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ON THE ZERO BOUNDS OF POLYNOMIALS AND REGULAR FUNCTIONS OF A QUATERNIONIC VARIABLE

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In this manuscript, we are concerned with the problem of locating the zeros of some special quaternionic polynomials and regular functions with restricted coefficients; namely quaternionic coefficients whose real components or their moduli satisfy suitable inequalities. The obtained results for this subclass of quaternionic polynomials and regular functions produce extensions of the classical Eneström-Kakeya theorem and its various variants from complex to the quaternionic setting.

1. INTRODUCTION AND PRELIMINARIES

The task of finding the regions containing all the zeros of a polynomial on using various methods of the geometric function theory is a classical topic in analysis. In addition to having numerous applications, this study has been the inspiration for much more research both from the theoretical point of view, as well as from the practical point of view. Since the zeros of a polynomial are continuous functions of its coefficients, in general, it is quite complicated to derive bounds or the norm of zeros of a general algebraic polynomial. Therefore, in order to attain better and sharp zero bounds, it is desirable to put some restrictions on the coefficients of the polynomial. In the literature, we can see a large body of research concerning the regions containing all the zeros of a polynomial in terms of coefficients of the polynomial. The following elegant result concerning the distribution of zeros

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of a polynomial when its coefficients are restricted is known in the literature as Eneström-Kakeya theorem [**12**].

Theorem 1 (Eneström-Kakeya Theorem)**.** *If* $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ $(z \in \mathbb{C})$ *is a polynomial of degree n with real coefficients and satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(z)$ *lie in* $|z| \leq 1$.

The above classical result is particularly important in the study of stability of numerical methods for differential equations and is the starting point of a rich literature concerning its extensions, generalizations and improvements in several directions, see, e.g., the papers [**1**], [**2**], [**4**], [**10**] to mention only a few.

In this form, it has been extensively studied and extended in various ways, even to complex coefficients with their moduli satisfying suitable inequalities. For an exhaustive survey of its extensions and refinements, we refer the readers to the comprehensive books of Marden [**12**], Milovanović et al. [**15**] and Gardner et al. [**6**].

We can get the following equivalent form of Theorem 1 by applying it to the polynomial $z^n T(1/z)$.

Theorem 2. *If* $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ $(z \in \mathbb{C})$ *is a polynomial of degree n with real coefficients and satisfying*

$$a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n > 0,$$

then $T(z)$ *does not vanish in* $|z| < 1$.

From the above results on polynomials, analogues specifying a zero free disk of a power series analytic in a given region can be deduced. In this connection, an extension of Theorem 2 to a class of related analytic functions was established by Aziz and Mohammad [**1**] in the form of the following result.

Theorem 3. *Let* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \not\equiv 0$ *be analytic in* $|z| \leq t$, $t > 0$. *If*

$$a_\nu > 0 \quad \textit{and} \quad a_{\nu-1} - t a_\nu \geq 0, \quad \nu = 1, 2, \ldots,$$

then $f(z)$ *does not vanish in* $|z| < t$.

In this paper, we are interested to extend the above results and their various generalizations in the quaternionic setting. We begin with some preliminaries on quaternions and regular functions of a quaternionic variable which will be useful in the sequel. Quaternions are essentially a generalization of complex numbers to four dimensions (one real and three imaginary parts) and were first studied and developed by Sir Rowan William Hamilton in 1843. This number system of quaternions is denoted by $\mathbb{H}$ in honor of Hamilton. This theory of quaternions is by now very well developed in many different directions, and we refer the reader to [**19**] for the basic features of quaternionic functions (see also recent papers [**3**], [**13**], [**14**]).

The set of quaternions denoted by $\mathbb{H}$ is a noncommutative division ring. It consists of elements of the form $q = \alpha + \beta i + \gamma j + \delta k,\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$, where the imaginary units i, j, k satisfy

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Every element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ is composed by the real part $\mathrm{Re}(q) = \alpha$ and the imaginary part $\mathrm{Im}(q) = \beta i + \gamma j + \delta k$.

The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = \alpha - \beta i - \gamma j - \delta k$ and the norm of q is

$$|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

The inverse of each non zero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$.

