UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 678 — № 715 (1980), 48—53.

687. THE BEST CONSTANT IN SOME INTEGRAL INEQUALITIES OF OPIAL TYPE*

Gradimir V. Milovanović and *Igor Ž. Milovanović*

Using some results from P. R. Beesack's paper [1] we will determine best constants on some concrete integral inequalities of Opial type (see [2]).

Let $W_r^2[a, b]$ be the space of all functions u which are locally absolutely continuous on (a, b) with $\int_a^b r u'^2 \, dx < +\infty$, where $x \mapsto r(x)$ is a given weight positive function.

Lemma 1. *Let* $-\infty \leqq a < b \leqq +\infty$, *and let* p *be positive and continuous on* (a, b) *with* $\int_a^b p(t)\, dt = P < +\infty$. *Set* $r(x) = \frac{1}{p(x)}$. *Then, if* f *is an integral on* $[a, b)$ *with* $f(a) = 0$ *and* $\int_a^b r f'^2 \, dx < +\infty$, *we have*

$$\int_a^b \Big(\int_a^x p(t)\, dt\Big) |f'(x) f(x)| \, dx \leqq \frac{P^2}{2A^2} \int_a^b r(x) |f'(x)|^2 \, dx, \tag{1}$$

with equality if and only if f *is given by*

$$f(x) = B \sinh\Big(\frac{A}{P} \int_a^x p(t)\, dt\Big), \tag{2}$$

where B *is arbitrary constant and* A *positive solution of the equation* $\coth x = x$.

---

* Presented May 15, 1980 by D. W. BOYD.

Proof. The hypotheses on f imply that $f(x)=\int_a^x f'\,dt$ for $a\leq x<b$. If a is finite this is equivalent to saying that $f(a)=0$ and f is locally absolutely continuous on $[a, b)$. To prove (1), set $u=yz$, where $y(x)=\sinh\left(\sqrt{\lambda_0}\int_a^x p(t)\,dt\right)$ for $x\in[a, b)$ with $\lambda_0=A^2/P^2$ ($\coth A=A$, $A>0$).

It is easy to verify that $(ry')'=\lambda_0 py$ on (a, b).

Now, if $a<\alpha<\beta<b$, we have

$$\int_\alpha^\beta ru'^2\,dx\geq 2\int_\alpha^\beta ryy'\,zz'\,dx+\int_\alpha^\beta r(y'z)^2\,dx$$

$$=ryy'z^2\Big|_\alpha^\beta-\lambda_0\left(\int_a^x p\,dt\right)(yz)^2\Big|_\alpha^\beta+2\lambda_0\int_\alpha^\beta\left(\int_a^x p\,dt\right)u'u\,dx.$$

In the other hand

$$\int_\alpha^\beta\left(\int_a^x p(t)\,dt\right)|f'(x)f(x)|\,dx\leq\int_\alpha^\beta\left(\int_a^x p(t)\,dt\right)|f'(x)|\left(\int_a^x|f'(t)|\,dt\right)dx.$$

If we put $u=\int_a^x|f'(t)|\,dt$, then from above we obtain

$$\int_\alpha^\beta\left(\int_a^x p(t)\,dt\right)|f'(x)f(x)|\,dx \tag{3}$$

$$\leq\frac{1}{2\lambda_0}\int_\alpha^\beta rf'^2(x)\,dx-\frac{1}{2\lambda_0}\left\{r\left(\frac{y'}{y}\right)-\lambda_0\int_a^x p\,dt\right\}f(x)^2\Big|_\alpha^\beta.$$

It is easy to verify that $f(\alpha)^2\coth\left(\sqrt{\lambda_0}\int_a^\alpha p(t)\,dt\right)\to 0$ as $\alpha\to a+$.

From (3), when $\alpha\to a+$ and $\beta\to b-$, and since $\coth A=A$, we obtain the inequality (1).

The above proof shows that equality can hold in (1) only if $z'=0$, or $f=By$ for some constant B. Moreover, for any such f, we do have $f(x)=\int_a^x f'(t)\,dt$, and $f\in W_r^2[a, b]$ as one easily verifies, so that f is an admissible function. By direct substitution one sees that equality does hold in (1) for such f.

The idea for the proof was obtained from the results in reference [1]. Also, this idea was used in paper [3].

The proof of Lemma 1 can be derived using a result from [4], changing variables $s=s(x)=\int_a^x p(t)\,dt$ in the integrals in (1). Prof. D. W. BOYD has pointed to this fact.

REMARK. The approximative value of the constant A, with seven exact decimals, is 1.1996786.

