ON A FUNCTIONAL EQUATION HAVING
DETERMINANT FORM*

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In this paper we shall determine general continuous solutions of the functional equation

\[
\begin{vmatrix}
F_0(x) & F_1(x) & \cdots & F_n(x) \\
 a_0 x_1 & a_1 x_1 & \cdots & a_n x_1 \\
\vdots & \vdots & \ddots & \vdots \\
a_0 x_n & a_1 x_n & \cdots & a_n x_n \\
\end{vmatrix} = 0,
\]

under the following assumptions:

1° Unknown functions \( F_i : \mathbb{R} \to \mathbb{R} \),

2° \( 0 < a_0 < \cdots < a_n \),

3° \( x = \sum_{i=1}^{n} x_i \).

If by \( \Delta_i \) \((i=0,1,\ldots,n)\) we denote the determinant

\[
\Delta_i(x_1,\ldots,x_n) = \begin{vmatrix}
 x_1 & a_1 x_1 & \cdots & a_i x_1 & a_{i+1} x_1 & \cdots & a_n x_1 \\
 x_2 & a_1 x_2 & \cdots & a_i x_2 & a_{i+1} x_2 & \cdots & a_n x_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 x_n & a_1 x_n & \cdots & a_i x_n & a_{i+1} x_n & \cdots & a_n x_n \\
\end{vmatrix},
\]

i.e.

\[
\Delta_i(x_1,\ldots,x_n) = \begin{vmatrix}
 e^{k_0 x_1} & e^{k_1 x_1} & \cdots & e^{k_{i-1} x_1} & e^{k_{i+1} x_1} & \cdots & e^{k_n x_1} \\
 e^{k_0 x_2} & e^{k_1 x_2} & \cdots & e^{k_{i-1} x_2} & e^{k_{i+1} x_2} & \cdots & e^{k_n x_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 e^{k_0 x_n} & e^{k_1 x_n} & \cdots & e^{k_{i-1} x_n} & e^{k_{i+1} x_n} & \cdots & e^{k_n x_n} \\
\end{vmatrix},
\]

where we have put \( k_i = \log a_i \) \((i=0,1,\ldots,n)\), the functional equation (1) becomes

\[
\sum_{i=0}^{n} (-1)^i F_i(x) \Delta_i(x_1,\ldots,x_n) = 0.
\]

Let \( S = \{k_0, k_1, \ldots, k_n\} \) and \( T_i \) be an operator for which

\[
T_i S = (k_0, k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) \quad (i = 0, 1, \ldots, n).
\]

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Note that $T_iS$ is an arranged $n$-tuple. Let us by $(T_iS)_r$ ($r=1, \ldots, n!$) denote the $r$-th permutation of $T_iS=(T_iS)_1$. $(T_iS)_r$ is also an arranged $n$-tuple for each $r$.

If $(T_iS)_r$ and $X=(x_1, \ldots, x_n)$ are understood as vectors, the coordinates of which are given by the corresponding arranged $n$-tuple, we shall denote their scalar product by $(T_iS)_r X$.

With regard to the notations introduced, determinant (2) may be represented in the form

$$\Delta_i(x_1, \ldots, x_n) = \sum_{r=1}^{n!} (-1)^r \exp \{(T_iS)_r X\},$$

where $s_r$ is the number of inversions in permutation $(T_iS)_r$ in relation to the original permutation $T_iS$.

Then equation (3) becomes

$$\sum_{i=0}^{n} (-1)^i F_i(x) \left\{ \sum_{r=1}^{n!} (-1)^r \exp \{(T_iS)_r X\} \right\} = 0,$$

i.e.

(4) $$\sum_{r=1}^{n!} (-1)^r \left\{ \sum_{i=0}^{n} (-1)^i F_i(x) \exp \{(T_iS)_r X\} \right\} = 0,$$

that is

$$\sum_{r=1}^{n!} (-1)^r \left\{ \sum_{i=0}^{n} (-1)^i F_i(x) \exp \{(T_iS)_r X - (T_jS)_j X\} = 0 \quad (j\in\{1, \ldots, n\}) \right\}$$

or

(5) $$\sum_{r=1}^{n!} (-1)^r \left\{ \sum_{i=0}^{n} (-1)^i F(i, j, r, X) \right\} = 0 \quad (j\in\{1, \ldots, n\}),$$

where the expressions $F(i, j, r, X)$ are determined by

$$F(i, j, r, X) = F_i(x) \exp \{(T_iS)_r X - (T_jS)_j X\}$$

or

$$F(i, j, r, X) = F_i(x) \exp \left( \sum_{m=1}^{n} \lambda_m (i, j, r) x_m \right) \quad (\lambda_m \in \mathbb{R}).$$

With regard to the values taken by the numbers $\lambda_m$ we shall distinguish three kinds of the expression $F(i, j, r, X)$:

1° Expressions which are the functions of $x$. Those are the ones for which $\lambda_m=0$ ($m=1, \ldots, n$). Such a single expression is $F(j, j, 1, X)$.

