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ON A GENERALIZATION OF MODIFIED BIRKHOFF-YOUNG QUADRATURE FORMULA*

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Dedicated to Professor D. S. Mitrinović on the occation of his 75th birthday

Nine-point quadrature formula of interpolatory type for analitical functions is obtained. The efficiency of this quadrature has been checked on some examples.

1. Introduction. Let $z \mapsto f(z)$ be a regular analytic function in the square S, whose vertices are $1, i, -1, -i$. In [1] D. Đ. TOŠIĆ has proved the following formula for numerical integration

$$\int_{-1}^{1} f(z)\,dz = Af(0) + B\big(f(k)+f(-k)\big) + C\big(f(ik)+f(-ik)\big) + R, \tag{1}$$

where

$$A = 2\left(1 - \frac{1}{5k^4}\right),\quad B = \frac{1}{6k^2} + \frac{1}{10k^4},\quad C = -\frac{1}{6k^2} + \frac{1}{10k^4} \qquad (k>0)$$

and where the error-term was given by the expansion

$$R = \left(-\frac{2}{3\cdot 6!}k^4 + \frac{2}{7!}\right) f^{(6)}(0) + \left(\frac{2}{9!} - \frac{2}{5\cdot 8!}k^4\right) f^{(8)}(0) + \cdots$$

For $k=1$ the formula (1) reduces to BIRKHOFF-YOUNG formula ([2])

$$\int_{-1}^{1} f(z)\,dz = \frac{8}{5} f(0) + \frac{4}{15}\big(f(1)+f(-1)\big) - \frac{1}{15}\big(f(i)+f(-i)\big) + R_5(f).$$

The remainder $R_5(f)$ satisfies (see [3, p. 136] and [4])

$$|R_5(f)| \leq \frac{1}{1890} \max_{z\in S} |f^{(6)}(z)|.$$

For $k=\sqrt{0.6}$ the formula (1) reduces to three-point Gauss-Legendre formula (then $C=0$). In special case for $k=\sqrt[4]{3/7}$, the formula (1) reduces to the modified BIRKHOFF-YOUNG formula of maximum accuracy (named MF in [1])

$$\int_{-1}^{1} f(z)\,dz = \frac{16}{15} f(0) + \frac{1}{6}\left(\frac{7}{5} + \sqrt{\frac{7}{3}}\right)\left(f\left(\sqrt[4]{\frac{3}{7}}\right) + f\left(-\sqrt[4]{\frac{3}{7}}\right)\right)$$
$$+ \frac{1}{6}\left(\frac{7}{5} - \sqrt{\frac{7}{3}}\right)\left(f\left(i\sqrt[4]{\frac{3}{7}}\right) + f\left(-i\sqrt[4]{\frac{3}{7}}\right)\right) + R_{MF},$$

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with error-term

$$R_{MF}=\frac{1}{793800}f^{(8)}(0)+\frac{1}{61122600}f^{(10)}(0)+\cdots.$$

In the paper [5] formulas of the following forms were investigated:

$$\int_{-1}^{1} f(z)\,dz=Af(0)+B\big(f(x_1)+f(-x_1)\big)+C\big(f(ix_2)+f(-ix_2)\big)+R$$

dn

$$\int_{-1}^{1} f(z)\,dz=Af(0)+B\big(f(x_1)+f(-x_1)\big)+C\big(f(ix_2)+f(-ix_2)\big)$$
$$+D\big(f(x_2)+f(-x_2)\big)+R.$$

2. Nine-point quadrature formula. In this paper we shall consider the following quadrature formula of interpolatory type

$$\int_{-1}^{1} f(z)\,dz=Af(0)+C_{11}\big(f(r_1)+f(-r_1)\big)+C_{12}\big(f(r_2)+f(-r_2)\big)$$

(2)

$$+C_{21}\big(f(ir_1)+f(-ir_1)\big)+C_{22}\big(f(ir_2)+f(-ir_2)\big)+R(f),$$

where $0<r_2<r_1<1$.

It is easy to notice that the formula (2) is valid for every function of form $f(z)=z^{2m-1}\ (m\in\mathbf{N})$.