For $r > 0$, we define the ball $B(0, r) = \big\{q \in \mathbb{H} : |q| < r\big\}$. By $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centred at the origin, i.e.,

$$\mathbb{B} = \big\{q = \alpha + \beta i + \gamma j + \delta k : \alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 1\big\},$$

and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \big\{q = \beta i + \gamma j + \delta k : \beta^2 + \gamma^2 + \delta^2 = 1\big\}.$$

The angle between two quaternions $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$ is given by

$$\measuredangle(q_1, q_2) = \cos^{-1}\left(\frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + \delta_1\delta_2}{|q_1||q_2|}\right).$$

The functions we consider in this paper are regular functions as polynomials of the form

$$T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$$

and power series of the form

$$f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$$

of the quaternionic variable q on the left and with quaternionic coefficients a_ν on the right.

Two quaternionic polynomials of this kind can be multiplied according to the convolution product (Cauchy multiplication rule): given $T_1(q) = \sum_{i=0}^{n} q^i a_i$ and $T_2(q) = \sum_{j=0}^{m} q^j b_j$, we define

$$(T_1 * T_2)(q) := \sum_{\substack{i=0,1,\ldots,n \\ j=0,1,\ldots,m}} q^{i+j} a_i b_j.$$

If T_1 has real coefficients, the so called $*$ multiplication coincides with the usual pointwise multiplication. Notice that the $*$ product is associative and not, in general, commutative. Given two quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ and $g(q) = \sum_{\nu=0}^{\infty} q^\nu b_\nu$ with radii of convergence greater than R, we define the regular product of f and g as the series

$$(f * g)(q) = \sum_{\nu=0}^{\infty} q^\nu c_\nu,$$

where $c_\nu = \sum_{k=0}^{\nu} a_k b_{\nu-k}$ for all ν. Further, as observed in ([**5**], [**9**]) for each quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$, there exists a ball $B(0,R) = \{q \in \mathbb{H} : |q| < R\}$ such that f converges absolutely and uniformly on each compact subset of $B(0,R)$ and where the sum function of f is regular.

The regular functions of a quaternionic variable $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ have been introduced and intensively studied in the past decade, and they have proven to be a fertile topic in analysis, and their rapid development has been largely driven by the applications to operator theory.

In the preliminary steps, the structure of the zero sets of a quaternionic regular function and the factorization property of zeros was described. In this regard, Gentili and Stoppato [**9**] (see also [**7**]) gave a necessary and sufficient condition for a regular quaternionic power series to have a zero at a point in the form of the following result.

Theorem 4. *Let $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ be a given quaternionic power series with radius of convergence R, and let $p \in B(0,R)$. Then $f(p) = 0$ if and only if there exists a quaternionic power series $g(q)$ with radius of convergence R such that*

$$f(q) = (q - p) * g(q).$$

This extends to quaternionic power series the theory presented in [**11**] for polynomials. The following result which completely describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors is from [**11**] (see also [**7**] and [**9**]).

Theorem 5. *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\left(f(q_0)^{-1} q_0 f(q_0)\right) = 0$.*

Gentili and Struppa [**8**] introduced a maximum modulus theorem for regular functions, which includes convergent power series and polynomials in the form of the following result.

Theorem 6 (Maximum Modulus Theorem)**.** *Let $B = B(0,r)$ be a ball in $\mathbb{H}$ with centre 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is a constant on B.*

In [**7**]–[**9**] the structure of the zeros of polynomials was used and a topological proof of the Fundamental Theorem of Algebra was established. We point out that the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbb{H}$ was already proved by Niven (for reference, see [**16**], [**17**]) by using different techniques. This lead to the complete identification of the zeros of polynomials in terms of their factorization, for reference see [**18**]. Thus it became an interesting perspective to think about the regions containing some or all the zeros of a regular polynomial of quaternionic variable.

Very recently, Carney et al. [**3**] extended the Eneström-Kakeya theorem and its various generalizations from complex polynomials to quaternionic polynomials by making use of Theorems 5 and 6. Firstly, they established the following quaternionic analogue of Theorem 1.

