Orthogonal Polynomials on Radial Rays in the Complex Plane

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Abstract

We consider some classes of polynomials orthogonal on radial rays in the complex plane with respect to the Hermitian and Non-Hermitian inner products, as well as some applications of such polynomials. Some applications of such polynomials could be done, including an electrostatic interpretation of their zeros.

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Introduction

Orthogonal polynomials play a very important role in applications not only in mathematics, but in many other computational and applied sciences, physics, chemistry, engineering, economics, etc. The most important orthogonal polynomials are ones which are orthogonal on the real line with respect to the inner product

$$(p,q)=\int_{\mathbb{R}} f(t)g(t)\mathrm{d}\mu(t) \quad \big(p,q\in L^2(\mathbb{R};\mathrm{d}\mu\big)\,,$$

where $\mathrm{d}\mu$ is a positive measure on $\mathbb{R}$ with finite or unbounded support, for which all moments $\mu_k=\int_{\mathbb{R}} t^k\,\mathrm{d}\mu(t)$, $k=0,1,\ldots$, exist and are finite, and $\mu_0>0$ (cf. [4], [8]). Because of the property $(tp,q)=(p,tq)$, these orthogonal polynomials $\pi_k(\,\cdot\,)=\pi_k(\mathrm{d}\mu;\,\cdot\,)$ satisfy a three–term recurrence relation

$$\pi_{k+1}(t)=(t-\alpha_k)\pi_k(t)-\beta_k\pi_{k-1}(t), \quad k=0,1,2\ldots, \tag{1}$$

with $\pi_0(t)=1$ and $\pi_{-1}(t)=0$, where the sequences of recursion coefficients α_k and β_k depend on the measure $\mathrm{d}\mu$. Only for certain narrow classes of measures, e.g., for the *classical measures* (Jacobi, generalized Laguerre, Hermite), these coefficients α_k and β_k are known in the explicit form (for a characterization of the classical orthogonal polynomials see [1]). Orthogonal polynomials for which the recursion coefficients are not known we call *strongly non-classical polynomials*.

In eighties of the last century Walter Gautschi developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, which includes effective algorithms for numerical generating orthogonal polynomials with respect to an arbitrary measure, strong stability analysis of such algorithms, necessary software for implementing such algorithms and applications (cf. [3], [4], [5]).

This constructive theory opened the door for extensive computational work on orthogonal polynomials and many their applications (construction of many new classes of strongly non-classical polynomials, development of other types of orthogonality, s

and σ-orthogonality, Sobolev type of orthogonality, multiple orthogonality, orthogonality on some curves in the complex plane (circle, semicircle [6, 7, 9], circular arc), orthogonality on radial rays [10, 11, 12, 13], etc.), applications in diverse areas of applied and numerical analysis (numerical integration, interpolation, integral equations, ...), approximation theory (e.g., moment-preserving spline approximation), integration of fast oscillating functions, summation of slowly convergent series, integration of fast oscillating functions, etc.

Orthogonal Polynomials on Radial Rays

Let $M \in \mathbb{N}$, $a_s > 0$, $s = 1, \ldots, M$, and $0 \le \theta_1 \le \cdots \le \theta_M < 2\pi$. Putting $\varepsilon_s = \mathrm{e}^{\mathrm{i}\theta_s}$, $s = 1, \ldots, M$, we consider M points in the complex plane, $z_s = a_s \varepsilon_s \in \mathbb{C}$, $s = 1, \ldots, M$, with arguments θ_s (see Fig. 1). Some of a_s (or all) may coincide and also can be ∞.

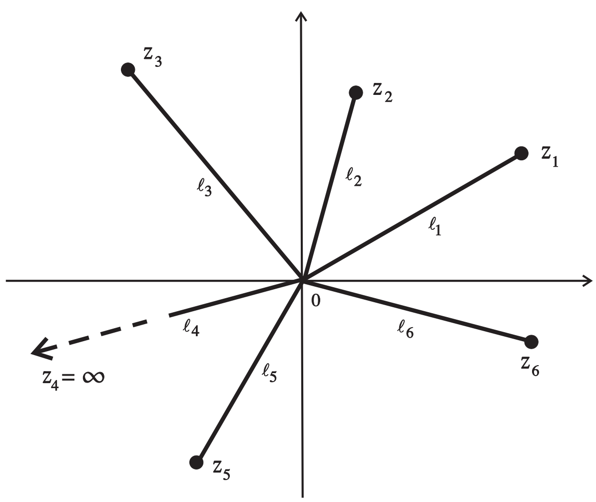

Figure 1: The case of six rays ($M = 6$)

The inner product can be introduced so that it is *hermitian*,

$$(f, g) = \sum_{s=1}^{M} \mathrm{e}^{-\mathrm{i}\theta_s} \int_{\ell_s} f(z)\overline{g(z)}\, |w_s(z)| \mathrm{d}z,$$

where $x \mapsto \omega_s(x) = |w_s(x\varepsilon_s)| = |w_s(z)|$ $(z \in \ell_s;\ s = 1, \ldots, M)$ are weight functions on $(0, a_s)$, i.e., they are nonnegative on $(0, a_s)$ and $\int_0^{a_s} \omega_s(x)\mathrm{d}x > 0$. It can be represented as

$$(f, g) = \sum_{s=1}^{M} \int_0^{a_s} f\big(x\varepsilon_s\big)\overline{g\big(x\varepsilon_s\big)}\, \omega_s(x)\mathrm{d}x,$$

and we can see that $(f, f) > 0$, except when $f(z) = 0$. Polynomials orthogonal with respect to this inner product can be considered. In the symmetric case with even numbers of rays ($M = 2m$) we can obtained analytic results for the recurrence coefficients for all classical weight functions (Jacobi, generalized Laguerre, Hermite).

