THE GENERALIZATION OF AN INEQUALITY FOR A FUNCTION AND ITS DERIVATIVES

Gradimir V. Milovanović and Miomir S. Stanković

In monograph [1, p. 362] the following result is given:

Let function \( f \) be defined in an interval \((a, b)\). Let us assume that there exists \( f''' \) and that it is an increasing function in the interval \((a, b)\). Then, if \( x \in (a + 1, b - 1) \), we have

\[
(1) \quad f''(x) < f(x + 1) - 2f(x) + f(x - 1).
\]

Let us introduce an operator \( \Delta \) by means of

\[
\Delta^n f(x) = \Delta^{n-1} f(x + 1) - \Delta^{n-1} f(x), \quad \Delta^0 f(x) = f(x) \quad (n \in \mathbb{N}).
\]

It may be shown that

\[
(2) \quad \Delta^n f(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + n - i).
\]

In this paper we shall use the following denotation

\[
(3) \quad \lambda_n = \Delta^n f(x - k) - f^{(0)} (x) \quad (n \in \mathbb{N}),
\]

where \( k = \left[ \frac{n}{2} \right] \), by which the inequality (1) has the form

\[
0 < \lambda_2.
\]

Lemma 1. If \( 0 \leq m \leq n \), then

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x + n - i)^m = \begin{cases} 0 & (0 \leq m < n) \\ n! & (m = n) \end{cases}
\]

Proof. Let \( x \mapsto x^m \ (0 \leq m \leq n) \). Then, using (2),

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x + n - i)^m = \Delta^n (x^m).
\]

Since

\[
\Delta^n (x^m) = \begin{cases} 0 & (0 \leq m < n) \\ n! & (m = n) \end{cases}
\]

(see [2]), the proof is completed.

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Let the function $f$ be defined and be $(n+1)$-times differentiable in $(a, b)$. Let us introduce the operators $T$ and $R$ by means of

$$
T(n, l; f) = \sum_{m=0}^{n} \frac{f^{(m)}(x)}{m!} l^m \quad \text{and} \quad R(n, l; f) = \frac{1}{(n+1)!} l^r f^{(n+1)}(x+1) \theta_1,
$$

where $0 \leq \theta_1 \leq 1$. Then the development of the function $f$ into the Taylor expansion in the neighbourhood of point $x \in (a, b)$, is given by

$$
f(x+l) = T(n, l; f) + R(n, l; f) \quad \text{for} \quad x+1 \in (a, b).
$$

**Lemma 2.** Equality

$$
\lambda_n = (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} R(n, j-k; f)
$$

is valid.

**Proof.** Since

$$
f(x+(n-k-i)) = T(n, n-k-i; f) + R(n, n-k-i; f),
$$

using (2), we have

$$
\Delta^n f(x-k) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} T(n, n-k-i; f) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} R(n, n-k-i; f).
$$

Since

$$
\sum_{i=0}^{n} (-1)^i \binom{n}{i} T(n, n-k-i; f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{m=0}^{n} \frac{f^{(m)}(x)}{m!} (n-k-i)^m
$$

$$
= \sum_{m=0}^{n} \frac{f^{(m)}(x)}{m!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-k-i)^m,
$$

using Lemma 1, it follows

$$
\sum_{i=0}^{n} (-1)^i \binom{n}{i} T(n, n-k-i; f) = f^{(n)}(x).
$$

According to (5), the equality (3) becomes

$$
\lambda_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} R(n, n-k-i; f).
$$

Placing in (6) $i=n-j$, we obtain (4), and thus the proof is completed.

For a sequence of functions $F=(F_1, F_2, \ldots, F_k)$, let us define the terms $D(F)$ and $G(F)$ as follows:

$$
D(F) = \sum_{i=1}^{\left[\frac{k}{2}\right]} F_{k-2i+1}, \quad G(F) = \sum_{i=0}^{\left[\frac{k-1}{2}\right]} F_{k-2i}.
$$

In further discussion we shall define the upper and the lower limit for $\lambda_n$, under condition that $f^{(n+1)}$ is a nondecreasing function.

We shall distinguish the cases when $n$ is even and when $n$ is odd.
1. Case \( n = 2k \). Let sequence \( F \) be defined by
\[
F_m = F_m(k, x; f) = \binom{2k}{k-m} m^{2k+1} \left[ f^{(2k+1)}(x+m) - f^{(2k+1)}(x-m) \right] (m=1, 2, \ldots, k).
\]

**Theorem 1.** If \( f^{(2k+1)} \) is a nondecreasing function in \((a, b)\) \((b-a \geq 2k)\), then
\[
- \frac{D(F)}{(2k+1)!} \leq \lambda_{2k} \leq \frac{G(F)}{(2k+1)!} \quad (a+k < x < b-k; k \in \mathbb{N}).
\]

**Proof.** Since \( R(2k, 0; f) = 0 \), we have
\[
\begin{align*}
\lambda_{2k} &= \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} R(2k, j-k; f) \\
&= \frac{1}{(2k+1)!} \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} (k-j)^{2k+1} \left[ f^{(2k+1)}(x+\theta_{k-j}(k-j)) - f^{(2k+1)}(x-\theta_{k-j}(k-j)) \right],
\end{align*}
\]

i.e.
\[
\lambda_{2k} = \frac{1}{(2k+1)!} \sum_{m=1}^{k} (-1)^{k-m} s_m,
\]
where
\[
s_m = \binom{2k}{k-m} m^{2k+1} \left[ f^{(2k+1)}(x+\theta_{m}m) - f^{(2k+1)}(x-\theta_{m}m) \right].
\]

Since \( f^{(2k+1)} \) is a nondecreasing function in \((a, b)\), \( 0 \leq \theta_{m} \leq 1 \) and \( 0 \leq \theta_{-m} \leq 1 \), we deduce that \( s_m \in [0, F_m] \) when \( x \in (a+m, b-m) \).