Theorem 7. *If $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$, is a polynomial of degree n (where q is a quaternionic variable) with real coefficients and satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(q)$ lie in $|q| \leq 1$.

In the same paper, Carney et al. [**3**] also established the following generalization of Theorem 7 to quaternionic coefficients in the form of the following result.

Theorem 8. *If $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ is a quaternionic polynomial of degree n, where $a_{\nu} = \alpha_{\nu} + \beta_{\nu} i + \gamma_{\nu} j + \delta_{\nu} k$ for $\nu = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 \geq 0, \quad \alpha_n \neq 0,$$

then all the zeros of $T(q)$ lie in

$$|q| \leq 1 + \frac{2}{\alpha_n} \sum_{\nu=0}^{n} \big(|\beta_{\nu}| + |\gamma_{\nu}| + |\delta_{\nu}|\big).$$

In the meantime, Tripathi ([**20**, Corollary 3.3]) established the following generalization of Theorem 7 in the form of the following result.

Theorem 9. *If $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \left(|a_0| + \sum_{\nu=1}^{n} |a_{\nu} - a_{\nu-1}| \right) = \frac{1}{|a_n|} \left(|a_0| - a_0 + a_n \right).$$

The study of regular functions of a quaternionic variable are now a days a widely studied topic, important especially in replicating many properties of holomorphic functions of a complex variable. As remarked in the beginning, the main purpose of this paper is to extend various results of Eneström-Kakeya type from complex to the quaternionic setting and to obtain zero free regions of some special regular functions of a quaternionic variable with restricted coefficients. We shall make use of the recently established maximum modulus theorem (Theorem 6), the structure of the zero sets of regular product of two polynomials (Theorem 5) and factorization theorem (Theorem 4) to get the desired results. The obtained results also produce various generalizations of Theorems 7, 8 and 9.

2. MAIN RESULTS

In this section, we state our main results and their proofs are given in the next section. We start with the following generalization of Theorem 9. As a consequence, it also provides a generalization of Theorem 7.

Theorem 10. *Let $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ be a polynomial of degree n (where q is a quaternionic variable) with real coefficients. If for some real numbers k_0 and k_1,*

$$k_0 + a_n \geq k_1 + a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $T(q)$ lie in

$$\left| q + \frac{k_0 - k_1}{a_n} \right| \leq \frac{1}{|a_n|} \big(a_n + k_0 - a_0 + |a_0| + |k_1| \big).$$

Remark 11. The above theorem is applicable to locate the zeros of all those polynomials of a quaternionic variable with real coefficients whose first three coefficients do not satisfy the condition of monotonicity and some suitable weights k_0 and k_1 are added to the coefficients a_n and a_{n-1} respectively to support the monotonicity condition.

Taking $k_1 = 0$ and $k_0 = (\lambda - 1)a_n$ with $\lambda \geq 1$ in Theorem 10, we get the following generalization of Theorem 9.

Corollary 12. *Let $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ be a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$\lambda a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0,$$

for some $\lambda \geq 1$, then all the zeros of $T(q)$ lie in

$$\big| q + \lambda - 1 \big| \leq \frac{1}{|a_n|} \big(\lambda a_n - a_0 + |a_0| \big).$$

Remark 13. The above Corollary 12 extends a results of Aziz and Zargar [**2**, Theorem 2] from complex to the quaternionic setting.

Instead of proving Theorem 10, we prove the following more general result for a quaternionic polynomial with quaternionic coefficients when we have information only about the real parts its coefficients. We recover Theorem 10 and other related results as special cases from this theorem.

Theorem 14. *Let* $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ *be a quaternionic polynomial of degree* n, *where* $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$ *for* $\nu = 0, 1, \ldots, n$. *If for some real numbers* k_0 *and* k_1,

$$k_0 + \alpha_n \geq k_1 + \alpha_{n-1} \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

then all the zeros of $T(q)$ *lie in*

$$\left| q + \frac{k_0 - k_1}{a_n} \right| \leq \frac{1}{|a_n|} \Big(\alpha_n + k_0 - \alpha_0 + |\alpha_0| + |k_1| + M_0 \Big),$$

where

$$M_0 = \sum_{\nu=0}^{n} \big(|\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}| \big), \quad \beta_{-1} = \gamma_{-1} = \delta_{-1} = 0.$$

Remark 15. Taking $\beta_\nu = \gamma_\nu = \delta_\nu = 0$ for $\nu = 0, 1, 2, \ldots, n$ in Theorem 14, we recover Theorem 10.

