548. ON SOME GENERALIZATIONS OF ZMOROVIĆ'S INEQUALITY*

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In [1] V. A. Zmorošić has proved the following theorem:

**Theorem A.** If the function $f: [a-h, a+h] \to \mathbb{R}$ is twice continuously-differentiable, then

$$\int_{a-h}^{a+h} (f''(x))^2 \, dx \geq \frac{3}{2h^2} [f(a+h) - 2f(a) + f(a-h)]^2,$$

with equality if and only if $f$ is given by

$$f(x) = \begin{cases} C_1 \{(h-a+x)^3 + 6h^2(a-x)\} + C_2x + C_3 & (x \in [a-h, a]) \\ C_1(h+a-x)^3 + C_2x + C_3 & (x \in [a, a+h]), \end{cases}$$

where $C_1, C_2, C_3$ are arbitrary real constants.

The mentioned Zmorošić's result is an improvement of the inequality

$$\int_{a-h}^{a+h} (f''(x))^2 \, dx \geq \frac{1}{2h^2} [f(a+h) - 2f(a) + f(a-h)]^2,$$

which was, through geometric considerations, obtained by M. A. Lavrent'ev (see [1]), under stronger conditions. Namely, Lavrent'ev has proved the last inequality under the condition that $f$ is a four times continuously-differentiable function on $[a-h, a+h]$.

A similar result may be found in [2] (Theorem 264):

**Theorem B.** If

$$f(-1) = -1, \quad f(1) = 1, \quad f'(-1) = f'(1) = 0$$

and $k$ is a positive integer, then

$$\int_{-1}^{1} (f''(x))^{2k} \, dx \geq 2 \left( \frac{4k-1}{2k-1} \right)^{2k-1},$$

with inequality unless

$$f(x) = \frac{4k-1}{2k} x - \frac{2k-1}{2k} x^{(4k-1)/(2k-1)}.$$

This paper will give some generalizations of Theorem A.

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Let us first introduce the notation
\[
\Delta = \frac{1}{h^2} [f(a+h) - 2f(a) + f(a-h)]
\]
and let us note that
\[
(1) \quad \int_0^h (h-t) (f''(a-t) + f''(a+t)) \, dt = h^2 \Delta.
\]

**Theorem 1.** Let the function \( f: [a-h, a+h] \to \mathbb{R} \) be twice continuously-differentiable and \( g: [a-h, a+h] \to \mathbb{R}^+ \) continuous.

Then
\[
(2) \quad \int_{a-h}^{a+h} g(x) \left| f''(x) \right|^\gamma \, dx \geq \frac{h^2 \gamma}{\delta(\gamma-1)} \left| \Delta \right| \gamma \quad (\gamma > 1),
\]
where
\[
(3) \quad \delta(\gamma) = \int_0^h (h-t)^{\gamma-1} \left( g(a-t)^{\frac{1}{\gamma}} + g(a+t)^{\frac{1}{\gamma}} \right) \, dt.
\]

Equality in (2) holds if and only if the function \( f \) is given by
\[
(4) \quad f(x) = \begin{cases} 
  A_1 \int_a^x (x-t) \left| \frac{h-a+t}{g(t)} \right|^{\gamma-1} \, dt + A_2 x + A_3 & (x \in [a-h, a]) \\
  A_1 \int_a^x (x-t) \left| \frac{h+a-t}{g(t)} \right|^{\gamma-1} \, dt + A_2 x + A_3 & (x \in [a, a+h]),
\end{cases}
\]
where \( A_1, A_2, A_3 \) are arbitrary real constants.

**Proof.** Let us put \( \gamma(t) = \left( g(a-t)^{\frac{1}{\gamma}} + g(a+t)^{\frac{1}{\gamma}} \right)^{1-\gamma} \). Then (3) becomes
\[
(5) \quad \delta(\gamma) = \int_0^h \left( \frac{h-t}{\gamma(t)} \right)^{\gamma-1} \, dt.
\]

Since
\[
\int_{a-h}^{a+h} g(x) \left| f''(x) \right|^\gamma \, dx = \int_0^h (g(a-t)^{\frac{1}{\gamma}} + g(a+t)^{\frac{1}{\gamma}})^{1-\gamma} \, dt,
\]
putting
\[
p_1 = g(a-t), \quad p_2 = g(a+t), \quad z_1 = f''(a-t), \quad z_2 = f''(a+t),
\]
and using inequality (see [3], [4])
\[
(6) \quad |z_1 + \cdots + z_n|^{\gamma} \leq \left( \sum_{i=1}^n p_i^{\frac{1}{\gamma-1}} \right)^{\gamma-1} (p_1 |z_1|^\gamma + \cdots + p_n |z_n|^\gamma)
\]
\((z_i \in \mathbb{C}, p_i > 0 (i = 1, \ldots, n), \gamma > 1)\),
we have

\[ \int_{a-h}^{a+h} g(x) |f''(x)|^r \, dx \geq \frac{1}{\delta(t)^{r-1}} \left( \int_0^h (\gamma(t)|f''(a-t)+f''(a+t)|^r \, dt \right)^{\frac{r}{r-1}} \left( \int_0^h \left( \frac{h-t}{\gamma(t)} \right)^{r-1} \, dt \right)^{\frac{1}{r-1}}. \]

Applying Hölder's inequality to the right side in (7), we obtain

\[ \int_{a-h}^{a+h} g(x) |f''(x)|^r \, dx \geq \frac{1}{\delta(r)^{r-1}} \left( \int_0^h (h-t) |f''(a-t)+f''(a+t)| \, dt \right)^r. \]

from where, with regard to (1), follows (2).

