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624. SOME CONSIDERATIONS ON FUNCTIONAL INEQUALITIES (I)

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Theorem 1. *Let* $x \mapsto f(x)$ *be a solution of the functional inequality*

$$f(x+y) \leqq f(x)\, f(y) \tag{1}$$

such that $f(0)=1$ *and* f'' *exists and is continuous in a neighbourhood of* $x=0$, *and let* f_1 *and* f_2 *be the even and odd parts of* f *respectively. Then, for every real* x

$$|f_2(x)| \leqq \sqrt{f_1(x)^2-1} \quad \textit{and} \quad 1 \leqq f_1(x) \leqq \operatorname{ch} \sqrt{k}\, x,$$

where $k=|f''(0)|$.

Proof. Since $f(x)=f_1(x)+f_2(x)$, the inequality (1) becomes

$$f_1(x+y)+f_2(x+y) \leqq f_1(x) f_1(y)+f_1(x) f_2(y)+f_2(x) f_1(y)+f_2(x) f_2(y). \tag{2}$$

If we put $x+y=0$ in (2) we obtain

$$\begin{aligned} f_1(0)+f_2(0) \leqq\, & f_1(x) f_1(-x)+f_1(x) f_2(-x) \\ & +f_2(x) f_1(-x)+f_2(x) f_2(-x), \end{aligned}$$

i.e.,

$$1 \leqq f_1(x)^2-f_2(x)^2, \tag{3}$$

since $f_1(0)=f(0)=1$, $f_2(0)=0$, $f_1(x)=f_1(-x)$, $f_2(x)=-f_2(-x)$.

Now, if we replace x and y by $-x$ and $-y$ respectively in (2), then

$$f_1(x+y)-f_2(x+y) \leqq f_1(x) f_1(y)-f_1(x) f_2(y)-f_2(x) f_1(y)+f_2(x) f_2(y). \tag{4}$$

Adding (2) and (4), we obtain

$$f_1(x+y) \leqq f_1(x) f_1(y)+f_2(x) f_2(y). \tag{5}$$

Substituting y for $-y$, the last inequality becomes

$$f_1(x-y) \leqq f_1(x) f_1(y)-f_2(x) f_2(y). \tag{6}$$

Finally, from (5) and (6) immediately follows:

$$f_1(x+y)+f_1(x-y) \leqq 2 f_1(x) f_1(y).$$

Under the mentioned conditions for f, this functional inequality has the solution (see [1])

$$f_1(x) \leqq \operatorname{ch} \sqrt{k}\, x \quad (k = |f_1''(0)| = |f''(0)|).$$

On the basis of the above, we conclude that f_1 and f_2 satisfy the inequalities

$$|f_2(x)| \leqq \sqrt{f_1(x)^2 - 1} \quad \text{and} \quad 1 \leqq f_1(x) \leqq \operatorname{ch} \sqrt{k}\, x$$

for every real x.

This completes the proof.

Theorem 2. *Let $x \mapsto g(x)$ be a given function differentiable at $x = 0$ with $g(0) = 1$ and $g'(0) = b$ $(b \in \mathbf{R})$. The functional inequality*

$$f(x+y) \geqq f(x)\, g(y)\, e^{2axy} \quad (a \in \mathbf{R}) \tag{7}$$

has the general solution f determined by

1° $f(x) = A^2 e^{ax^2+bx}$, *if* $g(x) \geqq e^{ax^2+bx}$;

2° $f(x) = -B^2 e^{ax^2+bx}$, *if* $g(x) \leqq e^{ax^2+bx}$;

3° $f(x) \equiv 0$, *otherwise,*

where A and B are arbitrary real constants.

Proof. Suppose that $x \mapsto f(x)$ is a solution of (7). Then, for small positive h, we have (see proof of Theorem 3 in [1])

$$\frac{f(x+h) - f(x)}{h} \geqq f(x)\, \frac{g(h)\, e^{2axh} - 1}{h}$$

and

$$f(x)\, \frac{\dfrac{e^{2ah(x+h)}}{g(-h)} - 1}{h} \geqq \frac{f(x+h) - f(x)}{h},$$

i.e.,

$$\frac{1}{h}\left(\frac{e^{2ah(x+h)}}{g(-h)} - 1\right) f(x) \geqq \frac{1}{h}\bigl(f(x+h) - f(x)\bigr) \geqq \frac{1}{h}\bigl(g(h)\, e^{2axh} - 1\bigr) f(x). \tag{8}$$

Note that for small negative h, the reverse inequalities in (8) hold.

Since

$$\lim_{h \to 0} \frac{1}{h}\bigl(g(h)\, e^{2axh} - 1\bigr) = \lim_{h \to 0} \frac{1}{h}\left(\frac{e^{2ah(x+h)}}{g(-h)} - 1\right) = 2ax + b,$$

we conclude that f' exists and

$$f'(x) = (2ax + b) f(x),$$

from where, by integration, we obtain

$$f(x)=Ce^{ax^2+bx},$$

where C is a constant.