It is easy to verify that

$$M_0 \leq 2 \sum_{\nu=0}^{n} \big(|\beta_\nu| + |\gamma_\nu| + |\delta_\nu| \big).$$

Using this and taking $k_1 = 0$ in Theorem 14, we get the following generalization of Theorem 8.

Corollary 16. *Let* $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ *be a quaternionic polynomial of degree* n, *where* $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$ *for* $\nu = 0, 1, 2, \ldots, n$. *If for some non-negative real number* k_0,

$$k_0 + \alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 \geq 0 \quad (\alpha_n \neq 0),$$

then all the zeros of $T(q)$ *lie in*

$$\left| q + \frac{k_0}{a_n} \right| \leq 1 + \frac{1}{\alpha_n} \left(k_0 + 2 \sum_{\nu=0}^{n} (|\beta_\nu| + |\gamma_\nu| + |\delta_\nu|) \right).$$

Remark 17. For $k_0 = 0$, Corollary 16 reduces to Theorem 8.

We now turn to study the zeros of some special regular functions of the form $\sum_{\nu=0}^{\infty} q^\nu a_\nu$ with restricted coefficient, regular in the ball $B(0, R)$, $R > 0$. in this direction, we first prove the following which gives quaternionic analogue of Theorem 3 as a special case.

Theorem 18. *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ for all $q \in B(0,R)$. Let b be any non-zero quaternion such that $\measuredangle(a_\nu, b) \le \theta \le \pi/2$, $\nu = 0,1,2,\ldots$, and for some finite non-negative integer λ, we have*

$$|a_0| \le t|a_1| \le \cdots \le t^\lambda |a_\lambda| \ge t^{\lambda+1}|a_{\lambda+1}| \ge \cdots ,$$

where $0 < t < R$, then $f(q)$ does not vanish in

$$|q| < \frac{t}{M},$$

where

$$M = \left(2t^\lambda \left|\frac{a_\lambda}{a_0}\right| - 1\right)\cos\theta + \sin\theta + \frac{2\sin\theta}{|a_0|}\sum_{\nu=1}^{\infty} t^\nu |a_\nu|.$$

Taking $\lambda = 0$ in Theorem 18, we get the following result.

Corollary 19. *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ for all $q \in B(0,R)$. Let b be any quaternion such that $\measuredangle(a_\nu, b) \le \theta \le \pi/2$, $\nu = 0,1,2,\ldots$, and*

$$|a_0| \ge t|a_1| \ge t^2|a_2| \ge \cdots ,$$

where $0 < t < R$, then $f(q)$ does not vanish in $|q| < t/M_0$, where

$$M_0 = \cos\theta + \sin\theta + \frac{2\sin\theta}{|a_0|}\sum_{\nu=1}^{\infty} t^\nu |a_\nu|.$$

Remark 20. Taking $\theta = 0$ and assume b to be a positive real number in Corollary 19, we get the quaternionic analogue of Theorem 3.

Finally, we shall prove the following result for regular power series with quaternionic coefficients that gives a generalization of Corollary 19.

Theorem 21. *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ for all $q \in B(0,R)$. If a_ν, $\nu = 0,1,2,\ldots$, are such that for some $\lambda_0, \lambda_1 \ge 1$, we have*

$$\lambda_0|a_0| \ge \lambda_1 t|a_1| \ge t^2|a_2| \ge t^3|a_3| \ge \cdots ,$$

where $0 < t < R$. Let b be any non-zero quaternion such that $\measuredangle(a_\nu, b) \le \theta \le \pi/2$, $\nu = 0,1,2,\ldots$, then $f(q)$ does not vanish in

$$\left|q - \frac{(\lambda_0 - 1)t}{E^2 - (\lambda_0-1)^2}\right| < \frac{Et}{E^2 - (\lambda_0 - 1)^2},$$

where

$$E = \lambda_0(\cos\theta + \sin\theta) + 2t(\lambda_1 - 1)\left|\frac{a_1}{a_0}\right| + \frac{2\sin\theta}{|a_0|}\left(\lambda_1 t|a_1| + \sum_{\nu=2}^{\infty} t^\nu |a_\nu|\right). \tag{1}$$

Remark 22. On setting $\lambda_0 = \lambda_1 = 1$ in Theorem 21, we recover Corollary 19.

