaux Formels.* Lecture Notes in Mathematics, Springer-Verlag, 1983.

12. M. Frontini, G. V. Milovanović: *Moment-preserving spline approximation on finite intervals and Turán quadratures.* Facta Univ. Ser. Math. Inform. **4** (1989) 45–56.

13. M. Frontini, W. Gautschi, G. V. Milovanović: *Moment-preserving spline approximation on finite intervals.* Numer. Math. **50** (1987) 503–518.

14. C. F. Gauss: *Methodus nova integralium valores per approximationem inveniendi.* Commentationes Societatis Regiae Scientiarum Göttingensis Recentiores **3**, (1814). Also in Werke III, 163–196.

15. W. Gautschi: *Spline approximation and quadrature formulae.* Atti Sem. Mat. Fis. Univ. Modena **40** (1992) 169–182.

16. W. Gautschi: *High-precision Gauss-Turán quadrature rules for Laguerre and Hermite weight functions.* Numer. Algorithms **67** (2014), 59–72.

17. W. Gautschi, S. Li: *The remainder term for analytic functions of Gauss-Radau and Gauss-Lobatto quadrature rules with multiple points.* J. Comput. Appl. Math. **33** (1990) 315–329.

18. W. Gautschi, G.V. Milovanović: *Spline approximations to spherically symmetric distributions.* Numer. Math. **49** (1986) 111–121.

19. W. Gautschi, G. V. Milovanović: *S-orthogonality and construction of Gauss–Turán-type quadrature formulae.* J. Comput. Appl. Math. **86** (1997) 205–218.

20. W. Gautschi, R. S. Varga: *Error bounds for Gaussian quadrature of analytic functions.* SIAM J. Numer. Anal. **20** (1983) 1170–1186.

21. W. Gautschi, E. Tychopoulos and R. S. Varga: *A note on the contour integral representation of the remainder term for a Gauss-Chebyshev quadrature rule.* SIAM J. Numer. Anal. **27** (1990) 219–224.

22. A. Ghizzetti, A. Ossicini: *Su un nuovo tipo di sviluppo di una funzione in serie di polinomi.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **43** (1967) 21–29.

23. A. Ghizzetti, A. Ossicini: *Quadrature Formulae.* Akademie Verlag, Berlin, 1970.

24. A. Ghizzetti, A. Ossicini: *Polinomi s-ortogonali e sviluppi in serie ad essi collegati.* Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (4), **18** (1974) 1–16.

25. A. Ghizzetti, A. Ossicini: *Generalizzazione dei polinomi s-ortogonali e dei corrispondenti sviluppi in serie.* Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **109** (1975) 371–379.

26. A. Ghizzetti, A. Ossicini: *Sull' esistenza e unicità delle formule di quadratura gaussiane.* Rend. Mat. (6) **8** (1975) 1–15.

27. G. H. Golub, J. Kautsky: *Calculation of Gauss quadratures with multiple free and fixed knots.* Numer. Math. **41** (1983) 147–163.

28. G. H. Golub, J. H. Welsch: *Calculation of Gauss quadrature rules.* Math. Comp. **23** (1969) 221–230.

29. V. L. Gončarov: *Theory of Interpolation and Approximation of Functions.* GITTL, Moscow, (1954) (in Russian).

30. L. Gori, M. L. Lo Cascio: *A note on a class of Turán type quadrature formulas with generalized Gegenbauer weight functions.* Studia Univ. Babeş-Bolyai Math. **37** (1992) 47–63.

31. L. Gori, C. A. Micchelli: *On weight functions which admit explicit Gauss-Turán quadrature formulas.* Math. Comp. **65** (1996) 1567–1581.

32. L. Gori, E. Santi: *Quadrature rules based on s-orthogonal polynomials for evaluating integrals with strong singularities.* Internat. Series Numer. Math. **131** (1999) 109–119.

33. L. Gori, M. L. Lo Cascio, G. V. Milovanović: *The σ-orthogonal polynomials: a method of construction.* In: C. Brezinski, L. Gori, A. Ronveaux (Eds.), Orthogonal Polynomials and Their Applications, IMACS Ann. Comput. Appl. Math., vol. 9, J.C. Baltzer AG, Scientific Publ. Co., Basel, 1991, pp. 281–285.

34. E. Heine: *Anwendungen der Kugelfunctionen und der verwandten Functionen.* 2nd ed., Reimer, Berlin, 1881. [I. Theil: Mechanische Quadratur, 1–31]

35. C. Hermite: *Sur la formule d'interpolation de Lagrange.* J. Reine Angew. Math. **84**, 70–79 (1878) [Oeuvres III, 432-443]

36. D. K. Kahaner, G. Monegato: *Nonexistence of extended Gauss-Laguerre and Gauss-Hermite quadrature rules with positive weights.* Z. Angew. Math. Phys. **29** (1978) 983–986.

