THE STEFFENSEN INEQUALITY FOR CONVEX FUNCTION OF ORDER $n$

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0. J. F. Steffensen has proved the following results (see [1], [2], [3]):

**Theorem A.** Assume that two integrable functions $f$ and $g$ are defined on the interval $(a, b)$, that $f$ never increases and $0 \leq g(t) \leq 1$ in $(a, b)$. Then

\begin{equation}
\int_a^b f(t) \, dt \leq \int_a^b f(t) g(t) \, dt \leq \int_a^b f(t) \, dt,
\end{equation}

where

$$\lambda = \int_a^b g(t) \, dt.$$

**Theorem B.** Let $g_1$ and $g_2$ be functions defined on $[a, b]$ such that

\begin{equation}
\int_a^x g_1(t) \, dt \leq \int_a^x g_2(t) \, dt \quad (\forall \ x \in [a, b]) \quad \text{and} \quad \int_a^b g_1(t) \, dt = \int_a^b g_2(t) \, dt.
\end{equation}

Let $f$ be an nondecreasing function on $[a, b]$, then

\begin{equation}
\int_a^b f(x) g_1(x) \, dx \leq \int_a^b f(x) g_2(x) \, dx.
\end{equation}

If $f$ is a nonincreasing function on $[a, b]$, the reverse inequality holds.

In paper [4], M. Marjanović gave the elegant proof of Theorem A: Let in Theorem B be $g_2(x) = g(x)$, $\lambda = \int_a^b g(x) \, dx$ and $g_1(x) = 1 \ (x \in [a, a + \lambda])$ and $g_1(x) = 0 \ (x \in (a + \lambda, b])$. Then, we have

$$\int_a^{a+\lambda} f(x) \, dx = \int_a^b f(x) g_1(x) \, dx \geq \int_a^b f(x) g(x) \, dx,$$

which gives the second inequality in (0.1). First inequality in (0.1) is obtained similarly.
Let us notice that the quoted proof holds even with the weaker condition for function $g$, i.e. if

\[(0.4) \int_a^x g(x) \, dx \leq x - a \quad (\forall \, x \in [a, a + \lambda]) \quad \text{and} \quad \int_x^b g(x) \, dx \geq 0 \quad (\forall \, x \in [a + \lambda, b]).\]

From (0.4) it follows

\[(0.5) \int_x^b g(x) \, dx = \int_x^a g(x) \, dx - \int_a^x g(x) \, dx \geq \lambda - (x - a) \geq 0 \quad \text{for} \quad x \in [a, a + \lambda],\]

and

\[(0.6) \int_a^x g(x) \, dx = \int_a^b g(x) \, dx - \int_x^b g(x) \, dx \leq \lambda \leq x - a \quad \text{for} \quad x \in [a + \lambda, b].\]

Combining (0.4), (0.5), (0.6), we obtain that implication (0.4) $\Rightarrow$ (0.7) holds, where

\[(0.7) \int_a^x g(x) \, dx \leq x - a \quad \text{and} \quad \int_x^b g(x) \, dx \geq 0 \quad (\forall \, x \in [a, b]).\]

Since, evidently, (0.7) $\Rightarrow$ (0.4), we conclude that (0.4) $\Leftrightarrow$ (0.7) is valid.

On the basis of the above, we can formulate the following results:

**Theorem A1.** Assume that two integrable functions $f$ and $g$ on $[a, b]$, that $f$ is noninc-reasing and that (0.7) holds. Then the second inequality in (0.1) is valid.

**Theorem A2.** Let function $f$ fulfills conditions as in Theorem A1. If

\[ \int_x^b g(x) \, dx \leq b - x \quad \text{and} \quad \int_a^x g(x) \, dx \geq 0 \quad (\forall \, x \in [a, b]),\]

then the first inequality in (0.1) is valid.

