A GENERALIZATION OF A PROBLEM GIVEN
BY D. S. MITRINOVIC

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D. S. MITRINOVIC (see [1]) has stated the problem:
Let \( a, b, c, d \) be positive numbers and \( m \) a natural number. Then
\[
\left( \sqrt[m]{a^m + b^m} - \sqrt[m]{c^m + d^m} \right) \leq \min \left( |a - c| + |b - d|, \ |a - d| + |b - c| \right).
\]
Prove (1) and determine the corresponding estimation for
\[
\left( \sqrt[m]{\sum_{k=1}^{n} a_k^m} - \sqrt[m]{\sum_{k=1}^{n} b_k^m} \right),
\]
where \( a_k \) and \( b_k \) (\( k = 1, \ldots, n \)) are positive numbers and \( m \) a natural number.

Solving this problem, D. D. ADAMOVIĆ ([2], [3]), proved a more general result:

Theorem A. \( 1^\circ \) If \( p \geq 1 \) is a real number and \( a_k > 0, b_k > 0 \) (\( k = 1, \ldots, n \)), then
\[
\left| \left( \sum_{k=1}^{n} a_k^p \right)^{1/p} - \left( \sum_{k=1}^{n} b_k^p \right)^{1/p} \right| \leq \min \left\{ \sum_{k=1}^{n} |a_k - b_{\pi(k)}| : \pi \in \mathcal{P} \right\},
\]
where \( \mathcal{P} \) denotes the set of all permutations of the set \( \{1, \ldots, n\} \).

\( 2^\circ \) Excluding trivial cases \( p = 1 \) and \( n = 1 \), the inequality (2) turns in equality if and only if its right side is equal to zero.

\( 3^\circ \) With additional conditions
\[
\sum_{k=1}^{+\infty} a_k^p < +\infty, \quad \sum_{k=1}^{+\infty} b_k^p < +\infty,
\]
the statements \( 1^\circ \) and \( 2^\circ \) are true also for \( n = +\infty \).

In this paper we will prove a more general result than the statement \( 1^\circ \) in Theorem A. Namely, under certain conditions for function \( f \) and sequences \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), p = (p_1, \ldots, p_n) \) we will find a bound for
\[
\left| f^{-1} \left( \sum_{k=1}^{n} p_k f(a_k) \right) - f^{-1} \left( \sum_{k=1}^{n} p_k f(b_k) \right) \right|.
\]

First, we give the following definition.
Definition. A convex function \( f: [0, +\infty] \rightarrow [0, +\infty] \) belongs to the class \( M \) if \( f(0) = 0 \) and if the function \( F, \) defined by \( F(t) = \log f(e^t) \), is convex for all real \( t \).

The following auxiliary result generalizes a result of H. P. Mulholland (see [4]):

**Theorem 1.** Let \( f \in M \) and let the sequences of nonnegative numbers \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), p = (p_1, \ldots, p_n), p_n \geq 1 \) \((k = 1, \ldots, n)\) be given; then the inequality

\[
\frac{1}{f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k + b_k)\right)} \leq f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k)\right) + f^{-1}\left(\sum_{k=1}^{n} p_k f(b_k)\right)
\]

is valid.

**Proof.** Similarly as in proof of Theorem 1 from [4], now we shall introduce the function \( g \) and the sequence \( s = (s_1, \ldots, s_n) \), by means of \( f(x) = xg(x) \) and \( s_k = a_k + b_k \) \((k = 1, \ldots, n)\), respectively. Put

\[
G_f(a, p) = f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k)\right).
\]

In order to prove the inequality (3) it is enough to prove that the inequality

\[
\sum_{k=1}^{n} \frac{p_k x_k g(s_k)}{G_f(x, p)} \leq \sum_{k=1}^{n} \frac{p_k s_k g(s_k)}{G_f(s, p)}
\]

holds for all sequences \( x = (x_1, \ldots, x_n) \) for which \( 0 \leq x_k \leq s_k \) \((k = 1, \ldots, n)\).

If we assume that the left hand side of the inequality (5) is a functional \( U(x) \), then (5) can be represented in the form

\[
U(x) \leq U(s).
\]

The last inequality, in case \( n = 2 \), can be easily proved by the same method which is applied in proof of Theorem 1 in the paper [4].

