ON A GENERALIZATION OF MODIFIED BIRKHOFF-YOUNG QUADRATURE FORMULA*

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Dedicated to Professor D. S. Mitrinović on the occasion of his 75th birthday

Nine-point quadrature formula of interpolatory type for analytical functions is obtained. The efficiency of this quadrature has been checked on some examples.

1. Introduction. Let \( z \mapsto f(z) \) be a regular analytic function in the square \( S \), whose vertices are \( 1, i, -1, -i \). In [1] D. D. Tošić has proved the following formula for numerical integration

\[
\int_{-1}^{1} f(z) \, dz = A f(0) + B \left( f(k) + f(-k) \right) + C \left( f(ik) + f(-ik) \right) + R,
\]

where

\[
A = 2 \left( 1 - \frac{1}{5} k^4 \right), \quad B = \frac{1}{6} k^2 + \frac{1}{10} k^4, \quad C = -\frac{1}{6} k^2 + \frac{1}{10} k^4 \quad (k > 0)
\]

and where the error-term was given by the expansion

\[
R = \left( -\frac{2}{3 \cdot 6!} k^4 + \frac{2}{7!} \right) f^{(6)}(0) + \left( \frac{2}{9!} - \frac{2}{5 \cdot 8!} k^4 \right) f^{(8)}(0) + \ldots
\]

For \( k = 1 \) the formula (1) reduces to BIRKHOFF-YOUNG formula ([2])

\[
\int_{-1}^{1} f(z) \, dz = \frac{8}{5} f(0) + \frac{4}{15} \left( f(1) + f(-1) \right) - \frac{1}{15} \left( f(i) + f(-i) \right) + R_5(f).
\]

The remainder \( R_5(f) \) satisfies (see [3, p. 136] and [4])

\[
| R_5(f) | \leq \frac{1}{1890} \max_{z \in S} | f^{(6)}(z) |.
\]

For \( k = \sqrt{0.6} \) the formula (1) reduces to three-point Gauss-Legendre formula (then \( C = 0 \)). In special case for \( k = \sqrt{3/7} \), the formula (1) reduces to the modified BIRKHOFF-YOUNG formula of maximum accuracy (named MF in [1])

\[
\int_{-1}^{1} f(z) \, dz = \frac{16}{15} f(0) + \frac{1}{6} \left( \frac{7}{5} + \sqrt{\frac{7}{3}} \right) \left( f\left(\frac{4}{\sqrt{7}}\right) + f\left(-\frac{4}{\sqrt{7}}\right) \right)
\]

\[
+ \frac{1}{6} \left( \frac{7}{5} - \sqrt{\frac{7}{3}} \right) \left( f\left(i\sqrt{\frac{3}{7}}\right) + f\left(-i\sqrt{\frac{3}{7}}\right) \right) + R_{MF},
\]

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with error-term

\[ R_{MF} = \frac{1}{793800} f^{(8)}(0) + \frac{1}{61122600} f^{(10)}(0) + \cdots. \]

In the paper [5] formulas of the following forms were investigated:

\[
\begin{align*}
\int_{-1}^{1} f(z) \, dz &= A f(0) + B \left( f(x_1) + f(-x_1) \right) + C \left( f(ix_2) + f(-ix_2) \right) + R \\
\int_{-1}^{1} d(f(z) \, dz &= A f(0) + B \left( f(x_1) + f(-x_1) \right) + C \left( f(ix_2) + f(-ix_2) \right) \\
&\quad + D \left( f(x_2) + f(-x_2) \right) + R.
\end{align*}
\]

2. Nine-point quadrature formula. In this paper we shall consider the following quadrature formula of interpolatory type

\[
\begin{align*}
\int_{-1}^{1} f(z) \, dz &= A f(0) + C_{11} \left( f(r_1) + f(-r_1) \right) + C_{12} \left( f(r_2) + f(-r_2) \right) \\
&\quad + C_{21} \left( f(\sqrt{r_1}) + f(-\sqrt{r_1}) \right) + C_{22} \left( f(\sqrt{r_2}) + f(-\sqrt{r_2}) \right) + R(f),
\end{align*}
\]

where \( 0 < r_2 < r_1 < 1. \)

It is easy to notice that the formula (2) is valid for every function of form \( f(z) = z^{2m-1} (m \in \mathbb{N}). \)

