THE BEST CONSTANT IN SOME INTEGRAL INEQUALITIES OF OPIAL TYPE*

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Using some results from P. R. Beesack's paper [1] we will determine best constants on some concrete integral inequalities of Opial type (see [2]).

Let \( W^1[a, b] \) be the space of all functions \( u \) which are locally absolutely continuous on \((a, b)\) with \( \int_a^b r u'^2 \, dx < +\infty \), where \( x \mapsto r(x) \) is a given weight positive function.

**Lemma 1.** Let \(-\infty \leq a < b \leq +\infty\), and let \( p \) be positive and continuous on \((a, b)\) with \( \int_a^b p(t) \, dt = P < +\infty \). Set \( r(x) = \frac{1}{p(x)} \). Then, if \( f \) is an integral on \([a, b]\) with \( f(a) = 0 \) and \( \int_a^b r f'^2 \, dx < +\infty \), we have

\[
\int_a^b \left( \int_a^x p(t) \, dt \right) |f'(x)| f(x) \, dx \leq \frac{P^2}{2 A^2} \int_a^b r(x)|f'(x)|^2 \, dx \tag{1}
\]

with equality if and only if \( f \) is given by

\[
f(x) = B \sinh \left( \frac{A}{P} \int_a^x p(t) \, dt \right), \tag{2}
\]

where \( B \) is arbitrary constant and \( A \) positive solution of the equation

\[\coth x = x.\]

proof. The hypotheses on $f$ imply that $f(x) = \frac{\partial}{\partial x} \int_a^x g(t) \, dt$ for $a \leq x < b$. If $a$ is finite this is equivalent to saying that $f(a) = 0$ and $f$ is locally absolutely continuous on $[a, b)$. To prove (1), set $u = yz$, where $y(x) = \sinh(\sqrt{\lambda_0} \int_a^x p(t) \, dt)$ for $x \in [a, b)$ with $\lambda_0 = A^2/P^2$ (coth $A = A$, $A > 0$).

It is easy to verify that $(ry')' = \lambda_0 py$ on $(a, b)$.

Now, if $a < x < b$, we have

$$
\int_a^b r u'^2 \, dx \geq 2 \int_a^b r y y' z z' \, dx + \int_a^b r (y' z)^2 \, dx
$$

$$\begin{align*}
= r y y' z^2 \bigg|_a^b - \lambda_0 \left( \int_a^x p(t) \, dt \right) (yz)^2 \bigg|_a^b + 2\lambda_0 \int_a^b \left( \int_a^x p(t) \, dt \right) u' u \, dx.
\end{align*}
$$

In the other hand

$$
\int_a^b \left( \int_a^x p(t) \, dt \right) f'(x) f(x) \, dx \leq \int_a^b \left( \int_a^x p(t) \, dt \right) \left( \int_a^x f'(t) \, dt \right) \, dx.
$$

If we put $u = \int_a^x f'(t) \, dt$, then from above we obtain

$$
(3) \quad \int_a^b \left( \int_a^x p(t) \, dt \right) f'(x) f(x) \, dx
$$

$$\leq \frac{1}{2\lambda_0} \int_a^b r f''(x) \, dx - \frac{1}{2\lambda_0} \left\{ \left( \frac{y'}{y} \right)' - \lambda_0 \int_a^x p(t) \, dt \right\} f(x)^2 \bigg|_a^b.
$$

It is easy to verify that $f(x)^2 \coth(\sqrt{\lambda_0} \int_a^x p(t) \, dt) \to 0$ as $x \to a^+$. 

From (3), when $a \to a^+$ and $b \to b^-$, and since $\coth A = A$, we obtain the inequality (1).

The above proof shows that equality can hold in (1) only if $z' = 0$, or $f = By$ for some constant $B$. Moreover, for any such $f$, we do have $f(x) = \int_a^x f'(t) \, dt$, and $f \in W^2[a, b]$ as one easily verifies, so that $f$ is an admissible function. By direct substitution one sees that equality does hold in (1) for such $f$. 