Taking $\theta = 0$, assume b to be a positive real number and $\lambda_1 = 1$ in Theorem 21, we get the following corollary.

Corollary 23. *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ for all $q \in B(0,R)$. If a_ν, $\nu = 0,1,2,\ldots$, are real and positive satisfying*

$$\lambda_0 a_0 \geq t a_1 \geq t^2 a_2 \geq \cdots$$

for some $\lambda_0 \geq 1$ and $0 < t < R$, then $f(q)$ does not vanish in

$$\left| q - \frac{(\lambda_0 - 1)t}{2\lambda_0 - 1} \right| < \frac{\lambda_0 t}{2\lambda_0 - 1}.$$

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 14. Consider the polynomial

$$\begin{aligned} T(q) * (1-q) &= a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= \sum_{\nu=0}^{n-1} q^\nu (a_\nu - a_{\nu-1}) + q^n[(k_0 + a_n) - (k_1 + a_{n-1})] \\ &\qquad - q^n(k_0 - k_1) - q^{n+1}a_n \qquad\qquad (a_{-1} = 0) \\ &= f(q) - q^n(q a_n + k_0 - k_1), \end{aligned}$$

where

$$\begin{aligned} f(q) &= \sum_{\nu=0}^{n-1} q^\nu (a_\nu - a_{\nu-1}) + q^n[(k_0 + a_n) - (k_1 + a_{n-1})] \\ &= \sum_{\nu=0}^{n-2} q^\nu (\alpha_\nu - \alpha_{\nu-1}) - q^{n-1}k_1 + q^{n-1}(k_1 + \alpha_{n-1} - \alpha_{n-2}) \\ &\quad + q^n[(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})] \\ &\quad + \sum_{\nu=0}^{n} q^\nu [(\beta_\nu - \beta_{\nu-1})i + (\gamma_\nu - \gamma_{\nu-1})j + (\delta_\nu - \delta_{\nu-1})k], \end{aligned}$$

where $\alpha_{-1} = \beta_{-1} = \gamma_{-1} = \delta_{-1} = 0$.

By Theorem 5, $T(q) * (1-q) = 0$ if and only if either $T(q) = 0$ or $T(q) \neq 0$ implies $T(q)^{-1} q T(q) - 1 = 0$, that is $T(q)^{-1} q T(q) = 1$. Thus, if $T(q) \neq 0$, this implies $q = 1$, so the only zero of $T(q) * (1 - q)$ are $q = 1$ and the zeros of $T(q)$.

We first note that

$$\begin{aligned}|a_\nu - a_{\nu-1}| &= |(\alpha_\nu - \alpha_{\nu-1}) + (\beta_\nu - \beta_{\nu-1})i + (\gamma_\nu - \gamma_{\nu-1})j + (\delta_\nu - \delta_{\nu-1})k| \\ &\le |\alpha_\nu - \alpha_{\nu-1}| + |\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}|.\end{aligned}$$

For $|q| = 1$, we have

$$\begin{aligned}|f(q)| \le \sum_{\nu=0}^{n-2} |q^\nu(\alpha_\nu - \alpha_{\nu-1})| &+ |q^{n-1}k_1| + |q^{n-1}(k_1 + \alpha_{n-1} - \alpha_{n-2})| \\ &+ \left|q^n[(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})]\right| \\ &+ \sum_{\nu=0}^{n} \left|q^\nu[(\beta_\nu - \beta_{\nu-1})i + (\gamma_\nu - \gamma_{\nu-1})j + (\delta_\nu - \delta_{\nu-1})k]\right|,\end{aligned}$$

i.e.,

$$\begin{aligned}|f(q)| &\le |\alpha_0| + \sum_{\nu=0}^{n-2}(\alpha_\nu - \alpha_{\nu-1}) + |k_1| + (k_1 + \alpha_{n-1} - \alpha_{n-2}) \\ &\qquad + [(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})] + M_0 \\ &= |\alpha_0| - \alpha_0 + |k_1| + k_0 + \alpha_n + M_0,\end{aligned}$$

where

$$M_0 = \sum_{\nu=0}^{n} \big(|\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}|\big).$$

