Bounds of the error of Gauss–Turán-type quadratures, II

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Article history:
Received 4 September 2008
Received in revised form 2 August 2009
Accepted 8 August 2009
Available online 19 August 2009

MSC:
primary 65D30, 65D32
secondary 41A55

Keywords:
Gauss–Turán quadrature formula
Error bound
Remainder term for analytic functions
Contour integral representation

1. Introduction

Let \( w \) be an integrable (nonnegative) weight function on the interval \((-1, 1)\), \( n \in \mathbb{N} \) and \( s \in \mathbb{N}_0 \). It is well known that Gauss–Turán quadrature formula with multiple nodes,

\[
\int_{-1}^{1} f(t)w(t)\, dt = \sum_{i=1}^{n} \sum_{\nu=0}^{2s} \lambda_{i,\nu} f^{(i)}(\tau_{\nu}) + R_{n,s}(f),
\]

is exact for all algebraic polynomials of degree at most \( 2(s+1)n-1 \). The nodes \( \tau_{\nu} \) in (1.1) must be zeros of the corresponding \( s \)-orthogonal polynomials \( \pi_n = \pi_{n,s} \) satisfying the following orthogonality conditions

\[
\int_{-1}^{1} \pi_n(t)^{2s+1} t^k w(t)\, dt = 0, \quad k = 0, 1, \ldots, n - 1.
\]

Gauss–Turán quadrature formulae, or quadrature formulae with the highest degree of algebraic precision with multiple nodes, have extensively been studied in the last decades from both an algebraic and numerical point of view. Numerically

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doi:10.1016/j.apnum.2009.08.002
stable methods for constructing nodes τn and coefficients λk,ν can be found in [12] and [17]. Some interesting theoretical results concerning this theory have recently been obtained (see [16] (and references therein), [7,15]).

Let Γ be a simple closed curve in the complex plane surrounding the interval [−1, 1] and let D be its interior. If integrand f is analytic on D and continuous on ∂D, then the remainder term Rn,s in (1.1) admits the contour integral representation

$$R_{n,s}(f) = \frac{1}{2\pi i} \oint_{\Gamma} K_{n,s}(z) f(z) \, dz.$$  

(1.2)

The kernel is given by

$$K_{n,s}(z; w) = \frac{\rho_{n,s}(z; w)}{[\pi n_s(z)]^{2s+1}}, \quad z \not\in [-1, 1],$$

where

$$\rho_{n,s}(z; w) = \int_{-1}^{1} \frac{[\pi n_s(t)]^{2s+1}}{z - t} w(t) \, dt.$$  

(1.3)

The modulus of the kernel is symmetric with respect to both axes, i.e., |K_{n,s}(2z)| = |K_{n,s}(z)|. If the weight function w is even, the modulus of the kernel is symmetric with respect to both axes, i.e., |K_{n,s}(-z)| = |K_{n,s}(z)| (see [8, Lemma 2.1]).

The integral representation (1.2) leads to a general error estimate, by using Hölder’s inequality,

$$|R_{n,s}(f)| \leq \frac{1}{2\pi} \left( \oint_{\Gamma} |K_{n,s}(z)|^r \, |dz| \right)^{1/r} \left( \oint_{\Gamma} |f(z)|^r \, |dz| \right)^{1/r'},$$

(1.4)

i.e.,

$$|R_{n,s}(f)| \leq \frac{1}{2\pi} \|K_{n,s}\|_r \|f\|_{r'},$$

(1.5)

where 1 ≤ r ≤ +∞, 1/r + 1/r' = 1, and

$$\|f\|_r := \begin{cases} \left( \oint_{\Gamma} |f(z)|^r \, |dz| \right)^{1/r}, & 1 \leq r < +\infty, \\ \max_{z \in \Gamma} |f(z)|, & r = +\infty. \end{cases}$$

The case r = +∞ (r' = 1) gives

$$|R_{n,s}(f)| \leq \frac{1}{2\pi} \left( \max_{z \in \Gamma} |K_{n,s}(z)| \right) \|f\|_1,$$

(1.6)

whereas for r = 1 (r' = +∞) we have

$$|R_{n,s}(f)| \leq \frac{1}{2\pi} \left( \oint_{\Gamma} |K_{n,s}(z)| \, |dz| \right) \|f\|_\infty.$$  

(1.7)

It is possible to obtain error bounds of the type (1.6) and (1.7) analytically (i.e., to calculate max_{z \in \Gamma} |K_{n,s}(z)| or \oint_{\Gamma} |K_{n,s}(z)| \, |dz|) only for weight functions which admit explicit Gauss–Turán quadrature formulae, i.e., in the cases when explicit formulae for corresponding ω-orthogonal polynomials are known. There are only a couple of them.