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The idea for the proof was obtained from the results in reference [1]. Also, this idea was used in paper [3].

The proof of Lemma 1 can be derived using a result from [4], changing variables \( s = s(x) = \int_a^x p(t) \, dt \) in the integrals in (1). Prof. D. W. BOYD has pointed to this fact.

**Remark.** The approximative value of the constant \( A \), with seven exact decimals, is 1.1996786.

The following result can be proved similarly.

**Lemma 2.** Let \(-\infty \leq a < b \leq +\infty\), and let \( p \) be positive and continuous on \((a, b)\) with \( \int_a^b p(t) \, dt = P < +\infty \). Set \( r(x) = \frac{1}{p(x)} \). If \( f \) is an integral on \((a, b)\) with \( f(b) = 0 \), and \( \int_a^b rf'' \, dx < +\infty \), we have

\[
\int_a^b \left( \int_a^b p(t) \, dt \right) |f'(x)| f(x) \, dx \leq \frac{P^2}{2 A^2} \int_a^b r(x) |f'(x)|^2 \, dx,
\]

with equality if and only if \( f \) is given by \( f(x) = B \sinh \left( \frac{A}{P} \int_a^b p(t) \, dt \right) \), where \( B \) is arbitrary constant, and \( A \) positive solution of the equation \( \coth x = x \).

We now state:

**Theorem 1.** Let \( p \) be positive and continuous on \((a, b)\) with \( \int_a^b p(t) \, dt = P < +\infty \).

Set \( r(x) = \frac{1}{p(x)} \), and let \( a < \xi < b \). Then for all \( F \in W^2_1[a, b] \) the inequality

\[
\int_a^b s(x) F'(x) (F(x) - F(\xi)) \, dx \leq \frac{1}{2 A^2} \max \left\{ \left( \int_a^\xi p(t) \, dt \right)^2, \left( \int_\xi^b p(t) \, dt \right)^2 \right\} \int_a^b rF''(x)^2 \, dx,
\]

holds, where

\[
s(x) = \begin{cases} 
\int_a^x p(t) \, dt & (a \leq x \leq \xi), \\
\int_x^\xi p(t) \, dt & (\xi \leq x \leq b), 
\end{cases}
\]

and the number \( A \) is the same as in Lemmas 1 and 2.
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Equality holds in (4) if and only if

\[
F(x) = B_2 + \begin{cases} 
B_1 h(q) \sinh \left( \frac{\int p(t) \, dt}{x} \right) & (a \leq x \leq \xi) \\
B_1' h(-q) \sinh \left( \frac{\int p(t) \, dt}{b} \right) & (\xi \leq x \leq b) 
\end{cases}
\]

where \( B_1, B_1', B_2 \) are arbitrary constants, \( h(q) \) is Heaviside's function and \( q = \int p(t) \, dt - \int p(t) \, dt \).

Proof. Let \( a < \xi < b \). Applying Lemma 2 and Lemma 1 on the right hand side of the equality

\[
\int_a^b |F'(x)(F(x) - F(\xi))| \, dx = \int_a^\xi \left( \int_a^x |F'(x)(F(x) - F(\xi))| \, dx \right) \, dx \\
+ \int_\xi^b \left( \int_\xi^x |F'(x)(F(x) - F(\xi))| \, dx \right) \, dx
\]

we obtain (4). Notice that \( x \mapsto f(x) = F(x) - F(\xi) \) has the required behaviour at \( x = \xi \).

