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547. ON AN INEQUALITY OF IYENGAR*

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0. K. S. K. IYENGAR [1] has proved the following:

Theorem A. *Let f be a differentiable function on $[a, b]$ and $|f'(x)|\leqq M$. Then*

$$(0.1)\qquad \left|\int_a^b f(x)\,\mathrm{d}x-\frac{1}{2}(b-a)\big(f(a)+f(b)\big)\right|\leqq\frac{M(b-a)^2}{4}-\frac{1}{4M}\big(f(b)-f(a)\big)^2.$$

Similar inequalities can be found in the book [2] by D. S. MITRINOVIĆ.
Inequality (0.1) can be written in the form

$$(0.2)\qquad \left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{2}\big(f(a)+f(b)\big)\right|\leqq\frac{M(b-a)}{4}(1-q^2),$$

where

$$(0.3)\qquad q=\frac{|f(b)-f(a)|}{M(b-a)}.$$

REMARK. If in Theorem A we replace the condition $|f'(x)|\leqq M$ by $m\leqq f'(x)\leqq M$, we obtain the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{2}\big(f(a)+f(b)\big)\right|\leqq\frac{(M-m)(b-a)}{2}\left(\frac{1}{4}-\frac{\left(\dfrac{f(b)-f(a)}{b-a}-\dfrac{M+m}{2}\right)^2}{(M-m)^2}\right).$$

In this paper we shall give some generalizations of Theorem A.

1. We use the following notation

$$(1.1)\qquad A(f;p)=\frac{\displaystyle\int_a^b p(x)f(x)\,\mathrm{d}x}{\displaystyle\int_a^b p(x)\,\mathrm{d}x}.$$

Theorem 1. *Let $x\mapsto f(x)$ be a differentiable function defined on $[a, b]$ and $|f'(x)|\leqq M$ for every $x\in(a, b)$. If $x\mapsto p(x)$ is an integrable function on (a, b) such that*

$$0<c\leqq p(x)\leqq\lambda c\qquad(\lambda\geqq 1,\ x\in[a, b]),$$

* Presented June 29, 1976 by D. S. MITRINOVIĆ.

the following inequality holds

$$(1.2)\qquad \left|A(f;p)-\frac{1}{2}\big(f(a)+f(b)\big)\right|\leqq\frac{M(b-a)}{2}\cdot\frac{(\lambda+q)(1-q^2)+2(\lambda-1)q}{2\lambda(1+q)-(\lambda-1)(1+q^2)},$$

where A and q are defined by (1.1) *and* (0.3) *respectively.*

Proof. From $|f'(x)|\leqq M$ $(\forall x\in(a,b))$ it follows

$$-M(x-a)\leqq f(x)-f(a)\leqq M(x-a)$$

and

$$-M(b-x)\leqq f(b)-f(x)\leqq M(b-x),$$

i.e.,

$$f(a)-M(x-a)\leqq f(x)\leqq f(x)+M(x-a)$$

and

$$f(b)-M(b-x)\leqq f(x)\leqq f(b)+M(b-x),$$

wherefrom

$$(1.3)\qquad \max\big(f(a)-M(x-a),\ f(b)-M(b-x)\big)\leqq f(x)$$
$$\leqq\min\big(f(a)+M(x-a),\ f(b)+M(b-x)\big).$$

Since, for every $\alpha,\beta\in\mathbf{R}$,

$$\min(\alpha,\beta)=\frac{1}{2}(\alpha+\beta-|\beta-\alpha|)\ \text{ and }\ \max(\alpha,\beta)=\frac{1}{2}(\alpha+\beta+|\beta-\alpha|),$$

inequalities (1.3) become

$$(1.4)\qquad -\frac{1}{2}\big(M(b-a)-g(x)\big)\leqq f(x)-\frac{1}{2}\big(f(a)+f(b)\big)$$
$$\leqq\frac{1}{2}\big(M(b-a)-h(x)\big),$$

where

$$g(x)=|M(2x-a-b)+f(b)-f(a)|\ \text{ and }\ h(x)=|M(2x-a-b)-f(b)+f(a)|.$$

If $p(x)\geqq 0$, it follows from (1.4):

$$(1.5)\qquad -\frac{1}{2}\left(M(b-a)\int_a^b p(x)\,dx-\int_a^b p(x)g(x)\,dx\right)$$
$$\leqq\int_a^b p(x)f(x)\,dx-\frac{1}{2}\big(f(a)+f(b)\big)\int_a^b p(x)\,dx$$
$$\leqq\frac{1}{2}\left(M(b-a)\int_a^b p(x)\,dx-\int_a^b p(x)h(x)\,dx\right),$$

i.e.,

$$(1.6)\qquad -\frac{1}{2}\big(M(b-a)-A(g;p)\big)\leqq A(f;p)-\frac{1}{2}\big(f(a)+f(b)\big)$$
$$\leqq\frac{1}{2}\big(M(b-a)-A(h;p)\big).$$