Theorem. *If the quadrature formula* (2) *is of interpolatory type, then coefficients of this formula are given by*

$$A=\frac{2}{r_1^4 r_2^4}\left(\frac{1}{9}-\frac{1}{5}(r_1^4+r_2^4)+r_1^4 r_2^4\right),$$

$$C_{11}=\frac{1}{2r_1^4(r_1^4-r_2^4)}\left(\frac{1}{9}+\frac{1}{7}r_1^2-r_2^4\left(\frac{1}{5}+\frac{1}{3}r_1^2\right)\right),$$

$$C_{12}=\frac{1}{2r_2^4(r_2^4-r_1^4)}\left(\frac{1}{9}+\frac{1}{7}r_2^2-r_1^4\left(\frac{1}{5}+\frac{1}{3}r_2^2\right)\right),$$

$$C_{21}=\frac{1}{2r_1^4(r_1^4-r_2^4)}\left(\frac{1}{9}-\frac{1}{7}r_1^2-r_2^4\left(\frac{1}{5}-\frac{1}{3}r_1^2\right)\right),$$

$$C_{22}=\frac{1}{2r_2^4(r_2^4-r_1^4)}\left(\frac{1}{9}-\frac{1}{7}r_2^2-r_1^4\left(\frac{1}{5}-\frac{1}{3}r_2^2\right)\right).$$

Proof. Having in mind that the nodes are 0, $i^k r_1$, $i^k r_2\ (k=0,1,2,3)$, we define polynomial $z\mapsto\omega(z)$ by

$$\omega(z)=z(z^4-r_1^4)(z^4-r_2^4)=z^9-(r_1^4+r_2^4)z^5+r_1^4r_2^4 z.$$

Then the coefficients of quadrature formula of interpolatory type can be determined by

$$A=\frac{1}{\omega'(0)}\int_{-1}^{\infty}\frac{\omega(z)}{z}\,dz,$$

$$C_{1k}=\frac{2}{\omega'(r_k)}\int_0^1 \frac{z\,\omega(z)}{z^2-r_k^2}\,dz, \quad C_{2k}=\frac{2}{\omega'(ir_k)}\int_0^1 \frac{z\,\omega(z)}{z^2+r_k^2}\,dz \quad (k=1, 2),$$

wherefrom, by integration, we get the statement of Theorem.

3. Error analysis and optimal choice of r_1 and r_2. Let

$$Q(f)=Af(0)+C_{11}(f(r_1)+f(-r_1))+C_{12}(f(r_2)+f(-r_2))$$
$$+C_{21}(f(ir_1)+f(-ir_1))+C_{22}(f(ir_2)+f(ir_2)),$$

where A, $C_{ij}(i, j=1, 2)$ are given as in Theorem. The quadrature formula

$$\int_{-1}^{1} f(z)\,dz=Q(f)+R(f), \tag{3}$$

generally has precision $p=9$. By means of choice of r_1 and r_2 we can enlarge the precision mentioned above. Thus, we determine:

$$R(z^{10})=\frac{2}{11}-\frac{2}{7}(r_1^4+r_2^4)+\frac{2}{3}r_1^4 r_2^4, \quad R(z^{12})=\frac{2}{13}-\frac{2}{9}(r_1^4+r_2^4)+\frac{2}{5}r_1^4 r_2^4,$$

$$R(z^{14})=\frac{2}{15}-r_1^4(r_1^4+r_2^4)\left(\frac{2}{7}-\frac{2}{3}r_2^4\right)-\frac{2}{7}r_2^8.$$

Under conditions that $R(z^{10})=R(z^{12})=0$, we shall determine what are r_1 and r_2, so that the formula (3) is of maximal precision, i.e. that $p=13$

Thus, we have

$$\frac{1}{7}(r_1^4+r_2^4)=\frac{1}{11}+\frac{1}{3}r_1^4 r_2^4, \quad \frac{1}{9}(r_1^4+r_2^4)=\frac{1}{13}+\frac{1}{5}r_1^4 r_2^4.$$

Solving this system of equations, we find

$$r_1^4=\frac{63+4\sqrt{114}}{143} \quad \text{and} \quad r_2^4=\frac{63-4\sqrt{114}}{143},$$

i.e.

$$r_1=0.927\,242\,386\,651\,532 \quad \text{and} \quad r_2=0.613\,755\,686\,975\,668. \tag{4}$$