**Theorem.** If the quadrature formula (2) is of interpolatory type, then coefficients of this formula are given by

\[
\begin{align*}
A &= \frac{2}{r_1^4 r_2^4} \left( \frac{1}{9} - \frac{1}{5} \left( r_1^4 + r_2^4 \right) + r_1^4 r_2^4 \right), \\
C_{11} &= \frac{1}{2 r_1^4 (r_1^4 - r_2^4)} \left( \frac{1}{9} + \frac{1}{7} r_1^2 - r_2^4 \left( \frac{1}{5} + \frac{1}{3} r_1^2 \right) \right), \\
C_{12} &= \frac{1}{2 r_2^4 (r_2^4 - r_1^4)} \left( \frac{1}{9} + \frac{1}{7} r_2^2 - r_1^4 \left( \frac{1}{5} + \frac{1}{3} r_2^2 \right) \right), \\
C_{21} &= \frac{1}{2 r_1^4 (r_1^4 - r_2^4)} \left( \frac{1}{9} - \frac{1}{7} r_1^2 - r_2^4 \left( \frac{1}{5} - \frac{1}{3} r_1^2 \right) \right), \\
C_{22} &= \frac{1}{2 r_2^4 (r_2^4 - r_1^4)} \left( \frac{1}{9} - \frac{1}{7} r_2^2 - r_1^4 \left( \frac{1}{5} - \frac{1}{3} r_2^2 \right) \right).
\end{align*}
\]

**Proof.** Having in mind that the nodes are \( 0, i^k r_1, i^k r_2 (k = 0, 1, 2, 3) \), we define polynomial \( z \mapsto \omega(z) \) by

\[
\omega(z) = z (z^4 - r_1^4) (z^4 - r_2^4) = z^9 - (r_1^4 + r_2^4) z^5 + r_1^4 r_2^4 z.
\]

Then the coefficients of quadrature formula of interpolatory type can be determined by

\[
A = \frac{1}{\omega'(0)} \int_{-1}^{1} \frac{\omega(z)}{z} \, dz,
\]

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\[ C_{1k} = \frac{2}{\omega'(r_k)} \int_0^1 \frac{z \omega(z)}{z^2-r_k^2} \, dz, \quad C_{2k} = \frac{2}{\omega'(ir_k)} \int_0^1 \frac{z \omega(z)}{z^2+r_k^2} \, dz \quad (k = 1, 2), \]

wherefrom, by integration, we get the statement of Theorem.

3. **Error analysis and optimal choice of \( r_1 \) and \( r_2 \).** Let

\[ Q(f) = Af(0) + C_{11}(f(r_1) + f(-r_1)) + C_{12}(f(r_2) + f(-r_2)) + C_{21}(f(ir_1) + f(-ir_1)) + C_{22}(f(ir_2) + f(ir_2)), \]

where \( A, C_{ij}(i, j = 1, 2) \) are given as in Theorem. The quadrature formula

\[ \int_{-1}^1 f(z) \, dz = Q(f) + R(f), \]

generally has precision \( p = 9 \). By means of choice of \( r_1 \) and \( r_2 \) we can enlarge the precision mentioned above. Thus, we determine:

\[ R(z^{10}) = \frac{2}{11} - \frac{2}{7} (r_1^4 + r_2^4) + \frac{2}{3} r_1^4 r_2^4, \quad R(z^{12}) = \frac{2}{13} - \frac{2}{9} (r_1^4 + r_2^4) + \frac{2}{5} r_1^4 r_2^4, \]

\[ R(z^{14}) = \frac{2}{15} - r_1^4 (r_1^4 + r_2^4) \left( \frac{2}{7} - \frac{2}{3} r_2^4 \right) - \frac{2}{7} r_2^8. \]

Under conditions that \( R(z^{10}) = R(z^{12}) = 0 \), we shall determine what are \( r_1 \) and \( r_2 \), so that the formula (3) is of maximal precision, i.e. that \( p = 13 \).

Thus, we have

\[ \frac{1}{7} (r_1^4 + r_2^4) = \frac{1}{11} + \frac{1}{3} r_1^4 r_2^4, \quad \frac{1}{9} (r_1^4 + r_2^4) = \frac{1}{13} + \frac{1}{5} r_1^4 r_2^4. \]

Solving this system of equations, we find

\[ r_1^4 = \frac{63 + 4 \sqrt{114}}{143}, \quad r_2^4 = \frac{63 - 4 \sqrt{114}}{143}, \]

i.e.

\[ (4) \quad r_1 = 0.927 242 386 651 532 \quad \text{and} \quad r_2 = 0.613 755 686 975 668. \]

With so found values of \( r_1 \) and \( r_2 \), we have \( R(z^{14}) = \frac{512}{165165} \), while the coefficients, which are computed with 14 significant digits, of the appropriate quadrature formula of maximum accuracy are:

\[ A = \frac{512}{675} \approx 0.758 \quad 518 \quad 518 \quad 518 \quad 52, \]

\[ C_{11} = 0.186 \quad 716 \quad 433 \quad 427 \quad 68, \quad C_{12} = 0.446 \quad 789 \quad 042 \quad 127 \quad 13, \]

\[ C_{21} = 0.649 \quad 000 \quad 354 \quad 960 \quad 38 \cdot 10^{-3}, \quad C_{22} = -0.134 \quad 137 \quad 351 \quad 690 \quad 30 \cdot 10^{-1}. \]

