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Some Properties of Boubaker Polynomials and Applications

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Abstract. Some new properties of Boubaker polynomials, as well as an application of these polynomials for obtaining approximate analytical solution of Love's integral equation are presented.

Keywords: Chebyshev polynomials; Boubaker polynomials; zeros; recurrence relation; Fredholm integral equation; Love's integral equation.
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INTRODUCTION

There are several papers on the so-called Boubaker polynomials and their applications in different problems in physics and other computational and applied sciences (cf. [1, 2, 3, 9] and references therein). Such polynomials are defined in a similar way as Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)$, which are orthogonal on $(-1,1)$ with respect to the weights functions $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, respectively.

The monic Boubaker polynomials are defined as

$$B_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \frac{n-4k}{n-k} x^{n-2k}, \qquad n \geq 1, \tag{1}$$

and $B_0(x) = 1$. Alternatively, they can be expressed by a three-term recurrence relation

$$B_{m+1}(x) = xB_m(x) - B_{m-1}(x), \quad m = 2,3,\dots\ , \tag{2}$$

where $B_0(x) = 1$, $B_1(x) = x$, $B_2(x) = x^2 + 2$.

The next seven members of this polynomial sequence are:

$$B_3(x) = x^3 + x,\ \ B_4(x) = x^4 - 2,\ \ B_5(x) = x^5 - x^3 - 3x,\ \ B_6(x) = x^6 - 2x^4 - 3x^2 + 2,$$
$$B_7(x) = x^7 - 3x^5 - 2x^3 + 5x,\ \ B_8(x) = x^8 - 4x^6 + 8x^2 - 2,\ \ B_9(x) = x^9 - 5x^7 + 3x^5 + 10x^3 - 7x.$$

Otherwise, the polynomials (1) can be expressed in terms of Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$. Namely, we can prove (e.g., by the mathematical induction) the following result:

Theorem 1. *For $m \geq 1$ the following formula $B_m(x) = 2T_m(x/2) + 4U_{m-2}(x/2)$ holds, where $U_{-1}(x) \equiv 0$.*

THREE-TERM RECURRENCE RELATION AND ZEROS

As we can see, the relation (2) is not true for $m = 1$. In order to provide a relation for each $m \in \mathbb{N}$, we can define a sequence $\{\beta_m\}_{m\in\mathbb{N}}$ by $\beta_1 = -2$ and $\beta_m = 1$ for $m \geq 2$, and then we have the three-term recurrence relation in the form

$$B_{m+1}(x) = xB_m(x) - \beta_m B_{m-1}(x), \quad m = 1,2,\dots\ , \ \text{with} \quad B_0(x) = 1,\ B_{-1}(x) = 0. \tag{3}$$

Using this relation for $m = 0,1,\dots,n-1$, and defining n-dimensional vectors $\mathbf{b}_n(x) = [B_0(x)\ B_1(x)\ \dots\ B_{n-1}(x)]^T$ and $\mathbf{e}_n = [0\ 0\ \dots\ 0\ 1]^T$ (the last coordinate vector), we obtain the equation

$$(xI_n - M_n)\mathbf{b}_n(x) = B_n(x)\mathbf{e}_n, \tag{4}$$

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where I_n is the identity matrix of order n and M_n is a tridiagonal matrix of order n,

$$M_n = \begin{bmatrix} 0 & 1 & & & O \\ \beta_1 & 0 & 1 & & \\ & \beta_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & \beta_{n-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & O \\ -2 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & 1 & 0 \end{bmatrix}.$$

According to (4) we conclude that the zeros of the polynomial $B_n(x)$ are also eigenvalues of the matrix M_n. Also, using Gerschgorin's theorem, it is easy to see that these eigenvalues are in the unit circle $|z| < 2$ (see also [9]).

It is well-known that for orthogonal polynomials on a symmetric interval $(-a,a)$, which satisfy a three-term recurrence relation of the form (3), it can be defined two new polynomial systems which are orthogonal on $(0,a^2)$ (cf. [7, pp. 102–103]). In a similar way, we can introduce here also two new (nonorthogonal) systems of (monic) polynomials $P = \{p_m(t)\}$ and $Q = \{q_m(t)\}$ via Boubaker polynomials $B_m(x)$, so that

$$B_{2m}(x) = p_m(x^2) \quad \text{and} \quad B_{2m+1}(x) = xq_m(x^2).$$

Theorem 2. *Let β_m, $m \geq 1$, be recursive coefficients in the recurrence relation* (3). *Then,*

$$p_{m+1}(t) = (t-a_m)p_m(t) - b_m p_{m-1}(t) \quad \textit{and} \quad q_{m+1}(t) = (t-c_m)q_m(t) - d_m q_{m-1}(t),$$

with $p_0(t) = q_0(t) = 1$, $p_{-1}(t) = q_{-1}(t) =$, where the recursive coefficients are given by

$$a_m = \beta_{2m} + \beta_{2m+1} = \begin{cases} -2, & m = 0, \\ 2, & m \geq 1, \end{cases} \qquad b_m = \beta_{2m}\beta_{2m-1} = \begin{cases} -2, & m = 1, \\ 1, & m \geq 2, \end{cases}$$

and

$$c_m = \beta_{2m+1} + \beta_{2m+2} = \begin{cases} -1, & m = 0, \\ 2, & m \geq 1, \end{cases} \qquad d_m = \beta_{2m}\beta_{2m+1} = 1, \ m \geq 1.$$