Notice that, we have

$$\max_{|q|=1} \left|q^n * f\left(\frac{1}{q}\right)\right| = \max_{|q|=1} \left|q^n f\left(\frac{1}{q}\right)\right| = \max_{|q|=1} \left|f\left(\frac{1}{q}\right)\right| = \max_{|q|=1} |f(q)|,$$

it is clear that $q^n * f(1/q)$ has the same bound on $|q| = 1$ as f, that is

$$\left|q^n * f\left(\frac{1}{q}\right)\right| \le |\alpha_0| - \alpha_0 + |k_1| + k_0 + \alpha_n + M_0 \quad \text{for} \quad |q| = 1.$$

Since $q^n * f(1/q)$ is a polynomial and hence is regular in $|q| \le 1$, it follows by the Maximum Modulus Theorem (Theorem 6), that

$$\left|q^n * f\left(\frac{1}{q}\right)\right| = \left|q^n f\left(\frac{1}{q}\right)\right| \le |\alpha_0| - \alpha_0 + |k_1| + k_0 + \alpha_n + M_0 \quad \text{for} \quad |q| \le 1.$$

Hence

$$\left|f\left(\frac{1}{q}\right)\right| \leq \frac{1}{|q^n|}\Big(|\alpha_0| - \alpha_0 + |k_1| + k_0 + \alpha_n + M_0\Big) \quad \text{for} \quad |q| \leq 1.$$

Replacing q by $1/q$, we see that

$$|f(q)| \leq \Big(|\alpha_0| - \alpha_0 + |k_1| + k_0 + \alpha_n + M_0\Big)|q|^n \quad \text{for} \quad |q| \geq 1. \tag{2}$$

For $|q| \geq 1$, we have

$$\begin{aligned} |T(q) * (1-q)| &= \big|f(q) - q^n(qa_n + k_0 - k_1)\big| \\ &\geq |q|^n |qa_n + k_0 - k_1| - |f(q)| \\ &\geq |q|^n \left[|qa_n + k_0 - k_1| - (\alpha_n + k_0 - \alpha_0 + |\alpha_0| + |k_1| + M_0)\right] \end{aligned}$$

by (2). Hence, if

$$\left|q + \frac{k_0 - k_1}{a_n}\right| > \frac{1}{|a_n|}\Big(\alpha_n + k_0 - \alpha_0 + |\alpha_0| + |k_1| + M_0\Big),$$

then $|T(q)*(1-q)| > 0$, that is $T(q)*(1-q) \neq 0$. Since the only zeros of $T(q)*(1-q)$ are $q = 1$ and the zeros of $T(q)$, therefore, $T(q) \neq 0$ for

$$\left|q + \frac{k_0 - k_1}{a_n}\right| > \frac{1}{|a_n|}\Big(\alpha_n + k_0 - \alpha_0 + |\alpha_0| + |k_1| + M_0\Big).$$

In other words, all the zeros of $T(q)$ lie in

$$\left|q + \frac{k_0 - k_1}{a_n}\right| \leq \frac{1}{|a_n|}\Big(\alpha_n + k_0 - \alpha_0 + |\alpha_0| + |k_1| + M_0\Big).$$

This completes the proof of Theorem 14. □

We need the following auxiliary result due to Carney et al. [**3**] for the proofs of Theorems 18 and 21.