In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial \( \hat{T}_n(t) = T_n(t)/2^{n-1} \) minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi n(t)|^{k+1}}{\sqrt{1-t^2}} \, dt \quad (k \geq 0).$$

This means that the Chebyshev polynomials \( T_n \) are ω-orthogonal on (−1, 1) for each \( s \geq 0 \). Ossicini and Rosati [14] found three other weight functions \( w_k(t) \) (k = 2, 3, 4),

\[ w_2(t) = (1 - t^2)^{1/2 + s}, \quad w_3(t) = \frac{(1+t)^{1/2 + s}}{(1-t)^{1/2}}, \quad w_4(t) = \frac{(1-t)^{1/2 + s}}{(1+t)^{1/2}}, \]

for which the ω-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind: \( U_n, V_n, \) and \( W_n, \) which are defined by
Let contour $I$ be an ellipse with foci at the points $\pm 1$ and a sum of semi-axes $\rho > 1$.

$$E_\varrho = \left\{ z \in \mathbb{C} : z = \frac{1}{2} (\xi + \xi^{-1}), \xi = \rho e^{i \theta}, 0 \leq \theta \leq 2 \pi \right\}.$$  \hfill (2.1)

As it is mentioned above we have that $\pi_{n,t}(t; w_{n,t}) = T_n(t)$. We use the following facts (see [14, Eqs. (4.1) and (4.2)])

$$\begin{align*}
\left[ T_n(t) \right]^{2s+1} &= 2^{-2s} \sum_{k=0}^{\infty} \binom{2s+1}{s-k} T_{n(2k+1)}(t), \\
\left(1 - t^2\right)^s \left[ U_n(t) \right]^{2s+1} &= 2^{-2s} \sum_{k=0}^{\infty} (-1)^k \binom{2s+1}{s-k} U_{n(2k+1)+2k}(t),
\end{align*}$$

and

$$I_{j,p} = \int_0^\pi \frac{\cos \theta}{z - \cos \theta} \sin \theta \, d\theta = \frac{2\pi}{\xi - \xi^{-1}} \frac{\xi^n - 1}{n \xi^{2j+p+1} + \xi^{2j+p+1} - \xi^{2j+p+1} - \xi^{2j+p+1}},$$

$$I_{j,p} = \frac{\pi}{2} \frac{\xi^{2j+p+1}}{\xi^n (\xi^{-1})^{n} + 1} + \frac{\text{sign}(p-j)}{\xi^{2j+p+1} n}.$$
Denoting \( k = \ell - 1 \), we get

\[
\rho_{n,s}(\xi; w_{n,\ell}) = \int_{-1}^{1} \frac{[U_{n-1}(t)]^{2\ell}}{n^{2\ell}} (1 - t^2)^{\ell - 1} \sqrt{1 - t^2} \frac{[T_n(t)]^{2s+1}}{z - t} dt
\]

\[
= \int_{-1}^{1} \frac{U_{n-1}(t) \sqrt{1 - t^2} \{ (1 - t^2)^k [U_{n-1}(t)]^{2k+1} \} [T_n(t)]^{2s+1}}{n^{2\ell} (z - t)} dt
\]

\[
= \int_{-1}^{1} \frac{U_{n-1}(t) \sqrt{1 - t^2}}{n^{2\ell} (z - t)} \left[ \frac{1}{2^{2k}} \sum_{p=0}^{k} (-1)^p \binom{2k+1}{k-p} U_n(2p+1,\ell-1)(t) \right] \left[ \frac{1}{2^{2s}} \sum_{j=0}^{s} \binom{2s+1}{s-j} T_n(2j+1)(t) \right] dt.
\]

By substituting \( t = \cos \theta \), we have, in view of \( T_n(\cos \theta) = \cos n \theta \) and \( U_{n-1}(\cos \theta) = \sin n \theta / \sin \theta \),

\[
\rho_{n,s}(\xi; w_{n,\ell}) = \frac{1}{n^{2\ell} 4k+s} \int_{0}^{\pi} \frac{\sin n \theta}{z - \cos \theta} \left[ \sum_{p=0}^{k} \sum_{j=0}^{s} (-1)^p \binom{2k+1}{k-p} \binom{2s+1}{s-j} \frac{\sin(2p+1)n \theta}{\sin \theta} \cos(2j+1)n \theta \right] \sin \theta d \theta
\]

\[
= \frac{1}{n^{2\ell} 4k+s} \sum_{p=0}^{k} \sum_{j=0}^{s} (-1)^p \binom{2k+1}{k-p} \binom{2s+1}{s-j} \int_{p-j=0}^{s+1} j \cdot p
\]

\[
= \frac{\pi}{2n^{2\ell} 4k+s} \frac{\xi^{2n-1}}{\xi^{2n} - \xi^{-1}} \left[ \frac{1}{\xi^{2(j+p+1)n}} + \frac{\sin(p-j)}{\xi^{(j-p)n}} \right]
\]