The following result can be proved similarly.

Lemma 2. *Let* $-\infty \leqq a<b \leqq +\infty$, *and let* p *be positive and continuous on* (a, b) *with* $\int_a^b p\,dt \quad P<+\infty$. *Set* $r(x)=\frac{1}{p(x)}$. *If* f *is an integral on* $(a, b]$ *with* $f(b)=0$, *and* $\int_a^b rf'^2\,dx<+\infty$, *we have*

$$\int_a^b \Big(\int_x^b p(t)\,dt\Big)|f'(x) f(x)|\,dx \leq \frac{P^2}{2A^2} \int_a^b r(x)|f'(x)|^2\,dx,$$

with equality if and only if f *is given by* $f(x)=B\sinh\Big(\frac{A}{P}\int_x^b p(t)\,dt\Big)$, *where* B *is arbitrary constant, and* A *positive solution of the equation* $\coth x=x$.

We now state:

Theorem 1. *Let* p *be positive and continuous on* (a, b) *with* $\int_a^b p\,dt=P<+\infty$. *Set* $r(x)=\frac{1}{p(x)}$, *and let* $a<\xi<b$. *Then for all* $F \in W_r^2[a, b]$ *the inequality*

$$(4) \quad \int_a^b s(x)|F'(x)(F(x)-F(\xi))|\,dx \leq \frac{1}{2A^2} \max\Big\{\Big(\int_a^\xi p\,dt\Big)^2, \Big(\int_\xi^b p\,dt\Big)^2\Big\} \int_a^b rF'(x)^2\,dx,$$

holds, where

$$s(x)=\begin{cases} \int_x^\xi p(t)\,dt & (a \leq x \leq \xi), \\ \int_\xi^x p(t)\,dt & (\xi \leq x \leq b), \end{cases}$$

and the number A *is the same as in Lemmas* 1 *and* 2.

Equality holds in (4) *if and only if*

$$F(x)=B_2+\begin{cases} B_1\, h(q)\sinh\left(A\dfrac{\int_x^{\xi} p(t)\,\mathrm{d}t}{\int_a^{\xi} p(t)\,\mathrm{d}t}\right) & (a\leqq x\leqq \xi)\\[2ex] B_1'\, h(-q)\sinh\left(A\dfrac{\int_{\xi}^{x} p(t)\,\mathrm{d}t}{\int_{\xi}^{b} p(t)\,\mathrm{d}t}\right) & (\xi\leqq x\leqq b)\end{cases}$$

where B_1, B_1', B_2 *are arbitrary constants,* $h(q)$ *is Heaviside's function and* $q=$ $=\int_a^{\xi} p(t)\,\mathrm{d}t-\int_{\xi}^{b} p(t)\,\mathrm{d}t$.

Proof. Let $a<\xi<b$. Applying Lemma 2 and Lemma 1 on the right hand side of the equality

$$\int_a^b s(x)\,|F'(x)(F(x)-F(\xi))|\,\mathrm{d}x=\int_a^{\xi}\Big(\int_x^{\xi} p(t)\,\mathrm{d}t\Big)|F'(x)(F(x)-F(\xi))|\,\mathrm{d}x$$
$$+\int_{\xi}^{b}\Big(\int_{\xi}^{x} p(t)\,\mathrm{d}t\Big)|F'(x)(F(x)-F(\xi))|\,\mathrm{d}x$$

we obtain (4). Notice that $x\mapsto f(x)=F(x)-F(\xi)$ has the required behaviour at $x=\xi$.

Corollary 1. *Let functions* p *and* r *satisfy conditions as in Theorem* 1 *and let* ξ *be such that*

$$\int_a^{\xi} p\,\mathrm{d}t=\int_{\xi}^{b} p\,\mathrm{d}t. \tag{5}$$

Then

$$\int_a^b s(x)\,|F'(x)(F(x)-F(\xi))|\,\mathrm{d}x\leqq\frac{P^2}{8A^2}\int_a^b r(x)\,F'(x)^2\,\mathrm{d}x, \tag{6}$$

with equality if and only if

$$F(x)=B_2+\begin{cases} B_1\sinh\left(\dfrac{2A}{P}\int_x^{\xi} p(t)\,\mathrm{d}t\right) & (a\leqq x\leqq \xi)\\[2ex] B_1'\sinh\left(\dfrac{2A}{P}\int_{\xi}^{x} p(t)\,\mathrm{d}t\right) & (\xi\leqq x\leqq b)\end{cases}$$

where B_1, B_1', B_2 *are arbitrary constants.*

Proof. Since

$$Q=\max\left\{\int_a^{\xi} p\mathrm{d}t,\ \int_{\xi}^{b} p\mathrm{d}t\right\}=\frac{1}{2}\left\{\int_a^{b} p\mathrm{d}t+\left|\int_{\xi}^{b} p\mathrm{d}t-\int_a^{\xi} p\mathrm{d}t\right|\right\},$$

with regard to (5), we have $Q=\frac{1}{2}P$. Then, Corollary 1 follows from Theorem 1.