37. S. Karlin, A. Pinkus: *Gaussian quadrature formulae with multiple nodes.* In: S. Karlin, C.A. Micchelli, A. Pinkus, I.J. Schoenberg (Eds.), Studies in Spline Functions and Approximation Theory, Academic Press, New York, 1976, pp. 113–141.

38. M. A. Kovačević, G. V. Milovanović: *Spline approximation and generalized Turán quadratures.* Portugal. Math. **53** (1996) 355–366.

39. A. S. Kronrod: *Integration with control of accuracy.* Soviet Phys. Dokl. **9** (1964) 17–19.

40. S. Li: *Kronrod extension of Turán formula.* Studia Sci. Math. Hungar. **29** (1994) 71–83.

41. C. A. Micchelli: *Monosplines and moment preserving spline approximation.* In: H. Brass, G. Hämmerlin (Eds.), Numerical Integration III, Birkhäuser, Basel, 1988, pp. 130–139.

42. C. A. Micchelli, T. J. Rivlin: *Turán formulae and highest precision quadrature rules for Chebyshev coefficients.* IBM J. Res. Develop. **16** (1972) 372–379.

43. C. A. Micchelli, A. Sharma: *On a problem of Turán: multiple node Gaussian quadrature.* Rend. Mat. (7) **3** (1983) 529–552.

44. G. V. Milovanović: *Numerical Analysis.* University of Niš, Niš, 1979 (Serbian).

45. G. V. Milovanović: *Construction of s-orthogonal polynomials and Turán quadrature formulae.* In: G.V. Milovanović (Ed.), Numerical Methods and Approximation Theory III (Niš, 1987), Univ. Niš, Niš, 1988, pp. 311–328.

46. G. V. Milovanović: *S-orthogonality and generalized Turán quadratures: construction and applications.* In: D.D. Stancu, Ch. Coman, W.W. Breckner, P. Blaga (Eds.), Approximation and Optimization, vol. I (Cluj-Napoca, 1996), Transilvania Press, Cluj-Napoca, Romania, 1997, pp. 91–106.

47. G. V. Milovanović: *Quadratures with multiple nodes, power orthogonality, and moment-preserving spline approximation.* In: Numerical analysis 2000, vol. V, Quadrature and orthogonal polynomials, (W. Gautschi, F. Marcellan, L. Reichel, eds.) J. Comput. Appl. Math. **127** (2001) 267–286.

48. G. V. Milovanović, A. S. Cvetković: *Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type.* Math. Balkanica **26** (2012), 169–184.

49. G. V. Milovanović, M. A. Kovačević: *Moment-preserving spline approximation and Turán quadratures.* In: R. P. Agarwal, Y. M. Chow, S. J. Wilson (Eds.), Numerical Mathematics (Singapore, 1988), INSM vol. 86, Birkhäuser, Basel, 1988, pp. 357–365.

50. G. V. Milovanović, M. M. Spalević: *A numerical procedure for coefficients in generalized Gauss-Turán quadratures.* FILOMAT (formerly Zb. Rad.) **9** (1995) 1–8.

51. G. V. Milovanović, M. M. Spalević: *Construction of Chakalov–Popoviciu's type quadrature formulae.* Rend. Circ. Mat. Palermo **52** (1998) 625–636.

52. G. V. Milovanović, M. M. Spalević: *Quadrature formulae connected to σ–orthogonal polynomials.* J. Comput. Appl. Math. **140** (2002) 619–637.

53. G. V. Milovanović, M. M. Spalević: *Error bounds for Gauss–Turán quadrature formulae of analytic functions.* Math. Comp. **72** (2003) 1855–1872.

54. G. V. Milovanović, M. M. Spalević: *Error analysis in some Gauss-Turán-Radau and Gauss-Turán-Lobatto quadratures for analytic functions.* J. Comput. Appl. Math. **164-165** (2004) 569–586.

55. G. V. Milovanović, M. M. Spalević: *An error expansion for some Gauss–Turán quadratures and L^1-estimates of the remainder term.* BIT Numer. Math. **45** (2005) 117–136.

56. G. V. Milovanović, M. M. Spalević: *Bounds of the error of Gauss–Turán-type quadratures.* J. Comput. Appl. Math. **178** (2005) 333–346.

57. G. V. Milovanović, M. M. Spalević: *Quadrature rules with multiple nodes for evaluating integrals with strong singularities.* J. Comput. Appl. Math. **189** (2006) 689–702.