By direct substitution of $f(x)$ in (7) statements 1° and 2° are obtained.

Suppose now that the curves $y=g(x)$ and $y=\Phi(x)=e^{ax^2+bx}$ have a crossing point $x=\alpha$. In the domain where $g(x)\geqq\Phi(x)$, the general solution of the functional inequality (7) is given by $f(x)=A^2\,\Phi(x)$ $(A\in\mathbf{R})$, and in the domain in which $g(x)\leqq\Phi(x)$, the general solution is $f(x)=-B^2\,\Phi(x)$ $(B\in\mathbf{R})$. Since f is continuous at $x=0$ (by differentiability) we conclude that $A=B=0$, i.e. $f(x)=0$.

This result is an extension of Theorem 3 from **[1]**.

From the Theorem 2 immediately follows:

Corollary 1. *The functions* $x\mapsto f(x)=e^{ax^2+bx}$ *and* $x\mapsto g(x)=e^{ax^2+bx}$ *are the only solution of the following system of functional inequalities*

$$f(x+y)\geqq f(x)\,g(y)\,e^{2axy},$$

$$g(x+y)\geqq g(x)\,f(y)\,e^{2axy},$$

with constraint conditions that $f(0)=1$, $g(0)=1$ *and* $g'(0)=b$ $(or\ f'(0)=b)$ *exist.*

Theorem 3. *Consider the functional inequality*

$$f(x+y)\geqq H(x,y)\,f(x)\,g(y), \tag{9}$$

where given functions $x\mapsto g(x)$ *and* $(x,y)\mapsto H(x,y)$ *satisfy the conditions*:

1° g *is differentiable at* $x=0$ *and* H *has the partial derivatives* $\dfrac{\partial H}{\partial x}$ *and* $\dfrac{\partial H}{\partial y}$ *on the x-axis*;

2° $g(0)\,H(x,0)=1,\quad \dfrac{\partial H(x,0)}{\partial x}=0.$

Then the most general solution to (9) *is necessarily of the form*

$$f(x)=Ce^{\int\psi(x)\,\mathrm{d}x},$$

where $\psi(x)=g'(0)\,H(x,0)+g(0)\,\dfrac{\partial H(x,0)}{\partial y}$ *and* C *is a constant.*

Consider a functional inequality which is connected with JENSEN-CARMICHAEL's functional equation (**[2]**)

$$f(x+y)\,f(x-y)=f(x)^2-f(y)^2. \tag{10}$$

Theorem 4. *Function* f *is the solution of the functional inequality*

$$f(x+y)\,f(x-y)\leqq f(x)^2-f(y)^2 \tag{11}$$

if and only if it is the solution of the functional equation (10).

Proof. If f is the solution of the functional equation (10), the statement is trivial.

Now suppose that f is the solution of the functional inequality (11). Then

$$(12) \qquad (f_1(x+y)+f_2(x+y))\,(f_1(x-y)+f_2(x-y)) \\ \leqq (f_1(x)+f_2(x))^2-(f_1(y)+f_2(y))^2,$$

where we put $f(x)=f_1(x)+f_2(x)$ (f_1 and f_2 are the even and odd parts of f respectively).

Substituting $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$, (12) becomes

$$(13) \qquad (f_1(x+y)+f_2(x+y))\,(f_1(x-y)-f_2(x-y)) \\ \leqq (f_1(y)+f_2(y))^2-(f_1(x)+f_2(x))^2.$$

Adding (12) and (13) we obtain inequality

$$f_1(x+y)f_1(x-y)+f_2(x+y)f_1(x-y)\leqq 0,$$

which, after substitution $\begin{pmatrix} x & y \\ -x & -y \end{pmatrix}$, becomes

$$f_1(x+y)f_1(x-y)-f_2(x+y)f_1(x-y)\leqq 0.$$

From the last two inequalities it follows

$$(14) \qquad f_1(x+y)f_1(x-y)\leqq 0.$$

Finally, if we put $y=0$, we obtain $f_1(x)^2\leqq 0$, i.e., $f_1(x)\equiv 0$, and we conclude that f is an odd function.

Now substitute $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ in inequality (11). Then

$$-f(x+y)f(x-y)\leqq f(y)^2-f(x)^2.$$

On the basis of (11) and the last inequality we conclude that f satisfies JENSEN-CARMICHAEL's equation (10). This completes the proof.

Corollary 2. *The general system of continuous solutions of the functional inequality* (11) *is*

$$f(x)=Cx, \quad f(x)=C\sin\alpha x, \quad f(x)=C\,\mathrm{sh}\,\alpha x,$$

where C and α are arbitrary constants.

REFERENCES

1. IH-CHING HSU: *On Some Functional Inequalities.* Aequationes Math. **9** (1973), 129—135.

2. J. ACZÉL: *Lectures on functional equations and their applications.* New York—London 1966.

3. J. E. WETZEL: *On the Functional Inequality $f(x+y)\geqq f(x)f(y)$.* Amer. Math. Monthly **74** (1967), 1065—1068.