Lemma 24. *Let $q_1, q_2 \in \mathbb{H}$, where $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$, $\measuredangle(q_1, q_2) \leq 2\theta$ and $|q_1| \leq |q_2|$. Then*

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

Proof of Theorem 18. Consider the power series

$$F(q) = (t - q) * f(q) = ta_0 - q\sum_{\nu=1}^{\infty} q^{\nu-1}(a_{\nu-1} - ta_\nu) = ta_0 - q\psi(q),$$

where

$$\psi(q) = \sum_{\nu=1}^{\infty} q^{\nu-1}(a_{\nu-1} - ta_\nu).$$

For $|q| = t$, we have

$$\begin{aligned}
|\psi(q)| &\le \sum_{\nu=1}^{\infty} \left|q^{\nu-1}(a_{\nu-1} - ta_\nu)\right| \\
&= \sum_{\nu=1}^{\infty} t^{\nu-1}\left|a_{\nu-1} - ta_\nu\right| \\
&\le \sum_{\nu=1}^{\infty} t^{\nu-1}\left\{ |t|a_\nu| - |a_{\nu-1}|| \cos\theta + (t|a_\nu| + |a_{\nu-1}|)\sin\theta \right\} \\
&= \sum_{\nu=1}^{\lambda} t^{\nu-1}(t|a_\nu| - |a_{\nu-1}|)\cos\theta + \sum_{\nu=\lambda+1}^{\infty} t^{\nu-1}(|a_{\nu-1}| - t|a_\nu|)\cos\theta \\
&\quad + 2\sin\theta \sum_{\nu=1}^{\infty} t^\nu |a_\nu| + |a_0|\sin\theta \\
&= (2t^\lambda|a_\lambda| - |a_0|)\cos\theta + +2\sin\theta \sum_{\nu=1}^{\infty} t^\nu|a_\nu| + |a_0|\sin\theta,
\end{aligned}$$

i.e.,

$$|\psi(q)| \le |a_0| \left[\left(2t^\lambda \left|\frac{a_\lambda}{a_0}\right| - 1 \right) \cos\theta + \frac{2\sin\theta}{|a_0|} \sum_{\nu=1}^{\infty} t^\nu|a_\nu| + \sin\theta \right] = |a_0|M.$$

Since $\psi(q)$ is regular in $|q| \le t$, it follows by the Maximum Modulus Theorem (Theorem 6), that

$$|\psi(q)| \le |a_0|M \quad \text{for} \quad |q| \le t. \tag{3}$$

For $|q| \le t$, we have

$$\begin{aligned}
|F(q)| &= |ta_0 - q\psi(q)| \\
&\ge t|a_0| - |q||\psi(q)| \\
&\ge |a_0|[t - |q|M] \qquad \text{(by (3))} \\
&> 0
\end{aligned}$$

if $|q| < t/M$.

Since by Theorem 4, the only zeros of $(t-q) * f(q)$ are $q = t$ and the zeros of $f(q)$, it follows that $f(q)$ does not vanish in $|z| < t/M$. This proves Theorem 18. □

Proof of Theorem 21. Again we consider the power series

$$\begin{aligned} F(q) &= (t-q) * f(q) \\ &= ta_0 - q(a_0 - ta_1) - q^2(a_1 - ta_2) - \cdots \\ &= ta_0 - q(\lambda_0 a_0 - t\lambda_1 a_1) + q\,[(\lambda_0 - 1)a_0 - (\lambda_1 - 1)ta_1] \\ &\quad - q^2[(\lambda_1 a_1 - ta_2) - (\lambda_1 - 1)a_1] - q^3(a_2 - ta_3) - \cdots \\ &= ta_0 + q\,[(\lambda_0 - 1)a_0 - (\lambda_1 - 1)ta_1] - q\psi(q), \end{aligned}$$

where

$$\psi(q) = (\lambda_0 a_0 - t\lambda_1 a_1) + q\,[(\lambda_1 a_1 - ta_2) - (\lambda_1 - 1)a_1] + q^2(a_2 - ta_3) + \cdots .$$