Upon summing up the intervals (see [3]), we obtain
\[
\lambda_{2k} \in I \quad \forall \, x \in (a+k, b-k),
\]
where the interval \( I \) is given by
\[
I = \frac{1}{(2k+1)!} \sum_{m=1}^{k} (-1)^{k-m} [0, F_m]
\]
\[
= \frac{1}{(2k+1)!} \left[ -F_{k-1} + F_{k-3} + \cdots, F_k + F_{k-2} + \cdots \right]
\]
\[
= \left[ - \frac{D(F)}{(2k+1)!}, \frac{G(F)}{(2k+1)!} \right].
\]

Thus, Theorem 1 is proved.

**Example 1.** If \( f''' \) is a nondecreasing function in \((a, b)\), (7) is reduced to
\[
0 \leq \lambda_2 \leq \frac{1}{3!} F_1 \quad \forall \, x \in (a+1, b-1),
\]
i.e.
\[
f''(x) \leq f(x+1) - 2f(x) + f(x-1) \leq f''(x) + \frac{1}{6} \left( f'''(x+1) - f'''(x-1) \right).
\]

The first inequality in (8) includes inequality (1).

Similarly, for \( k=2 \) and \( k=3 \), (7) is reduced respectively to
\[
- \frac{1}{30} \left( f^{(5)}(x+1) - f^{(5)}(x-1) \right) \leq \lambda_4 \leq \frac{4}{15} \left( f^{(5)}(x+2) - f^{(5)}(x-2) \right).
\]
and
\[-\frac{16}{105} (f^{(7)}(x + 2) - f^{(7)}(x - 2)) \leq \lambda_6 \leq \frac{243}{560} (f^{(7)}(x + 3) - f^{(7)}(x - 3)) + \frac{1}{336} (f^{(7)}(x + 1) - f^{(7)}(x - 1)).\]

2. Case \(n = 2k + 1\). Let us define sequences \(P = (P_1, P_2, \ldots, P_k)\) and \(Q = (Q_1, Q_2, \ldots, Q_k)\) by
\[
P_m = P_m(k, x; f) = m^{2k+2} \left[ \binom{2k+1}{m-1} f^{(2k+2)}(x) + \binom{2k+1}{m-1} f^{(2k+2)}(x-m) \right],
\]
\[Q_m = Q_m(k, x; f) = P_m(k, x+m; f) \quad (m = 1, 2, \ldots, k).
\]

**Theorem 2.** If \(f^{(2k+2)}\) is a nondecreasing function in \((a, b)\) \((b - a > 2k + 1)\), then
\[
\frac{(k+1)^{k+2} f^{(2k+2)}(x) + D(P) - G(Q)}{(2k+2)!} \leq \lambda_{2k+1} \leq \frac{(k+1)^{k+2} f^{(2k+2)}(x+k+1) + D(Q) - G(P)}{(2k+2)!}.
\]
\((a + k < x < b - k - 1; \ k \in \mathbb{N})\)

and
\[
\frac{1}{2} f''(x) \leq \lambda_1 \leq f''(x + 1) \quad (a < x < b - 1).
\]

The proof of Theorem 2 is similar to that of Theorem 1.

**Example.** If \(f''\) is nondecreasing function in \((a, b)\), then
\[
f'(x) + \frac{1}{2} f''(x) \leq f(x + 1) - f(x) \leq f'(x) + \frac{1}{2} f''(x + 1) \quad \forall x \in (a, b - 1).
\]

Note that similar inequality is given in [1]. Namely, if \(f\) is an increasing function, the inequality
\[
f'(x) < f(x + 1) - f(x) < f'(x + 1).
\]
is proved.

For \(k = 1\) and \(k = 2\), (9) is reduced to
\[
\frac{5}{8} f^{(4)}(x) - \frac{1}{8} f^{(4)}(x + 1) \leq \lambda_3 \leq \frac{2}{3} f^{(4)}(x + 1) - \frac{1}{8} f^{(4)}(x) - \frac{1}{24} f^{(4)}(x - 1)
\]
and
\[
\frac{1}{144} f^{(6)}(x - 1) + \frac{15}{16} f^{(6)}(x) - \frac{4}{9} f^{(6)}(x + 2) \leq \lambda_5 \leq \frac{81}{80} f^{(6)}(x + 3) + \frac{1}{72} f^{(6)}(x) - \frac{4}{45} f^{(6)}(x - 1).
\]

**REFERENCES**