The remainder can be given in the form of Taylor series at \( z = 0 \), where the leading term \( R_1 \) is given by

\[ R_1 = \frac{R(z^{14})}{14!} f^{(14)}(0) \approx 3.56 \cdot 10^{-14} f^{(14)}(0). \]
In application of the quadrature formula (2) on integration of real functions, it is sufficient to find the value of functions only in seven points, regarding that the formula (2) can be represented in the form

$$
\int_{-1}^{1} f(x) \, dx \approx Af(0) + C_{11}(f(r_1) + f(-r_1)) + C_{12}(f(r_2) + f(-r_2)) \\
+ 2 C_{21} \Re(f(i r_1)) + 2 C_{22} \Re(f(i r_2)).
$$

(5)

REMARK If $r_1$ and $r_2$ are determined from $C_{11} = C_{22} = 0$, i.e. that in formula (2) only real nodes take part, the formula (2) is reduced to five-point GAUSS-LEGENDRE quadrature formula.

4. Numerical results. We shall apply the found quadrature formula of maximum accuracy on calculation of the following two integrals:

$$I_1 = \int_{-1}^{1} e^x \, dx = 2.35040 \, 23872 \, 87602 \ldots$$

and

$$I_2 = \int_{0}^{1} \frac{1}{1 + x^4} \, dx = \frac{\pi + 2 \log(1 + \sqrt{2})}{4 \sqrt{2}} = 0.86697 \ldots.$$

For the first integral we obtained $I_1 \approx 2.35040 \, 23872 \, 87567$, where the error is $0.36 \times 10^{-13}$.

We note a interesting fact. If $f(x) = g(x^4)$, the formula (5) is reduced to the three-point rule

$$
\int_{0}^{1} f(x) \, dx \approx C_0 f(0) + C_1 f(r_1) + C_2 f(r_2),
$$

(6)

where

$$C_0 = \frac{1}{2} A = \frac{256}{675} \approx 0.37925 \, 92593,$$

$$C_1 = C_{11} + C_{21} = \frac{15922 - 591 \sqrt{114}}{51300} \approx 0.18736 \, 54338,$$

$$C_2 = C_{12} + C_{22} = \frac{15922 + 591 \sqrt{114}}{51300} \approx 0.43337 \, 53070.$$

Using the formula (6) on the calculation of the integral $I_2$, we get the approximate value which is given in the following table.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Approximate value</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIMPSON</td>
<td>0.87745</td>
<td>$1.05 \times 10^{-2}$</td>
</tr>
<tr>
<td>RADAU</td>
<td>0.86490</td>
<td>$-2.07 \times 10^{-3}$</td>
</tr>
<tr>
<td>GAUSS (3 point)</td>
<td>0.86752</td>
<td>$5.45 \times 10^{-4}$</td>
</tr>
<tr>
<td>GAUSS (5 point)</td>
<td>0.86746</td>
<td>$4.90 \times 10^{-4}$</td>
</tr>
<tr>
<td>RULE (6)</td>
<td>0.86651</td>
<td>$-4.63 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
Comparing to results (also, given in the Table) obtained by following rules

\[ \int_{0}^{1} f(x) \, dx \approx \frac{1}{6} \left( f(0) + 4f\left( \frac{1}{2} \right) + f(1) \right) \quad \text{(SIMPSON)} \]

\[ \approx \frac{1}{9} f(0) + \frac{16 + \sqrt{6}}{36} f\left( \frac{6 - \sqrt{6}}{10} \right) + \frac{16 - \sqrt{6}}{36} f\left( \frac{6 + \sqrt{6}}{10} \right) \quad \text{(RADAU)}, \]

\[ \approx \frac{5}{18} f\left( \frac{1}{2} \left( 1 - \sqrt{\frac{3}{5}} \right) \right) + \frac{1}{9} f\left( \frac{1}{2} \right) + \frac{5}{18} f\left( \frac{1}{2} \left( 1 + \sqrt{\frac{3}{5}} \right) \right) \quad \text{(GAUSS THREE-POINT)}, \]

\[ \approx \frac{256}{900} f(0) + \frac{322 + \sqrt{11830}}{900} f\left( \frac{\sqrt{35 - 280}}{63} \right) + \frac{322 - \sqrt{11830}}{900} f\left( \frac{\sqrt{35 + 280}}{63} \right) \]

(derived from five-point Gaussian rule for even function), we see that formula (6), in this case, has the best accuracy.

REFERENCES