Thus, this theorem gives two systems of polynomials: $P = \{1,\ t+2,\ t^2-2,\ t^3-2t^2-3t+2,\ t^4-4t^3+8t-2,\ \ldots\}$ and $Q = \{1,\ t+1,\ t^2-t-3,\ t^3-3t^2-2t+5,\ t^4-5t^3+3t^2+10t-7,\ \ldots\}$.

In order to investigate zeros of the polynomials $B_n(z)$ on the imaginary axis we put $z = \mathrm{i}y$ and consider $B_n(\mathrm{i}y)/\mathrm{i}^m$, $n \geq 2$, i.e., the sequence of polynomials $y^2-2,\ y(y^2-1),\ y^4-2,\ y(y^4+y^2-3),\ y^6+2y^4-3y^2-2,\ \ldots$.

For $t > 0$ we introduce two sequences of polynomials $e_m(t)$ and $o_m(t)$, $m = 1,2,\ldots$, by

$$e_m(t) = (-1)^m B_{2m}(\mathrm{i}\sqrt{t}) \quad \text{and} \quad o_m(t) = (-1)^m \frac{B_{2m+1}(\mathrm{i}\sqrt{t})}{\mathrm{i}\sqrt{t}}.$$

According to (1) and Theorem 2, it is clear that

$$e_m(t) = (-1)^m p_m(-t) = \sum_{k=0}^{m} \binom{2m-k}{k} \frac{2m-4k}{2m-k} t^{m-k}, \ \ o_m(t) = (-1)^m q_m(-t) = \sum_{k=0}^{m} \binom{2m-k}{k} \frac{2m-4k+1}{2m-2k+1} t^{m-k}. \quad (5)$$

Theorem 3. *For any $m \in \mathbb{N}$ the polynomials $e_m(t)$ and $o_m(t)$ have only one positive zero.*

Proof. In the proof we use the number of sign variations (differences) between consecutive nonzero coefficients of a polynomial ordered by descending variable exponent. We note that the coefficients in (5) are positive for $k < m/2$ and negative for $k > m/2$, so that we have only one sign variation. According to Descartes' Rule the number of positive zeros is either equal to the number of sign differences between consecutive nonzero coefficients, or less than it by a multiple of 2. Since $e_m(0) = -2 < 0$ and $e_m(T) > 0$ for each sufficiently large positive T, we conclude that $e_m(t)$ has only one positive zero. A similar proof can be done for polynomials $o_m(t)$. ☐

Using this theorem one can prove the following result on the zero distribution:

Theorem 4. *Every polynomial $B_n(x)$, $n \geq 2$, has two complex conjugate zeros $\pm\mathrm{i}\sqrt{\gamma_n}$, $\gamma_n > 0$, and other zeros are real and symmetrically distributed in $(-2,2)$, where $\lim_{n\to+\infty} \gamma_n = 4/3$.*

Thus,

$$B_{2m}(x) = (x^2+\gamma_{2m})\prod_{\nu=1}^{m}(x^2-\tau_{2m,\nu}), \quad B_{2m+1}(x) = x(x^2+\gamma_{2m+1})\prod_{\nu=1}^{m}(x^2-\tau_{2m+1,\nu}),$$

where $4 > \tau_{n,1} > \ldots > \tau_{n,m} > 0$ and $n = 2m$ or $n = 2m+1$.

APPLICATIONS

The polynomials $\{B_{4m}(x)\}$ plays important role in applications. Solutions to several applied physics problems based on the so-called Boubaker Polynomials Expansion Scheme (BPES) (cf. [9] and references therein). It is easy to prove that these polynomials satisfy the relation (cf. [3]) $B_{4(m+1)}(x) = (x^4-4x^2+2)B_{4m}(x) - \beta_m B_{4(m-1)}(x)$, $m \geq 1$, where β_m is defined before.

Recently, for example, Kumar [4] has presented a method for obtaining an analytical solution of Love's integral equation (see [5, 6])

$$f(x) - \mu\int_{-1}^{1}\frac{r}{r^2+(x-y)^2}f(y)\,\mathrm{d}y = 1, \quad -1 < x < 1, \tag{6}$$

for a particular physical (electrostatical) system, based on the Boubaker polynomials expansion scheme (BPES). An approximation to the solution of (6), in the case $r = 1$ and $\mu = 1/\pi$, was given by Love [6],

$$f(x) \approx f_L(x) = 1.919200 - 0.311717x^2 + 0.015676x^4 + 0.019682x^6 - 0.000373x^8. \tag{7}$$