\[
= A_{n,s,k} \frac{\xi^{2n-1}}{\xi^{2n} - \xi^{-1}} V_{n,s,k}(\xi),
\]

where

\[
A_{n,s,k} = \frac{\pi}{2n^{2k+2}4k+s},
\]

\[
V_{n,s,k}(\xi) = \sum_{\lambda=0}^{s+k} F_{s,k}(\lambda) \frac{1}{\xi^{2\lambda n}},
\]

and

\[
F_{s,k}(\lambda) = \sum_{j+p=\lambda} (-1)^p \binom{2k+1}{k-p} \binom{2s+1}{s-j} + \sum_{|p-j|=\lambda+1} (-1)^p \text{sign}(p-j) \binom{2k+1}{k-p} \binom{2s+1}{s-j}.
\]

\( j = 0, 1, \ldots, s; \quad p = 0, 1, \ldots, k. \)

According to (1.3) and well-known fact \( T_n(z) = (\xi^n + \xi^{-n})/2 \) we get

\[
K_{n,s}(\xi; w_{n,\ell}) = B_{n,s,k} \frac{\xi^{2n-1}}{\xi^{2n} - \xi^{-1}} (\xi^n + \xi^{-n})^{2s+1} V_{n,s,k}(\xi),
\]

with \( B_{n,s,k} = 2^{2k+1} A_{n,s,k} = \pi/(n^{2k+2}4k). \) Further, using the well-known equalities

\[
|\xi^n + \xi^{-n}| = \left[ 2(a_{2n} + \cos 2n \theta) \right]^{1/2}, \quad |\xi^n - \xi^{-n}| = \left[ 2(a_{2n} - \cos 2n \theta) \right]^{1/2},
\]

where

\[
a_j = a_j(q) = \frac{1}{2} (q^j + q^{-j}), \quad j \in \mathbb{N},
\]

we get an explicit representation of \( K_{n,s}(\xi; w_{n,\ell}) \) in the form

\[
|K_{n,s}(\xi; w_{n,\ell})| = C_{n,s,k} \frac{(a_{2n} - \cos 2n \theta)^{1/2}}{(a_{2n} + \cos 2n \theta)^{1/2} (a_{2n} + \cos 2n \theta)^{1/2}} \left| V_{n,s,k}(\xi) \right|,
\]

where

\[
C_{n,s,k} = \frac{\pi}{n^{2k+2}2^{2k+1/2}q^{2n}}.
\]
Fig. 1. The function $\theta \mapsto |K_{n,3}(z; w_{n,2})|$ ($z \in \mathcal{E}_{1,1}$) when $n = 15, 20, 25, 30.$

Since $w_{n,\ell}$ is an even function, we have that $|K_{n,s}(z)|$, $z \in \mathcal{E}_Q$, is symmetric with respect to both axes. The graphs $\theta \mapsto |K_{n,s}(z; w_{n,\ell})|$ ($z \in \mathcal{E}_Q$) for certain values of $n, s, \ell$ and $Q$ are displayed in Fig. 1.

3. Error bounds of the type (1.7)

In this section we study the quantity $\frac{1}{2\pi} \int_{\mathcal{E}_Q} |K_{n,s}(z; w_{n,\ell})| \, |dz|$, where integrand $|K_{n,s}(z; w_{n,\ell})|$ is given by (2.6). We use the following integral from [5, Eq. 3.616.7]

$$J_j(a) = \int_0^\pi \frac{\cos j \theta}{(a + \cos \theta)^{2s+1}} \, d\theta = \frac{(-1)^j \pi 2^{2s+1} x^{j-(-j-1)/2}}{(x-1)^{4s+1}} \sum_{v=0}^{2s} \binom{2s+w}{v} \binom{2s+j}{v+j} (x-1)^{2s-v},$$

(3.1)

where $a = (x+1)/(2\sqrt{x})$ and $x > 1$. 

and (see [9, Lemma 4.1]),

$$|V_{n,s,k}(q^{\ell \theta})| = \left[ q^{-2n(s+k)} \sum_{j=0}^{s+k} \tilde{A}_j \cos 2jn\theta \right]^{1/2},$$

$$\tilde{A}_0 = \frac{1}{x^{(s+k)/2}} \sum_{v=0}^{s+k} F_{s,k}(s+k-v)^2 x^v, \quad x = q^{4n},$$

$$\tilde{A}_j = \frac{2}{x^{(s+k-j)/2}} \sum_{v=0}^{s+k-j} F_{s,k}(s+k-v)F_{s,k}(s+k-v-j)x^v, \quad j = 1, \ldots, s.$$  