2° Expressions which may be the functions of $x$. Those are the ones for which $\lambda_m>0$ ($m=1, \ldots, n$).

Such a single expression is $F(0, n, 1, X)$. It is the function of $x$ if and only if all $\lambda_m$ are mutually equal, i.e. if and only if

$$k_1-k_0=k_2-k_1=\cdots=k_n-k_{n-1}.$$ 

3° Other expressions.
Theorem 1. If $a_0, a_1, \ldots, a_n$ do not make a geometric progression the functional equation (1) in the set of continuous functions has only the trivial solutions $F_i(x) = 0$ ($i = 0, 1, \ldots, n$).

Proof. Instead of equation (1), let us observe its equivalent form (5).

Let $j \in \{1, \ldots, n-1\}$. In that case from all the expressions $F(i, j, r, X)$ only the expression $F(j, j, 1, X)$, given by

$$F(j, j, 1, X) = F_j(x),$$

is the function of $x$. With regard that each of the remaining expressions contains the exponential factor which is not the function of $x$, we deduce that $F_j(x) = 0$. Since $j$ is an arbitrary element of the set $\{1, \ldots, n-1\}$, it follows

$$F_j(x) = 0 \quad (j = 1, \ldots, n-1). \quad (6)$$

Using (6), equation (5) for $j = n$, becomes

$$\sum_{r=1}^{n!} (-1)^r \left\{ F(0, n, r, X) + (-1)^n F(n, n, r, X) \right\} = 0.$$

$F(n, n, 1, X) = F_n(x)$ is the function of $x$.

Expressions $F(0, n, r, X)$ and $F(n, n, r, X)$ for $r = 2, 3, \ldots, n!$ are not the functions of $x$.

Since $a_i$ ($i = 0, 1, \ldots, n$) by assumption do not make a geometric progression, all $\lambda_m = k_m - k_{m-1}$ are not mutually equal, which means that neither

$$F(0, n, 1, X) = F_0(x) \exp \left\{ \sum_{m=1}^{n} (k_m - k_{m-1}) x_m \right\}$$

is the function of $x$.

From here, similarly as for $j \neq n$, we deduce that $F_n(x) = 0$.

Therefore, equation (5), and (3) respectively, become

$$F_0(x) \Delta_0 (x_1, \ldots, x_n) = 0.$$

Since $\Delta_0 \neq 0$ and $F_0$, as assumed, is a continuous function, it follows $F_0(x) = 0$.

Thus, the Theorem 1 is proved.

Theorem 2. If $a_0, a_1, \ldots, a_n$ make a geometric progression with quotient $q$, the functional equation (1) has the general solution determined by

$$F_0(x) = F(x),$$

$$F_j(x) = 0 \quad (j = 1, \ldots, n-1),$$

$$F_n(x) = (-1)^{n-1} q^* F(x),$$

where $F$ is an arbitrary continuous function with values in $\mathbb{R}$. 
**Proof.** Let us observe equation (1) in form (5). If $j = 1, \ldots, n-1$, as in the proof of Theorem 1 it follows $F_j(x) = 0$. Using this, equation (4) becomes

$$\sum_{r=1}^{n!} (-1)^r \{F_0(x) \exp \{(T_0 S)_r X\} + (-1)^n F_n(x) \exp \{(T_n S)_r X\}\} = 0,$$

i.e.

$$\sum_{r=1}^{n!} (-1)^r \exp \{(T_n S)_r X\} (F_0(x) \exp \{(T_0 S)_r X\} + (-1)^n F_n(x)) = 0. \quad (7)$$

With regard that $\exp \{(T_0 S)_r X\} = \exp \{x \log q\} = q^x$, equation (7) becomes

$$(F_0(x) q^x + (-1)^n F_n(x)) \sum_{r=1}^{n!} (-1)^r \exp \{(T_n S)_r X\} = 0,$$

i.e.

$$(F_0(x) q^x + (-1)^n F_n(x)) \Delta_n(x_1, \ldots, x_n) = 0. \quad (8)$$

Since $\Delta_n \neq 0$ and $F_0$ and $F_n$ are continuous functions, it follows

$$F_0(x) q^x + (-1)^n F_n(x) = 0,$$

wherefrom we obtain

$$F_0(x) = F(x), \quad F_n(x) = (-1)^{n-1} q^x F(x),$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary continuous function.