As an approximate solution solution in the set of polynomials of degree at most $4n$ (in notation $\mathcal{P}_{4n}$), Kumar [4] used the expansion $f_{4n}^{(1)}(x) = \sum_{m=1}^{n} c_m B_{4m}(x)$, but in his approach was an error. The corrected version of the method leads to the equation

$$\sum_{m=1}^{n} c_m B_{4m}(x) - \mu\int_{-1}^{1}\frac{r}{r^2+(x-y)^2}\sum_{m=1}^{n} c_m B_{4m}(y)\,\mathrm{d}y = \sum_{m=1}^{n}\left(B_{4m}(x) - \mu\int_{-1}^{1}\frac{rB_{4m}(y)}{r^2+(x-y)^2}\,\mathrm{d}y\right)c_m = 1.$$

Taking collocation points as the positive zeros of $T_{2n}(x)$ we get a system of linear equations for determining the coefficients c_m, $m = 1,\ldots,n$. In the same case $r = 1$ and $\mu = 1/\pi$, the corresponding solutions for $n = 1$ and $n = 2$, are $f_4^{(1)}(x) = -1.01362B_4(x)$ and $f_8^{(1)}(x) = -1.01062B_4(x) + 0.140162B_8(x)$, or in the expanding form

$$f_4^{(1)}(x) = 2.02725 - 1.01362x^4 \quad \text{and} \quad f_8^{(1)}(x) = 1.74091 + 1.1213x^2 - 1.01062x^4 - 0.560649x^6 + 0.140162x^8.$$

However, we can get better solutions taking the constant term $(B_0(x) = 1)$ in the corresponding expansion of the approximate polynomial solution, i.e., $f_{4n}^{(0)}(x) = \sum_{m=0}^{n} c_m B_{4m}(x)$. In that case, using the positive zeros of $T_{2n+2}(x)$ as collocation points, we obtain the following approximative solutions

$$f_4^{(0)}(x) = 1.32192B_0(x) - 0.279362B_4(x) = 1.88064 - 0.279362x^4$$

and

$$\begin{aligned} f_8^{(0)}(x) &= 1.63647B_0(x) - 0.106254B_4(x) - 0.0339144B_8(x) \\ &= 1.91681 - 0.271315x^2 - 0.106254x^4 + 0.135658x^6 - 0.0339144x^8. \end{aligned}$$

Moreover, in the previous set of polynomials we can get much better results if we take the complete basis of (even) polynomials. Thus, in order to find an approximate solution in the set $\mathcal{P}_{2n}$, we put $\tilde{f}_{2n}(x) = \sum_{m=0}^{n} c_m B_{2m}(x)$. For example, in this case we find $\tilde{f}_4(x) = 2.63989B_0(x) - 0.32014B_2(x) + 0.0400543B_4(x)$ and

$$\begin{aligned} \tilde{f}_8(x) &= 2.46662B_0(x) - 0.264159B_2(x) + 0.0160255B_4(x) + 0.000730762B_6(x) - 0.00565549B_8(x) \\ &= 1.91903 - 0.311595x^2 + 0.014564x^4 + 0.0233527x^6 - 0.00565549x^8. \end{aligned}$$

TABLE 1. Maximal relative errors of the approximate solutions

Approximate solution	Maximal relative errors			
	$n=1$	$n=2$	$n=3$	$n=4$
$f_L(x)$		1.69(−3)		
$f_{4n}^{(1)}(x)$	3.82(−1)	1.27(−1)	3.57(−2)	1.22(−2)
$f_{4n}^{(0)}(x)$	2.34(−2)	1.16(−3)	1.52(−4)	1.07(−5)
$\tilde{f}_{4n}(x)$	2.43(−4)	1.37(−6)	9.65(−9)	2.41(−10)

Maximal relative errors of the previous approximate solutions, including Love's solution (7), are displayed in Table 1, where we used as the exact solution one obtained by an efficient method for solving Fredholm integral equations of the second kind [8]. Numbers in parentheses indicate decimal exponents. The solutions $\tilde{f}_8(x)$ for $\mu = 1/\pi$ and $r = 0.1$, $r = 1$, and $r = 10$ are presented in Figure 1.

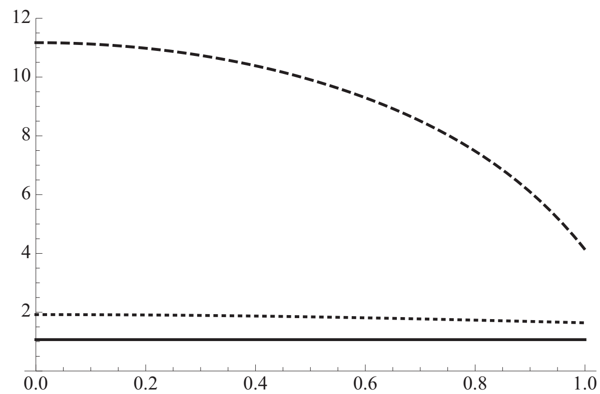

FIGURE 1. The solutions $\tilde{f}_8(x)$ of Love's equation (6) for $r = 1/10$ (dotted line), $r = 1$ (dashed line) and $r = 10$ (solid line)

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