Since, $x \in [a, b]$, we have

$$0 \leqq g(x) \leqq M(b-a)(1+q) \text{ and } 0 \leqq h(x) \leqq M(b-a)(1+q). \tag{1.7}$$

Also,

$$\mu = \frac{1}{b-a} \int_a^b g(x)\, \mathrm{d}x = \frac{1}{b-a} \int_a^b h(x)\, \mathrm{d}x = \frac{M(b-a)}{2}(1+q^2). \tag{1.8}$$

J. KARAMATA in [3] has proved the following result (transposed to the interval (a, b)):

If p and Φ are integrable functions on $[a, b]$ and

$$n \leqq \Phi(x) \leqq N, \quad \mu = \frac{1}{b-a} \int_a^b \Phi(t)\, \mathrm{d}t, \quad 0 < c \leqq p(x) \leqq \lambda c \qquad (\lambda \geqq 1),$$

then

$$\frac{\lambda n (N-\mu) + N(\mu-n)}{\lambda (N-\mu) + (\mu-n)} \leqq \frac{\int_a^b p(t)\Phi(t)\, \mathrm{d}t}{\int_a^b p(t)\, \mathrm{d}t} \leqq \frac{n(N-\mu) + \lambda N(\mu-n)}{(N-n) + \lambda(\mu-n)}. \tag{1.9}$$

Starting from this result and using (1.7) and (1.8) we get

$$A(g; p) \geqq \frac{M(b-a)(1+q)\dfrac{M(b-a)}{2}(1+q^2)}{\lambda M(b-a)(1+q) - (\lambda-1)\dfrac{M(b-a)}{2}(1+q^2)} = M(b-a)B(\lambda; q) \tag{1.10}$$

and

$$A(h; p) \geqq M(b-a)B(\lambda; q), \tag{1.11}$$

where

$$B(\lambda; q) = \frac{(1+q)(1+q^2)}{2\lambda(1+q) - (\lambda-1)(1+q^2)}. \tag{1.12}$$

Combining (1.6), (1.10) and (1.11), we obtain (1.2).

REMARK 1. The inequality (1.2) also holds if for f we suppose only that the LIPSCHITZ's condition:

$$|f(y) - f(x)| \leqq M|y-x| \qquad (\forall\, x, y \in [a, b])$$

is satisfied.

REMARK 2. If $p(x) \equiv 1$ ($\Rightarrow \lambda = 1$), inequality (1.2) reduces to (0.2).

2. Now, we shall use the following result from theory of convex functions (see [2, p. 18]):

Theorem B. 1° *Function f is convex on $[a, b]$ if and only if for every point $x_0 \in [a, b]$ function $x \mapsto \dfrac{f(x) - f(x_0)}{x - x_0}$ is nondecreasing on $[a, b]$.*

2° *Differentiable function f is convex if and only if f' is a nondecreasing function on $[a, b]$.*

3° *Twice differentiable function f is convex on $[a, b]$ if and only if $f''(x) \geqq 0$ for all $x \in (a, b)$.*

First, we shall prove the following:

Lemma 1. *Let $x \mapsto F(x)$ be a differentiable function defined on $[a, b]$. The inequalities*

$$-M \leqq F'(x) \leqq M \qquad (\forall\, x \in (a, b)) \tag{2.1}$$

hold if and only if

$$x \mapsto F(x) + M(x-a) \text{ is a nondecreasing function on } [a, b] \tag{2.2}$$

and

$$x \mapsto F(x) - M(x-a) \text{ is a nonincreasing function on } [a, b]. \tag{2.3}$$

Proof. (a) The conditions are necessary. Suppose that inequalities (2.1) hold and let $a \leqq x \leqq y \leqq b$. Then

$$-M(y-x) \leqq F(y) - F(x) \leqq M(y-x),$$

wherefrom

$$F(x) + M(x-a) \leqq F(y) + M(y-a) \text{ and } F(y) - M(y-a) \leqq F(x) - M(x-a).$$

(b) The conditions (2.2) and (2.3) are sufficient. Let the conditions (2.2) and (2.3) be fulfilled. Then

$$F'(x) + M \geqq 0 \text{ and } F'(x) - M \leqq 0,$$

i.e.,

$$-M \leqq F'(x) \leqq M.$$

This completes the proof.