Thus the proof is finished.

**Another proof of Theorem 2.** Putting $a_i = q^i a_0 (i = 1, \ldots, n)$, equation (1) becomes

$$a_0^x \begin{vmatrix} F_0(x) & F_1(x) & \cdots & F_n(x) \\ 1 & q^x & q^{nx} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q^{nx} & q^{nx_n} & \cdots & \end{vmatrix} = 0,$$

i.e. it is reduced to equation

$$\sum_{i=1}^{n!} (-1)^i F_i(x) D_i = 0, \quad (8)$$

where

$$D_i = \begin{vmatrix} 1 & q^{x_1} & q^{2x_1} & \cdots & q^{(l-1)x_1} & q^{(l+1)x_1} & \cdots & q^{nx_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ q^{x_n} & q^{2x_n} & q^{(l-1)x_n} & q^{(l+1)x_n} & \cdots & \cdots & \cdots & \end{vmatrix}. \quad (9)$$

Since (see [2], Problem 2.50.)

$$D_i = \sigma_{n-i} (q^{x_1}, \ldots, q^{nx}) V_n(q^{x_1}, \ldots, q^{nx}),$$

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where \( \sigma_p \) represents a symmetrical function of the order \( p \) of the elements \( q^1, \ldots, q^n \), and \( V_n \) the corresponding \textit{Vandermonde}\'s determinant, equation (8) becomes

\[
\sum_{i=0}^{n} (-1)^i F_i(x) \sigma_{n-i} V_n = 0,
\]
i.e. it reduces to the equation

(9) \[
\sum_{i=0}^{n} (-1)^i F_i(x) \sigma_{n-i} = 0,
\]
because \( V_n \neq 0 \), functions \( F_i \) being continuous, as it was supposed.

If we put

(10) \[
G_i(x) = (-1)^i F_i(x) \quad (i = 1, \ldots, n - 1), \quad G_n(x) = F_0(x) q^x + (-1)^n F_n(x),
\]
equation (9) becomes

(11) \[
\sum_{i=1}^{n} G_i(x) \sigma_{n-i} = 0.
\]

For \( x_2 = x_3 = \ldots = x_n = t \), \( x_i = x - (n - 1) t \), we have

\[
\sigma_i(q^1, \ldots, q^n) = \sigma_i(q^1, q^r, \ldots, q^n) = \binom{n-1}{i} q^{x^i} + \binom{n-1}{i-1} q^{x^i} q^{(i-1)t} (i = 1, \ldots, n - 1),
\]
so equation (11) becomes

\[
\sum_{i=1}^{n} G_i(x) \left[ \binom{n-1}{i-1} q^{x^i} q^{(i-1)t} + \binom{n-1}{i} q^{x^i} q^{-i} \right] = 0,
\]
or

(12) \[
\sum_{j=0}^{2n-1} H_j(x) q^j = 0,
\]
where

(13) \[
H_j(x) = \begin{cases} 
\binom{n-1}{j-1} q^x G_{n-j}(x) & (j = 0, 1, \ldots, n - 1) \\
\binom{n-1}{j-n} G_{n-j}(x) & (j = n, \ldots, 2n - 1).
\end{cases}
\]

Let now \( x = \alpha = \text{const} (\alpha \in \mathbb{R}) \). Then (12) is reduced to

(14) \[
\sum_{j=0}^{2n-1} H_j(\alpha) q^j = 0.
\]

Since \( q \neq 1 \), system of functions \( 1, q^r, q^2t, \ldots, q^{(2n-1)t} \) is linearly independent, thus from (14) it follows

\[
H_j(\alpha) = 0 \quad (j = 0, 1, \ldots, 2n - 1)
\]
for each real \( \alpha \), i.e. \( H_j(\alpha) \equiv 0. \)
On the basis of this result and equality (13) we deduce that

\[ G_i(x) = 0 \quad (i = 1, \ldots, n), \]

hence, according to (10), it follows

\[ F_i(x) = 0 \quad (i = 1, \ldots, n - 1), \quad F_0(x) q^n + (-1)^n F_n(x) = 0. \]

Here immediately follows that Theorem 2 holds.

REFERENCES


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