1. In this portion of paper we will generalize Theorems A1 and A2 in the case when function $f$ is convex of order $n$. There we will use result ([5]):

**Theorem C.** Let $x \mapsto f(x)$ be a convex function of order $n (n \geq 1)$ on $[a, b]$. Then, for every $c \in [a, b]$, the function $x \mapsto \frac{G(x)}{(x-c)^n}$ is nondecreasing on $[a, b]$, where

\[ G(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x-c)^k, \]

with $f^{(k)}(c)$ being the right derivative for $c = a$ ($f_+(k)(a)$) and the left derivative for $c = b$ ($f_-(k)(b)$).
The Steffensen inequality for convex function of order $n$

**Theorem 1.** Let the functions $f$ and $g$ satisfy the conditions:

1° $f$ is convex of order $n$ ($n \in \mathbb{N}$);

2° $f^{(k)}(a) = 0$ ($k = 0, 1, \ldots, n - 1$);

3° $\int_a^x (x - a)^n g(x) \, dx \leq \frac{(x-a)^{n+1}}{n+1}$ and $\int_a^b (x - a)^n g(x) \, dx \geq 0$ ($\forall \ x \in [a, b]$).

Then

\begin{equation}
\int_a^{a+\lambda_1} f(x) \, dx \leq \int_a^b f(x) g(x) \, dx,
\end{equation}

where

\begin{equation}
\lambda_1 = \left[ (n+1) \int_a^b (x-a)^n g(x) \, dx \right]^{\frac{1}{n+1}}.
\end{equation}

**Proof.** According to Theorem C for $c = a$, and with regard to the assumption for function $f$, the function $x \mapsto \frac{f(x)}{(x-a)^n}$ is nondecreasing.

Let the functions $g_1$ and $g_2$ satisfy the conditions

\begin{equation}
\int_a^x (t-a)^n g_1(t) \, dt \geq \int_a^x (t-a)^n g_2(t) \, dt \quad (\forall \ x \in [a, b])
\end{equation}

and

\begin{equation}
\int_a^b (t-a)^n g_1(t) \, dt = \int_a^b (t-a)^n g_2(t) \, dt.
\end{equation}

If we replace $f(x)$, $g_1(x)$, $g_2(x)$ by $\frac{f(x)}{(x-a)^n}$, $(x-a)^n g_1(x)$, $(x-a)^n g_2(x)$ respectively in the Theorem B, we obtain

\begin{equation}
\int_a^b f(x) g_1(x) \, dx \leq \int_a^b f(x) g_2(x) \, dx.
\end{equation}

Let be now $g_1(x) = 1$ ($x \in [a, a+\lambda_1]$), $g_1(x) = 0$ ($x \in (a+\lambda_1, b]$), $g_2(x) = g(x)$, where $\lambda_1$ is given by (1.2). It is easy to show that the conditions (1.3) and (1.4) are satisfied.

According to (1.5), we get

\begin{equation}
\int_a^{a+\lambda_1} f(x) \, dx = \int_a^b f(x) g_1(x) \, dx \leq \int_a^b f(x) g(x) \, dx,
\end{equation}

which proves the theorem.

The following result can be similarly proved:
Theorem 2. Let the functions $f$ and $g$ satisfy the conditions:

1° $f$ is convex of order $n$ ($n \in \mathbb{N}$);

2° $f^{(k)}(b) = 0$ ($k = 0, 1, \ldots, n - 1$);

3° \( \int_x^b (b - x)^n g(x) \, dx \leq \frac{(b-x)^{n+1}}{n+1} \) and \( \int_a^b (b - x)^n g(x) \, dx \geq 0 \) ($\forall x \in [a, b]$).

If $n$ is even number, the following inequality

\[ \int_a^b f(x) g(x) \, dx \leq \int_a^{b-\lambda_2} f(x) \, dx \]

holds, where

\[ \lambda_2 = \left( n + 1 \right) \int_a^b g(x) (b - x)^n \, dx \left( n+1 \right). \]

If $n$ is odd number, the reverse inequality holds.

Remark 1. If $0 \leq g(x) \leq 1$, the condition 3° in Theorem 1 (also in Theorem 2) is fulfilled.

References