Combining the inequalities, which are obtained from (6) for \( x = a \) and \( x = b \), we prove that the inequality (3) holds for \( n = 2 \).

Proof of the inequality (3) for arbitrary \( n \), we give by mathematical induction. Suppose that (3) holds for any \( n \geq 2 \). Then

\[
f^{-1}\left(\sum_{k=1}^{n+1} p_k f(a_k + b_k)\right) = f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k + b_k) + p_{n+1} f(a_{n+1} + b_{n+1})\right)
\]

\[
\leq f^{-1}\left(1 \cdot f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k)\right) + f^{-1}\left(\sum_{k=1}^{n} p_k f(b_k)\right) + p_{n+1} f(a_{n+1} + b_{n+1})\right)
\]

\[
\leq f^{-1}\left(\sum_{k=1}^{n+1} p_k f(a_k)\right) + f^{-1}\left(\sum_{k=1}^{n+1} p_k f(b_k)\right).
\]

Thus the proof is finished.
Theorem 2. If the conditions of Theorem 1 are fulfilled, the inequality

\[ |G_f(a, p) - G_f(b, p)| \leq \sum_{k=1}^{n} p_k |a_k - b_k| \]

holds, where \( G_f \) is defined by (4).

**Proof.** Since \( f \) is increasing function for \( x \geq 0 \), then

\[ f(a_k) = f((a_k - b_k) + b_k) \leq f(a_k - b_k) + f(b_k). \]

Applying Theorem 1 to the sequences \( c = (|a_1 - b_1|, \ldots, |a_n - b_n|) \) and \( b \) and using the property that \( f^{-1} \) is an increasing function we obtain

\[ G_f(a, p) \leq G_f(c + b, p) \leq G_f(c, p) + G_f(b, p), \]

i.e.

\[ G_f(a, p) - G_f(b, p) \leq G_f(c, p). \]

Permuting series \( a \) and \( b \), the inequality (8) becomes

\[ G_f(b, p) - G_f(a, p) \leq G_f(c, p). \]

Using the inequalities of P. M. Vasić (see [5], [3, pp. 368—369]) for convex functions we obtain

\[ \sum_{k=1}^{n} p_k f(c_k) \leq f\left( \sum_{k=1}^{n} p_k c_k \right), \]

as \( f(0) = 0 \). From (8), (9), (10) we obtain

\[ |G_f(a, p) - G_f(b, p)| \leq \sum_{k=1}^{n} p_k |a_k - b_k|, \]

which completes the proof.

**Remark 1.** Inequality (7) holds if the function \( f \) is defined by

\[ f(x) = x \exp\{H(\log x)\}, \]

where \( t \mapsto H(t) \) is a nondecreasing convex function in \((-\infty, +\infty)\).

For instance, the inequality (7) holds if

\[ f(x) = x^p e^{q x} \quad (p \geq 1, \ q \geq 0). \]

For \( q = 0 \), the inequality (7) can be reduced to

\[ \left| \left( \sum_{k=1}^{n} p_k a_k^p \right)^{\frac{1}{p}} - \left( \sum_{k=1}^{n} p_k b_k^p \right)^{\frac{1}{p}} \right| \leq \sum_{k=1}^{n} p_k |a_k - b_k|. \]

If \( p_k = \frac{1}{k} \) \((k = 1, \ldots, n)\), from the inequality (11) follows

\[ \left| \left( \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} - \left( \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}} \right| \leq \sum_{k=1}^{n} |a_k - b_k|. \]

wherefrom (2) is easily obtainable.
REMARK 2. If in (12) $a_k$ and $b_k$ are substituted with $p_k^{1/p}a_k$ and $p_k^{1/p}b_k$ respectively ($p_k \geq 0$) we obtain

$$
\left| \left( \sum_{k=1}^{n} p_k a_k^p \right)^{\frac{1}{p}} - \left( \sum_{k=1}^{n} p_k b_k^p \right)^{\frac{1}{p}} \right| \leq \sum_{k=1}^{n} p_k \left| a_k - b_k \right|.
$$

(13)

It is easy to show that for $p_k \geq 1$ ($k = 1, \ldots, n$), inequalities (11) and (13) are equivalent.

REFERENCES


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