Since in (6) equality holds if and only if

\[ p_1 |z_1|^{r-1} = \ldots = p_n |z_n|^{r-1} \quad \text{and} \quad z_k \bar{z}_j = 0 \quad (k, \ j = 1, \ldots, n), \]

and in Hölder's inequality

\[ \int_{\alpha}^{\beta} \left| \Phi(x) \Psi(x) \right| \, dx \leq \left( \int_{\alpha}^{\beta} |\Phi(x)|^p \, dx \right)^{1/p} \left( \int_{\alpha}^{\beta} |\Psi(x)|^q \, dx \right)^{1/q} \left( \frac{1}{p} + \frac{1}{q} = 1, \ p > 1 \right) \]

if and only if \( |\Phi(x)|^p = C \left| \Psi(x) \right|^q \), where \( C \) is a real constant (see [5], p. 54), we conclude that equality in (2) holds if and only if

\[ g(a-t)^{r-1} f''(a-t) = g(a+t)^{r-1} f''(a+t), \]

(8)

\[ \left( \gamma(t)(f''(a-t)+f''(a+t)) \right)^r = C \left( \frac{h-t}{\gamma(t)} \right)^{r-1} \quad (C \in \mathbb{R}). \]

From (8), if we put \( C = A_1 \), follows

\[ f''(a-t) = A_1 \left( \frac{h-t}{g(a-t)} \right)^{r-1}, \quad f''(a+t) = A_1 \left( \frac{h-t}{g(a+t)} \right)^{r-1} \quad (0 \leq t \leq h), \]

from where, by integration, we obtain (4).
This completes the proof.

**Corollary 1.** If the function \( f: [a-h, a+h] \rightarrow \mathbb{R} \) is twice continuously-differentiable, then

\[
\int_{a-h}^{a+h} \left| f''(x) \right|^r \, dx \geq \left( \frac{2r-1}{2r-2} \right)^{r-1} h^r |\Delta|^r \quad (r > 1).
\]

Equality in (9) holds if and only if the function \( f \) is given by

\[
f(x) = \begin{cases} 
C_1 \left( (h-a+x) \frac{2r-1}{r-1} + \frac{4r-2}{r-1} \frac{h}{r} (a-x) \right) + C_2 x + C_3 & (x \in [a-h, a]) \\
C_1 \left( (h+a-x) \frac{2r-1}{r-1} + C_2 x + C_3 \right) & (x \in [a, a+h]),
\end{cases}
\]

where \( C_i (i=1, 2, 3) \) are arbitrary real constants.

Putting \( g(x) = 1 \), the statement of the Corollary 1 follows from Theorem 1.

**Remark.** For \( r = 2 \), the Corollary 1 reduces to Theorem A.

Observing the left side of inequality (2) in the form

\[
\int_{a-h}^{a+h} g(x) \Phi \left( f''(x) \right) \, dx,
\]

where \( t \mapsto \Phi(t) = |t|^r \quad (r > 1) \), we conclude that \( \Phi \) is a convex function. That gave us the idea to generalize Theorem 1, for a more general function \( \Phi \).

First, we give the following definition.

**Definition.** Continuous function \( \Phi: \mathbb{R} \rightarrow \mathbb{R}^+ \) belongs to the class \( M \) if there is a convex function \( F: \mathbb{R} \rightarrow \mathbb{R}^+ \) and real numbers \( \lambda \) and \( m \) so that for each \( x \in \mathbb{R} \) the inequalities

\[
F(x) \leq \Phi(x)^{1/m} \leq \lambda F(x) \quad (\lambda \geq 1; \ m > 1)
\]

are valid.

In this paper we have taken convex functions to mean continuous and JENSEN convex functions, as defined in [5].

One of the possible generalizations of Theorem 1 is:

**Theorem 2.** Let the functions \( \Phi: \mathbb{R} \rightarrow \mathbb{R}^+, f: [a-h, a+h] \rightarrow \mathbb{R}, g: [a-h, a+h] \rightarrow \mathbb{R}^+ \) satisfy the conditions:

1° \( \Phi \in M; \)

2° \( f \) twice continuously-differentiable;

3° \( g \) continuous.