58. G. V. Milovanović, M. M. Spalević: *Gauss-Turán quadratures of Kronrod type for generalized Chebyshev weight functions.* Calcolo **43** (2006) 171–195.

59. G. V. Milovanović, M. M. Spalević: *A note on the bounds of the error of Gauss–Turán-type quadratures.* J. Comput. Appl. Math. **200** (2007) 276–282.

60. G. V. Milovanović, M. M. Spalević: *On monotony of the error term in Gauss-Turán quadratures for analytic functions.* ANZIAM J. **48** (2007) 567–581.

61. G. V. Milovanović, M. M. Spalević: *Kronrod extensions with multiple nodes of quadrature formulas for Fourier coefficients.* Math. Comp. **83** (2014) 1207–1231.

62. G. V. Milovanović, R. Orive, M. M. Spalević: *Quadrature with multiple nodes for Fourier-Chebyshev coefficients.* IMA J. Numer. Anal. doi: 10.1093/imanum/drx067.

63. G. V. Milovanović, A. V. Pejčev, M. M. Spalević: *A note on an error bound of Gauss-Turán quadrature with the Chebyshev weight function of the first kind.* FILOMAT **27** (2013) 1037–1042.

64. G. V. Milovanović, M. M. Spalević, A. S. Cvetković: *Calculation of Gaussian type quadratures with multiple nodes.* Math. Comput. Modelling. **39** (2004) 325–347.

65. G. V. Milovanović, M. M. Spalević, Lj. Galjak: *Kronrod extensions of Gaussian quadratures with multiple nodes.* Comput. Methods Appl. Math. **6** (2006) 291–305.

66. G. V. Milovanović, M. M. Spalević, M. S. Pranić: *Maximum of the modulus of kernels in Gauss–Turán quadratures with Chebyshev weights: The cases $s = 1, 2$.* Facta Univ. Ser. Math. Inform. **20** (2005) 123–128.

67. G. V. Milovanović, M. M. Spalević, M. S. Pranić: *Error bounds of some Gauss–Turán–Kronrod quadratures with Gori–Micchelli weights for analytic functions.* Kragujevac J. Math. **30** (2007) 221–234.

68. G. V. Milovanović, M. M. Spalević, M. S. Pranić: *Maximum of the modulus of kernels in Gauss–Turán quadratures.* Math. Comp. **77** (2008) 985–994.

69. G. V. Milovanović, M. M. Spalević, M. S. Pranić: *On the remainder term of Gauss-Radau quadratures for analytic functions.* J. Comput. Appl. Math. **218** (2008) 281–289.

70. G. V. Milovanović, M. M. Spalević, M. S. Pranić: *Error estimates for Gaussian quadratures of analytic functions.* J. Comput. Appl. Math. **233** (2009) 802–807.

71. G. V. Milovanović, M. M. Spalević, M. S. Pranić: *Error estimates for Gauss–Turán quadratures and their Kronrod extensions.* IMA J. Numer. Anal. **29** (2009) 486–507.

72. G. V. Milovanović, M. M. Spalević, M. S. Pranić: *Bounds of the error of Gauss–Turán-type quadratures II.* Appl. Numer. Math. **60** (2010) 1–9.

73. R. M. Mutavdžic, A. V. Pejčev, M. M. Spalević: *Error bounds for Kronrod extension of generalizations of Micchelli-Rivlin quadrature formula for analytic functions.* Electron. Trans. Numer. Anal., to appear.

74. A. Ossicini, F. Rosati: *Funzioni caratteristiche nelle formule di quadratura gaussiane con nodi multipli.* Boll. Un. Mat. Ital. (4) 11 (1975) 224–237.

75. A. Ossicini, F. Rosati: *Comparison theorems for the zeros of s-orthogonal polynomials.* Calcolo **16** (1979) 371–381 (in Italian).

76. A. Ossicini, F. Rosati: *Numeri di Christoffel e polinomi s-ortogonali.* In: P.L. Butzer, F. Fehér (Eds.), E.B. Christoffel, Birkhäuser, Basel, 1981, pp. 148–157.

77. A. Ossicini, F. Rosati: *S-orthogonal Jacobi polynomials.* Rend. Mat. Appl. (7) **12** (1992) 399–403 (in Italian).

78. A. V. Pejčev, M. M. Spalević: *Error bounds of Micchelli-Rivlin quadrature formula for analytic functions.* J. Approx. Theory **169** (2013) 23–34.

79. A. V. Pejčev, M. M. Spalević: *Error bounds of Micchelli-Sharma quadrature formula for analytic functions.* J. Comput. Appl. Math. **259** (2014) 48–56.