Corollary 1. Let functions \( p \) and \( r \) satisfy conditions as in Theorem 1 and let \( \xi \) be such that

(5)

\[
\int_a^\xi p(t) \, dt = \int_a^\xi p(t) \, dt.
\]

Then

(6)

\[
\int_a^b |s(x)| F'(x)(F(x) - F(\xi)) | \, dx \leq \frac{p^2}{8A^2} \int_a^b r(x) F'(x)^2 \, dx,
\]

with equality if and only if

\[
F(x) = B_2 + \begin{cases} 
B_1 \sinh \left( \frac{2A}{P} \int_a^x p(t) \, dt \right) & (a \leq x \leq \xi) \\
B_1' \sinh \left( \frac{2A}{P} \int_\xi^b p(t) \, dt \right) & (\xi \leq x \leq b) 
\end{cases}
\]

where \( B_1, B_1', B_2 \) are arbitrary constants.
Proof. Since
\[ Q = \max \left\{ \int_a^\xi p \, dt, \int_\xi^b p \, dt \right\} = \frac{1}{2} \left\{ \int_a^b p \, dt + \int_\xi^b p \, dt - \int_\xi^a p \, dt \right\}, \]
with regard to (5), we have \( Q = \frac{1}{2} P \). Then, Corollary 1 follows from Theorem 1.

Remark 1. Notice that (6) holds only for the single \( \xi \) such that (5) holds, and not for all \( \xi \).

From Theorem 1 it follows:

Corollary 2. For every \( F \in W_1^2 \) \([-1, 1]\) the inequality
\[ \int_{-1}^1 \left| x (F(x) - F(0)) F'(x) \right| \, dx \leq \frac{1}{2} A^2 \int_{-1}^1 F'(x)^2 \, dx, \]
holds, with equality if and only if
\[ F(x) = B_1 + \begin{cases} B \sinh (Ax) & (-1 \leq x \leq 0), \\ B' \sinh (Ax) & (0 \leq x \leq 1), \end{cases} \]
where \( B, B', B_1 \) are arbitrary constants.

Theorem 2. Let \( \Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a concave and nondecreasing function, and let functions \( p, r, f \) satisfy the conditions as in Lemma 1.

Then the inequality
\[ \int_0^a \left( \int_0^x p(t) \, dt \right) \Phi (|f'(x)f(x)|) \, dx \leq M_p \Phi \left( \frac{p_1}{2M_p A^2} \int_0^a r(x)f'(x)^2 \, dx \right) \]
holds, where \( P \) and \( A \) are as in Lemma 1 and \( M_p = \int_0^a (a - t) p(t) \, dt \).

Proof. Let \( \omega(x) = \int_0^x p(t) \, dt \). Using the JENSEN integral inequality for concave function \( \Phi \), we have
\[ \int_0^a \omega(x) \Phi (|f'(x)f(x)|) \, dx \leq \int_0^a \omega(x) \, dx \Phi \left( \frac{\int_0^a \omega(x)f'(x)f(x) \, dx}{\int_0^a \omega(x) \, dx} \right), \]
i.e.
\[
\int_0^a \omega(x) \Phi\left( \left| f'(x) f(x) \right| \right) \, dx \leq M_p \Phi \left( \frac{1}{M_p} \int_0^a \omega(x) f''(x) f(x) \, dx \right).
\]

Knowing that \( \Phi \) is a nondecreasing function applying Lemma 1 we obtain (7).

**Corollary 3.** If we take in Theorem 2 \( x \mapsto \Phi(x) = x^q \ (0 < q < 1) \), then the inequality (7) becomes

\[
\int_0^a \left( \int_0^x p(t) \, dt \right) f'(x) f(x)^q \, dx \leq M_p^{1-q} \left( \frac{p^2}{2 A^2} \right)^q \left( \int_0^a r(x) f'(x)^2 \, dx \right)^q.
\]

We obtain the inequality (1) when \( q \to 1 \) in the inequality (8). When \( q \to 0 \), the inequality (8) becomes an equality.

**REFERENCES**


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**NAJBOJLANA KONSTANTA U NEKIM INTEGRALNIM NEJEDNAKOSTIMA OPIALOVOG TIPA**

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Korišćenjem jednog rezultata P. R. Beesacka ([1]) u radu se određuju najbolje konstante u nekim konkretnim nejednakostima Opialovog tipa (videti [2]).