Then

\[
\int_{a-h}^{a+h} g(x) \Phi \left( f''(x) \right) \, dx \geq \frac{h^m}{\delta^m (m-1)} \Phi(\Delta),
\]

where \( \delta \) is given by (3).
On some generalizations of Zmorović's inequality

Proof. Since \( \Phi \subseteq M \), i.e. for each \( x \in \mathbb{R} \) the inequalities

\[
F(x) \leq \Phi(x)^{1/m} \leq \lambda F(x) \quad (\lambda \geq 1; \ m > 1)
\]

hold, where \( F \) is a convex function, we have

\[
\int_{a-h}^{a+h} g(x) \Phi \left( f''(x) \right) \, dx = \int_0^h \left\{ g(a-t) \Phi \left( f''(a-t) \right) + g(a+t) \Phi \left( f''(a+t) \right) \right\} \, dt
\]

\[
= \int_0^h \left[ g(a-t) F \left( f''(a-t) \right)^m + g(a+t) F \left( f''(a+t) \right)^m \right] \, dt.
\]

Applying (6) to the right side of the last inequality, we obtain

\[
\int_{a-h}^{a+h} g(x) \Phi \left( f''(x) \right) \, dx \geq \int_0^h \left\{ \gamma(t) \left( F \left( f''(a-t) \right) + F \left( f''(a+t) \right) \right) \right\} \, dt,
\]

where, now \( \gamma(t) = \frac{1}{2} \left( g(a-t) + g(a+t) \right) \left( \frac{1}{1-m} + 1 \right) \frac{1}{m} \).

According to Jensen's inequality we have

\[
\int_{a-h}^{a+h} g(x) \Phi \left( f''(x) \right) \, dx \geq 2m \int_0^h \left( \gamma(t) F \left( u(t) \right) \right)^m \, dt,
\]

where, with regard to the assumption for function \( f \), function \( t \mapsto u(t) = \frac{1}{2} \left( f''(a-t) + f''(a+t) \right) \) is continuous on \([0, h]\).

On the other hand, since \( x \mapsto F(x) \) is a convex function, using Jensen's integral inequality (see for example [6], Theorem 6, p. 228), we have

\[
\int_0^h \left( h-t \right) F \left( u(t) \right) \, dt \geq \int_0^h \left( h-t \right) \, dt \cdot F \left\{ \frac{1}{h} \int_0^h \left( h-t \right) u(t) \, dt \right\} = \frac{h^2}{2} F(\Delta),
\]

because \( \int_0^h \left( h-t \right) u(t) \, dt = \frac{1}{h^2} \left( f(a+h) - 2f(a) + f(a-h) \right) = \Delta. \)

Similarly to the proof of Theorem 1, using Hölder's inequality, from (11) follows

\[
\int_{a-h}^{a+h} g(x) \Phi \left( f''(x) \right) \, dx \geq \frac{2^m}{\delta(m-1)} \left( \int_0^h \left( h-t \right) F \left( u(t) \right) \, dt \right)^m,
\]

where \( \delta \) is a constant depending on \( m \).
from where, with regard to (12), we have

\[ \int_{a-h}^{a+h} g(x) \Phi(f''(x)) \, dx \geq \frac{2^m}{\delta(m)^{m-1}} \left( \frac{h^3}{2} F(\Delta) \right)^m. \]

Since, according to (10)

\[ F(x) \geq \frac{1}{\lambda} \Phi(x)^{1/m} \quad (m > 1), \]

we finally obtain

\[ \int_{a-h}^{a+h} g(x) \Phi(f''(x)) \, dx \geq \frac{h^3 m}{\delta(m)^{m-1}} \Phi(\Delta), \]

which completes the proof.

**REMARK.** If \( \Phi(x)^{1/m} = F(x) \) \((m > 1)\), where \( F \) is a convex function, the inequality

\[ \int_{a-h}^{a+h} g(x) \Phi(f''(x)) \, dx \geq \frac{h^3 m}{\delta(m)^{m-1}} \Phi(\Delta), \]

holds.

The last inequality reduces to (2) for \( \Phi(x) = |x|^r \) \((r > 1)\) and \( m = r \).

**Corollary 2.** If functions \( \Phi: \mathbb{R} \rightarrow \mathbb{R}^+ \) and \( f: [a-h, a+h] \rightarrow \mathbb{R} \) satisfy the conditions:

1° \( \Phi(x)^{1/m} = F(x) \) \((m > 1); F \) convex function;
2° \( f \) twice continuously-differentiable.

Then

\[ \int_{a-h}^{a+h} \Phi(f''(x)) \, dx \geq \frac{(2m-1)^{m-1}}{2m} h \Phi(\Delta). \]

The inequality (14) is stronger if \( m \) is greater.

If we put \( g(x) = 1 \) \((\forall x \in [a-h, a+h])\) in (13), we obtain (14). On the other, since \( m \mapsto C_m = \frac{(2m-1)^{m-1}}{2m} \) \((m > 1)\) is an increasing function, inequality (14) is stronger if \( m \) is greater. The possible maximum value for \( C_m \) is \( C_\infty = \lim_{m \rightarrow \infty} C_m = \sqrt{e} \) and it exists, for example for the function \( x \mapsto \Phi(x) = e^{\lambda x} \) \((\lambda \in \mathbb{R})\).

**REFERENCES**