80. A. V. Pejčev, M. M. Spalević: *Error bounds of a quadrature formula with multiple nodes for the Fourier-Chebyshev coefficients for analytic functions.* Sci. China Math., to appear. doi: https://doi.org/10.1007/s11425-016-9259-5.

81. F. Peherstorfer: *Positive quadrature formulas III: Asymptotics of weights.* Math. Comp. **77** (2008) 2241–2259.

82. F. Peherstorfer: *Interlacing and spacing properties of zeros of polynomials, in particular of orthogonal and L_q-minimal polynomials, $q \in [1,\infty]$.* J. Approx. Theory **160** (2009) 171–185.

83. F. Peherstorfer: *Gauss-Turán quadrature formulas: asymptotics of weights.* SIAM J. Numer. Anal. **47** (2009) 2638–2659.

84. F. Peherstorfer, K. Petras: *Ultraspherical Gauss-Kronrod quadrature is not possible for $\lambda > 3$.* SIAM J. Numer. Anal. **37** (2000) 927–948.

85. F. Peherstorfer, K. Petras: *Stieltjes polynomials and Gauss-Kronrod quadrature for Jacobi weight functions.* Numer. Math. **95** (2003) 689–706.

86. T. Popoviciu: *Sur une généralisation de la formule d'intégration numérique de Gauss.* Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. **6** (1955) 29–57 (in Romanian).

87. M. S. Pranić: *Error bounds of some Gauss type quadratures for analytic functions.* Ph.D. Dissertation, Univ. Kragujevac, 2007.

88. Y. G. Shi, G. Xu: *Construction of σ-orthogonal polynomials and Gaussian quadrature formulas.* Adv. Comput. Math. **27** (2007) 79–94.

89. M. M. Spalević: *Quadrature formulas of Radau and Lobatto type connected to s-orthogonal polynomials.* Zh. Vychisl. Mat. Mat. Fiz. **42** (2002) 615–626; Comput. Math. Math. Phys. **42** (2002) 587–598.

90. M. M. Spalević: *Calculation of Chakalov-Popoviciu quadratures of Radau and Lobatto type.* ANZIAM J. **43** (2002) 429–447.

91. M. M. Spalević: *Error bounds and estimates for Gauss-Turán quadrature formulae of analytic functions.* SIAM J. Numer. Anal. **52** (2014) 443–467.

92. M. M. Spalević, A. S. Cvetković: *Estimating the error of Gaussian quadratures with simple and multiple nodes by using their extensions with multiple nodes.* BIT Numer. Math. **56** (2016) 357–374.

93. M. M. Spalević, M. S. Pranić: *Error bounds of certain Gaussian quadrature formulae.* J. Comput. Appl. Math. **234** (2010) 1049–1057.

94. M. M. Spalević, M. S. Pranić: *The remainder term of Gauss-Turán quadratures for analytic functions.* Approximation and Computation: In Honor of Gradimir V. Milovanović, Springer Optimization and its Applications 42, W. Gautschi et al. (eds.), (2010) 253–266.

95. D. D. Stancu: *On certain general numerical integration formulas.* Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958) 209–216 (in Romanian).

96. A. H. Stroud, D. D. Stancu: *Quadrature formulas with multiple Gaussian nodes.* J. SIAM Numer. Anal. Ser. B **2** (1965) 129–143.

97. P. Turán: *On the theory of the mechanical quadrature.* Acta Sci. Math. Szeged 12 (1950) 30–37.

98. P. Turán: *On some open problems of approximation theory.* J. Approx. Theory **29** (1980) 23–85.

99. A. Marshall, I. Olkin: *Inequalities: Theory of Majorization and Its Applications.* Academic Press, New York, 1979.

100. G. Vincenti: *On the computation of the coefficients of s-orthogonal polynomials.* SIAM J. Numer. Anal. **23** (1986), 1290–1294.

Gradimir V. Milovanović
Serbian Academy of Sciences and Arts,
Beograd, Serbia
Faculty of Sciences and Mathematics,
University of Niš, Niš, Serbia
E-mail: *gvm@mi.sanu.ac.rs*

(Received 30.07.2018)
(Revised 25.09.2018)

Miroslav S. Pranić
Department of Mathematics and Informatics,
University of Banja Luka
Faculty of Science,
M. Stojanovića 2,
51000 Banja Luka
Bosnia and Herzegovina
E-mail: *miroslav.pranic@pmf.unibl.org*

Miodrag M. Spalević
Department of Mathematics,
University of Belgrade
Faculty of Mechanical Engineering,
Kraljice Marije 16
11120 Belgrade, Serbia
E-mail: *mspalevic@mas.bg.ac.rs*