From this lemma it follows:

Lemma 2. *Let $x \mapsto F(x)$ be a differentiable function defined on $[a, b]$. Inequalities* (2.1) *hold if ond only if*

$$x \mapsto -F(x) - M(b-x) \text{ is a nondecreasing function on } [a, b]$$

and

$$x \mapsto -F(x) + M(b-x) \text{ is a nonincreasing function on } [a, b].$$

The Theorem 1 can be generalised as follows:

Theorem 2. *Let $x \mapsto f(x)$ be a twice differentiable function defined on $[a, b]$ and let $|f''(x)| \leqq M$ for every $x \in (a, b)$ and $f'(a) = f'(b)$. If $x \mapsto p(x)$ is an integrable function on (a, b) such that*

$$0 < c \leqq p(x) \leqq \lambda c \qquad (\lambda \geqq 1,\ x \in [a, b]), \tag{2.4}$$

the following inequality

$$\left| A(f;\, p) - \frac{1}{2}\big(f(a) + f(b)\big) + \frac{1}{8}\left(f'(a) + \frac{f(b)-f(a)}{b-a}\right) A\left(\frac{a+b}{2} - x;\, p\right) \right| \tag{2.5}$$
$$\leqq \frac{M(b-a)^2}{8} \cdot \frac{(\lambda+q)(1-q^2) + 2(\lambda-1)q}{2\lambda(1+q) - (\lambda-1)(1+q^2)}$$

holds, where A is defined by (1.1), *and*

$$q=\frac{2}{M(b-a)}\left|\frac{f(b)-f(a)}{b-a}-f'(a)\right|.$$

Proof. Let

$$|f''(x)|\leqq M \qquad (\forall\, x\in(a, b)). \tag{2.6}$$

Then

$$x\mapsto f(x)+\frac{M}{2}(x-a)^2 \text{ is a convex function on } [a, b]$$

and

$$x\mapsto f(x)-\frac{M}{2}(x-a)^2 \text{ is a concave function on } [a, b],$$

from where, with regard to Theorem B, it follows that

$$x\mapsto\frac{f(x)-f(a)}{x-a}+\frac{M}{2}(x-a), \text{ is a nondecreasing function on } [a, b] \tag{2.7}$$

and

$$x\mapsto\frac{f(x)-f(a)}{x-a}-\frac{M}{2}(x-a), \text{ is a nonincreasing function on } [a, b]. \tag{2.8}$$

Using (2.7), (2.8) and Lemma 1, we conclude that function F defined on $[a, b]$ by

$$F(x)=\begin{cases}\dfrac{f(x)-f(a)}{x-a} & (x\neq a),\\ f'(a) & (x=a),\end{cases}$$

satisfies the conditions of Theorem 1, with $|F'(x)|\leqq\frac{M}{2}$ $(\forall\, x\in(a, b))$.

Substituting F in (1.5), i.e. in

$$-\frac{1}{2}\left(\frac{M}{2}(b-a)\int_a^b P(x)\,\mathrm{d}x-\int_a^b P(x)\,g(x)\,\mathrm{d}x\right) \tag{2.9}$$

$$\leqq\int_a^b P(x)\,F(x)\,\mathrm{d}x-\frac{1}{2}\big(F(a)+F(b)\big)\int_a^b P(x)\,\mathrm{d}x$$

$$\leqq\frac{1}{2}\left(\frac{M}{2}(b-a)\int_a^b P(x)\,\mathrm{d}x-\int_a^b P(x)\,h(x)\,\mathrm{d}x\right),$$

where

$$g(x)=\left|\frac{M}{2}(2x-a-b)+F(b)-F(a)\right|$$

and

$$h(x)=\left|\frac{M}{2}(2x-a-b)-F(b)+F(a)\right|,$$

and $P(x)=(x-a)p(x)$ $(p(x)>0)$, we obtain

$$-\frac{1}{2}\left(\frac{M}{2}(b-a)A(x-a;p)-A(g_a;p)\right) \tag{2.10}$$

$$\leqq A(f;p)-f(a)-\frac{1}{2}\left(f'(a)+\frac{f(b)-f(a)}{b-a}\right)A(x-a;p)$$

$$\leqq\frac{1}{2}\left(\frac{M}{2}(b-a)A(x-a;p)-A(h_a;p)\right),$$

where $g_a(x)=(x-a)g(x)$ and $h_a(x)=(x-a)h(x)$.