REMARK 1. Notice that (6) holds only for the single ξ such that (5) holds, and not for all ξ.

From Theorem 1 it follows:

Corollary 2. *For every* $F\in W_1^2\ [-1,\ 1]$ *the inequality*

$$\int_{-1}^{1} |x(F(x)-F(0))F'(x)|\,\mathrm{d}x\leqq\frac{1}{2A^2}\int_{-1}^{1} F'(x)^2\,\mathrm{d}x,$$

holds, with equality if and only if

$$F(x)=B_1+\begin{cases} -B\sinh(Ax) & (-1\leqq x\leqq 0),\\ B'\sinh(Ax) & (0\leqq x\leqq 1),\end{cases}$$

where B, B', B_1 *are arbitrary constants.*

Theorem 2. *Let* $\Phi:\mathbf{R}^+\to\mathbf{R}^+$ *be a concave and nondecreasing function, and let functions* p, r, f *satisfy the conditions as in Lemma* 1.

Then the inequality

$$(7)\qquad \int_0^a\left(\int_0^x p(t)\,\mathrm{d}t\right)\Phi(|f'(x)f(x)|)\,\mathrm{d}x\leqq M_p\,\Phi\left(\frac{P^2}{2M_pA^2}\int_0^a r(x)f'(x)^2\,\mathrm{d}x\right)$$

holds, where P *and* A *are as in Lemma* 1 *and* $M_p=\int_0^a (a-t)p(t)\,\mathrm{d}t.$

Proof. Let $\omega(x)=\int_0^x p\mathrm{d}t$. Using the JENSEN integral inequality for concave function Φ, we have

$$\int_0^a \omega(x)\,\Phi(|f'(x)f(x)|)\,\mathrm{d}x\leqq\int_0^a \omega(x)\,\mathrm{d}x\,\Phi\left(\frac{\left|\int_0^a \omega(x)f'(x)f(x)\,\mathrm{d}x\right.}{\int_0^a \omega(x)\,\mathrm{d}x}\right),$$

i.e.

$$\int_0^a \omega(x)\,\Phi(|f'(x)f(x)|)\,dx \leqq M_p\,\Phi\left(\frac{1}{M_p}\left|\int_0^a \omega(x)f'(x)f(x)\,dx\right|\right).$$

Knowing that Φ is a nondecreasing function applying Lemma 1 we obtain (7).

Corollary 3. *If we take in Theorem 2* $x \mapsto \Phi(x)=x^q$ $(0<q<1)$, *then the inequality* (7) *becomes*

$$(8) \qquad \int_0^a \left(\int_0^x p(t)\,dt\right)|f'(x)f(x)|^q\,dx \leqq M_p^{1-q}\left(\frac{P^2}{2A^2}\right)^q\left(\int_0^a r(x)f'(x)^2\,dx\right)^q.$$

We obtain the inequality (1) when $q\to 1$ in the inequality (8). When $q\to 0$, the inequality (8) becomes an equality.

REFERENCES

1. P. R. Beesack: *Integral inequalities involving a function and its derivative.* Amer. Math. Monthly **78** (1971), 705—741.
2. D. S. Mitrinović (with P. M. Vasić): *Analytic inequalities.* Berlin-Heidelberg-New York, 1970.
3. G. V. Milovanović and I. Ž. Milovanović: *On generalization of certain results by A. Ostrowski and A. Lupaş.* These Publications № **634**—№ **677** (1979), 62—69.
4. D. W. Boyd and J. S. W. Wong: *An Extension of Opial's Inequality.* J. Math. Anal. Appl. **19** (1967), 100—102.

Faculty of Electronic Engineering
Department of Mathematics
18000 Niš, Yugoslavia

NAJBOLJA KONSTANTA U NEKIM INTEGRALNIM NEJEDNAKOSTIMA OPIALOVOG TIPA

G. V. Milovanović i *I. Ž. Milovanović*

Korišćenjem jednog rezultata P. R. Beesacka ([1]) u radu se određuju najbolje konstante u nekim konkretnim nejednakostima Opialovog tipa (videti [2]).