In the simple symmetric (Legendre) case with four rays ($M = 4$) and

$$(f, g) = \int_0^1 \Big[f(x)\overline{g(x)} + f(\mathrm{i}x)\overline{g(\mathrm{i}x)} + f(-x)\overline{g(-x)} + f(-\mathrm{i}x)\overline{g(-\mathrm{i}x)}\Big]\mathrm{d}x,$$

we can prove the recurrence relation

$$\pi_{N+2}(z) = z^2\pi_N(z) - b_N\pi_{N-2}(z), \quad N \geq 2; \quad \pi_N(z) = z^N, \quad N \leq 3,$$

where the coefficient b_N $(N = 4n + \nu;\ n = [N/4])$ is given by

$$b_{4n+\nu} = \begin{cases} \dfrac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 0, 1, \\ \dfrac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 2, 3. \end{cases}$$

In the general case, using some kind of the discretized Stieltjes-Gautschi procedure, we can numerically construct the coefficients β_{kj} in the relation

$$\pi_k(z) = z\pi_{k-1}(z) - \sum_{j=1}^{k} \beta_{kj}\pi_{j-1}(z), \quad \beta_{kj} = \frac{(z\pi_{k-1}, \pi_{j-1})}{(\pi_{j-1}, \pi_{j-1})} \quad (1 \leq j \leq k).$$

The following result is related to the zero distribution of $\pi_N(z)$.

Theorem. *All the zeros of the orthogonal polynomial $\pi_N(z)$ lie in the convex hull of the rays $L = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_M$.*

Example. We consider an asymmetric case with five rays ($M = 5$), defined by points in the complex plane: $z_1 = 6$, $z_2 = 5\mathrm{e}^{9\pi\mathrm{i}/14}$, $z_3 = 2\mathrm{e}^{4\pi\mathrm{i}/5}$, $z_4 = 5\mathrm{e}^{6\pi\mathrm{i}/5}$, $z_5 = 5\mathrm{e}^{7\pi\mathrm{i}/4}$, with weight functions transformed to $(0, 1)$: $\omega_1(x) = 1$ (Legendre weight), $\omega_2(x) = 1/\sqrt{x(1-x)}$ (Chebyshev weight of the first kind), $\omega_3(x) = \sqrt{x(1-x)}$ Chebyshev weight of the second kind), $\omega_4(x) = \sqrt{x/(1-x)}$ (Chebyshev weight of the fourth kind), $\omega_5(x) = \sqrt{(1-x)/x}$ (Chebyshev weight of the third kind), respectively.

Zeros of $\pi_N(z)$ for $N = 20$ and $N = 100$ are presented in Figure 2.

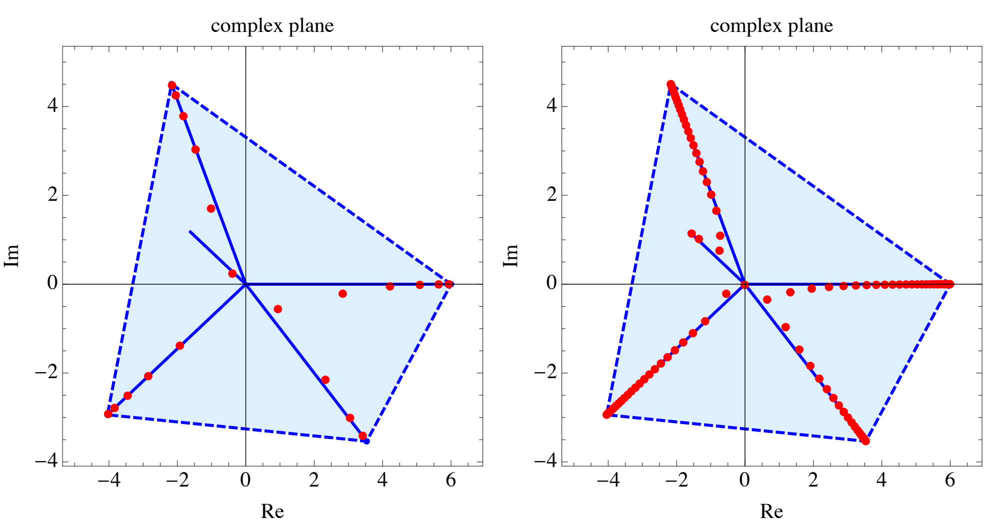

Figure 2: Zeros of $\pi_N(z)$ for $N = 20$ (left) and $N = 100$ (right)

In some symmetric cases, an electrostatic interpretation of the zeros of $\pi_N(z)$ can be done [11].

Orthogonal polynomials on radial rays with respect to a complex-valued moment functional

$$\mathcal{L}(p) = \sum_{s=1}^{M} \int_0^{a_s} p(x\varepsilon_s)\omega_s(x)\mathrm{d}x, \quad p \in \mathcal{P},$$

can be also considered, where $a_s > 0$ are given real numbers, and ε_s and ω_s are as before.

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