For $|q| = t$, we have

$$\begin{aligned} |\psi(q)| &\le |\lambda_0 a_0 - t\lambda_1 a_1| + |q||\lambda_1 a_1 - ta_2| + |q||\lambda_1 - 1||a_1| \\ &\quad + \sum_{\nu=3}^{\infty} |q|^{\nu-1}|a_{\nu-1} - ta_\nu| \\ &= |\lambda_0 a_0 - t\lambda_1 a_1| + |\lambda_1 a_1 t - t^2 a_\nu| + t(\lambda_1 - 1)|a_1| \\ &\quad + \sum_{\nu=3}^{\infty} t^{\nu-1}|a_{\nu-1} - ta_\nu| \\ &\le (\lambda_0|a_0| - t\lambda_1|a_1|)\cos\theta + (\lambda_0|a_0| + t\lambda_1|a_1|)\sin\theta \\ &\quad + (\lambda_1 t|a_1| - t^2|a_2|)\cos\theta + (\lambda_1 t|a_1| + t^2|a_2|)\sin\theta + t(\lambda_1 - 1)|a_1| \\ &\quad + \sum_{\nu=3}^{\infty} t^{\nu-1}\Big\{(|a_{\nu-1}| - t|a_\nu|)\cos\theta + (|a_{\nu-1}| + t|a_\nu|)\sin\theta\Big\}, \end{aligned}$$

i.e.,

$$|\psi(q)| \le \lambda_0|a_0|(\cos\theta + \sin\theta) + t(\lambda_1 - 1)|a_1| + 2\sin\theta\left(\lambda_1 t|a_1| + \sum_{\nu=2}^{\infty} t^\nu |a_\nu|\right).$$

It follows by the Maximum Modulus Theorem (Theorem 6), that

$$|\psi(q)| \le \lambda_0|a_0|(\cos\theta + \sin\theta) + t(\lambda_1 - 1)|a_1| + 2\sin\theta\left(\lambda_1 t|a_1| + \sum_{\nu=2}^{\infty} t^\nu |a_\nu|\right)$$

for $|q| \le t$.

Let E defined by (1). Now, for $|q| \le t$, we have

$$\begin{aligned}|F(q)| &= |a_0|\,|t+q(\lambda_0-1)| - (\lambda_1-1)t|a_1||q| - |q||\psi(q)| \\ &\ge |a_0|\,|t+q(\lambda_0-1)| - |a_0||q|\Bigg\{\lambda_0(\cos\theta+\sin\theta) \\ &\qquad + 2t(\lambda_1-1)\left|\frac{a_1}{a_0}\right| + \frac{2\sin\theta}{|a_0|}\left(\lambda_1 t|a_1| + \sum_{\nu=2}^{\infty} t^\nu |a_\nu|\right)\Bigg\} \\ &= |a_0|\,|t+q(\lambda_0-1)| - |a_0||q|E \\ &> 0,\end{aligned}$$

if $|q|E < |q(\lambda_0-1)+t|$, i.e., if

$$\alpha^2+\beta^2+\gamma^2+\delta^2 - \frac{2t(\lambda_0-1)\alpha}{E^2-(\lambda_0-1)^2} < \frac{t^2}{E^2-(\lambda_0-1)^2},$$

or

$$\left[\alpha - \frac{(\lambda_0-1)t}{E^2-(\lambda_0-1)^2}\right]^2 + \beta^2+\gamma^2+\delta^2 < \left(\frac{Et}{E^2-(\lambda_0-1)^2}\right)^2,$$

which is precisely the disk:

$$\left\{q : \left|q - \frac{(\lambda_0-1)t}{E^2-(\lambda_0-1)^2}\right| < \frac{Et}{E^2-(\lambda_0-1)^2}\right\}. \tag{4}$$

Since by Theorem 4, the only zeros of $(t-q)*f(q)$ are $q=t$ and the zeros of $f(q)$, it follows that $f(q)$ does not vanish in the disk defined by (4). This completes the proof of Theorem 21. □

4. CONCLUSION

The study of regular functions of a quaternionic variable are now a days a widely studied topic, important especially in replicating many properties of holomorphic functions of a complex variable and their rapid development has been largely driven by the applications to operator theory. In this paper, we study the properties of zeros of some special polynomials and regular functions of a quaternionic variable with restricted coefficients; namely quaternionic coefficients whose real components or their moduli satisfy suitable inequalities. We obtain zero free regions for these functions and also extend the well-known Eneström-Kakeya theorem and its various generalizations from complex to the quaternionic setting.

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