Similarly, it follows from (2.6) that

$$x\mapsto\frac{f(b)-f(x)}{b-x}-\frac{M}{2}(b-x) \text{ is a nondecreasing function on } [a,b]$$

and

$$x\mapsto\frac{f(b)-f(x)}{b-x}+\frac{M}{2}(b-x) \text{ is a nonincreasing function on } [a,b],$$

whence, in respect of Lemma 2, we conclude that function G is given by

$$G(x)=\begin{cases}-\dfrac{f(b)-f(x)}{b-x} & (x\neq b),\\ -f'(b) & (x=b),\end{cases}$$

which also satisfies the conditions of Theorem 1, with $|G'(x)|\leqq\frac{M}{2}$ $(\forall x\in(a,b))$.

Since $f'(a)=f'(b)$, we have

$$\frac{G(b)-G(a)}{\frac{M}{2}(b-a)}=\frac{F(b)-F(a)}{\frac{M}{2}(b-a)}=\frac{2}{M(b-a)}\left(\frac{f(b)-f(a)}{b-a}-f'(a)\right).$$

If, we replace $F(x)$ and $P(x)$ by $G(x)$ and $(b-x)p(x)$ respectively, in (2.9), we obtain

$$-\frac{1}{2}\left(\frac{M}{2}(b-a)A(b-x;p)+A(g_b;p)\right) \tag{2.11}$$

$$\leqq A(f;p)-f(b)-\frac{1}{2}\left(f'(a)+\frac{f(b)-f(a)}{b-a}\right)A(b-x;p)$$

$$\leqq\frac{1}{2}\left(\frac{M}{2}(b-a)A(b-x;p)+A(h_b;p)\right).$$

Since

$$A(C_1f_1+C_2f_2;p)=C_1A(f_1;p)+C_2A(f_2;p) \text{ and } A(C;p)=C,$$

where C_1, C_2, C are arbitrary real constants, we find by adding (2.10) and (2.11)

$$(2.12)\quad -\frac{1}{4}\left(\frac{M}{2}(b-a)^2-(b-a)A(g;p)\right)$$
$$\leqq A(f;p)-\frac{1}{2}(f(a)+f(b))+\frac{1}{8}\left(f'(a)+\frac{f(b)-f(a)}{b-a}\right)A\left(\frac{a+b}{2}-x;\,p\right)$$
$$\leqq \frac{1}{4}\left(\frac{M}{2}(b-a)^2-(b-a)A(h;p)\right).$$

With respect to (2.4), and applying (1.9), we have

$$(2.13)\qquad A(g;p)\geqq\frac{M}{2}(b-a)B(\lambda;q)\ \text{ and }\ A(h;p)\geqq\frac{M}{2}(b-a)B(\lambda;q),$$

where B is defined by (1.12).

Finally, using (2.12) and (2.13), we obtain (2.5), which proves the Theorem 2.

From Theorem 2, we directly get the following theorem.

Theorem 3. *Let functions $x\mapsto f(x)$ satisfy the conditions as in Theorem 2 and let*

$$p\left(\frac{a+b}{2}-x\right)=p\left(\frac{a+b}{2}+x\right)\qquad\left(\forall\,x\in\left[-\frac{b-a}{2},\,\frac{b-a}{2}\right]\right).$$

Then

$$\left|A(f;p)-\frac{1}{2}(f(a)+f(b))\right|\leqq\frac{M(b-a)^2}{8}\cdot\frac{(\lambda+q)(1-q^2)+2(\lambda-1)q}{2\lambda(1+q)-(\lambda-1)(1+q^2)},$$

where q is given by

$$q=\frac{2}{M(b-a)}\left|\frac{f(b)-f(a)}{b-a}-f'(a)\right|.$$

A corollary of this theorem is:

Corollary. *Let $x\mapsto f(x)$ be a twice differentiable function defined on $[a,b]$ and such that $|f''(x)|\leqq M$ for every $x\in(a,b)$ and $f'(a)=f'(b)$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{2}(f(a)+f(b))\right|\leqq\frac{M(b-a)^2}{16}-\frac{1}{4M}\left(\frac{f(b)-f(a)}{b-a}-f'(a)\right)^2.$$

This result is a natural extension of Theorem A.

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