With so found values of r_1 and r_2, we have $R(z^{14})=\frac{512}{165165}$, while the coefficients, which are computed with 14 significant digits, of the appropriate quadrature formula of maximum accuracy are:

$$A=\frac{512}{675}=0.758\,518\,518\,518\,52,$$

$$C_{11}=0.186\,716\,433\,427\,68, \quad C_{12}=0.446\,789\,042\,127\,13,$$

$$C_{21}=0.649\,000\,354\,960\,38\cdot 10^{-3}, \quad C_{22}=-0.134\,137\,351\,690\,30\cdot 10^{-1}.$$

The remainder can be given in the form of TAYLOR series at $z=0$, where the leading term R_1 is given by

$$R_1=\frac{R(z^{14})}{14!}f^{(14)}(0)\cong 3.56\cdot 10^{-14} f^{(14)}(0).$$

In application of the quadrature formula (2) on integration of real functions, it is sufficient to find the value of functions only in seven points, regarding that the formula (2) can be represented in the form

$$\int_{-1}^{1} f(x)\,dx \simeq Af(0)+C_{11}\left(f(r_1)+f(-r_1)\right)+C_{12}\left(f(r_2)+f(-r_2)\right)$$

(5)

$$+2\,C_{21}\,\mathrm{Re}\left(f(ir_1)\right)+2\,C_{22}\,\mathrm{Re}\left(f(ir_2)\right).$$

REMARK If r_1 and r_2 are determined from $C_{21}=C_{22}=0$, i.e. that in formula (2) only real nodes take part, the formula (2) is reduced to five-point GAUSS-LEGENDRE quadrature formula.

4. Numerical results. We shall apply the found quadrature formula of maximum accuracy on calculation of the following two integrals:

$$I_1=\int_{-1}^{1} e^x\,dx=2.35040\,23872\,87602\ldots$$

and

$$I_2=\int_{0}^{1}\frac{1}{1+x^4}\,dx=\frac{\pi+2\log(1+\sqrt{2})}{4\sqrt{2}}=0.86697\ldots .$$

For the first integral we obtained $I_1\simeq 2.35040\,23872\,87567$, where the error is 0.36×10^{-13}.

We note a interesting fact. If $f(x)=g(x^4)$, the formula (5) is reduced to the three-point rule

(6) $$\int_{0}^{1} f(x)\,dx\simeq C_0 f(0)+C_1 f(r_1)+C_2 f(r_2),$$

where

$$C_0=\frac{1}{2}A=\frac{256}{675}\simeq 0.37925\,92593,$$

$$C_1=C_{11}+C_{21}=\frac{15922-591\sqrt{114}}{51300}\simeq 0.18736\,54338,$$

$$C_2=C_{12}+C_{22}=\frac{15922+591\sqrt{114}}{51300}\simeq 0.43337\,53070.$$

Using the formula (6) on the calculation of the integral I_2, we get the approximate alue which is given in the following table.

Rule	Approximate value	Error
SIMPSON	0.87745	1.05×10^{-2}
RADAU	0.86490	-2.07×10^{-3}
GAUSS (3 point)	0.86752	5.45×10^{-4}
GAUSS (5 point)	0.86746	4.90×10^{-4}
RULE (6)	0.86651	-4.63×10^{-4}

Comparing to results (also, given in the Table) obtained by following rules

$$\int_0^1 f(x)\,dx \simeq \frac{1}{6}\left(f(0)+4f\left(\frac{1}{2}\right)+f(1)\right) \qquad \text{(SIMPSON)}$$

$$\simeq \frac{1}{9} f(0)+\frac{16+\sqrt{6}}{36} f\left(\frac{6-\sqrt{6}}{10}\right)+\frac{16-\sqrt{6}}{36} f\left(\frac{6+\sqrt{6}}{10}\right) \qquad \text{(RADAU)},$$

$$\simeq \frac{5}{18} f\left(\frac{1}{2}\left(1-\sqrt{\frac{3}{5}}\right)\right)+\frac{4}{9} f\left(\frac{1}{2}\right)+\frac{5}{18} f\left(\frac{1}{2}\left(1+\sqrt{\frac{3}{5}}\right)\right) \text{(GAUSS THREE-POINT)},$$

$$\simeq \frac{256}{900} f(0)+\frac{322+\sqrt{11830}}{900} f\left(\sqrt{\frac{35-280}{63}}\right)+\frac{322-\sqrt{11830}}{900} f\left(\sqrt{\frac{35+280}{63}}\right)$$

(derived from five-point Gaussian rule for even function), we see that formula (6), in this case, has the best accuracy.

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