(2.7)
Theorem 3.1. If \( a_j, \bar{A}_j \) and \( J_j \) are defined by (2.5), (2.7) and (3.1), then we have
\[
\frac{1}{2\pi} \oint_{E_\phi} |K_{n,s}(z; w_{n,\ell})| \, |dz| \leq \frac{\pi^{1/2}}{2^{s+2\ell-1}n^{2s+\ell+1}} \left\{ \sum_{j=0}^{s+\ell-1} \bar{A}_j \left[ a_{2n}J_j(a_{2n}) - \frac{1}{2} \left( J_{j+1}(a_{2n}) + J_{j-1}(a_{2n}) \right) \right] \right\}^{1/2}.
\] (3.2)

Proof. Since \( z = \frac{1}{2}(\xi + \xi^{-1}) \), \( \xi = \rho e^{i\theta} \), we have \(|dz| = 2^{-1/2} \sqrt{\rho^2 - \cos 2\theta} \, d\theta \) and
\[
\oint_{E_\phi} |K_{n,s}(z; w_{n,\ell})| \, |dz| = \frac{2\pi}{\rho^{2s+2\ell+1}} \sum_{j=0}^{s+\ell-1} \bar{A}_j \cos 2j\theta \, d\theta, \tag{3.3}
\]
where
\[
D_{n,s,k} = \frac{C_{n,s,k}}{\sqrt{Q_n(\rho)}} = \frac{\pi}{n^{2s+2\ell+1}} \rho^{2s+\ell+2}.
\]
Using the fact
\[
\int_0^{2\pi} g(2n\theta) \, d\theta = 2\int_0^\pi g(\theta) \, d\theta
\]
and applying Cauchy’s inequality, we get
\[
\oint_{E_\phi} |K_{n,s,k}(z)| \, |dz| \leq 2\sqrt{\pi} D_{n,s,k} \left( \int_0^{\pi} \frac{1}{\rho^{2s+2\ell+1}} \sum_{j=0}^{s+\ell-1} \bar{A}_j |a_{2n}\cos j\theta - \cos \theta \cos j\theta| \, d\theta \right)^{1/2}
\]
\[
= 2\sqrt{\pi} D_{n,s,k} \left\{ \sum_{j=0}^{s+\ell-1} \bar{A}_j \left[ a_{2n}J_j(a_{2n}) - \frac{1}{2} \left( J_{j+1}(a_{2n}) + J_{j-1}(a_{2n}) \right) \right] \right\}^{1/2},
\]
where
\[
2\sqrt{\pi} D_{n,s,k} = \frac{\pi^{3/2}}{n^{2s+2\ell+1}} \rho^{2s+\ell+2}.
\]

For \( k = 0 \), the bound (3.2) coincides with the bound (2.10) from [11]. As is seen from Figs. 2 and 3, the bound (3.2) is very sharp, especially for larger values of \( n, s \) and \( \rho \).

Completing the examples corresponding to Figs. 2 and 3, we explicitly include the connected quadrature formulae. Namely, it is well known that the nodes in the quadrature formula (1.1), in the cases under consideration, are given by
\[
\tau_v = \cos \left( \frac{2v - 1}{2n} \right) \pi, \quad v = 1, \ldots, n.
\]
The coefficients \( \lambda_{i,v} \) are given by (see [18, Eq. (3.5)])
\[
\lambda_{0,v} = \frac{\pi Q_0}{2n},
\]
and, for \( i = 1, \ldots, n \), by
\[
\lambda_{i,v} = \frac{\pi}{2n} \sum_{j=\binom{i+1}{2}}^{i} \frac{(1 - \tau_v^2)^j}{(i - 1)!2j^n} \sum_{k=0}^{j} \left( \begin{array}{c} j \end{array} \right) \ell_k,
\]
where
\[
b_{k,v,j} = \frac{1}{j!} (L_v(t) - j)^{(k)}_{t=\tau_v}, \quad k \in \mathbb{N}_0, \quad v = 1, \ldots, n, \quad j \in \mathbb{N},
\]
\[
L_v(t) = \frac{\omega_{n}(t)}{\omega_{n}(\tau_v)(t - \tau_v)}, \quad \omega_{n}(t) = \prod_{i=1}^{n}(t - \tau_v).
\]
Fig. 2. $\log_{10} \frac{1}{2\pi} \| K_{n,s}(z; w_n, \ell) \|_1$ (solid line) and $\log_{10}$ of the bound (3.2) (dashed line) as functions of $\varrho$, when $n = 8, s = 1, \ell = 2$.

Fig. 3. $\log_{10} \frac{1}{2\pi} \| K_{n,s}(z; w_n, \ell) \|_1$ (solid line) and $\log_{10}$ of the bound (3.2) (dashed line) as functions of $\varrho$, when $n = 5, s = 4, \ell = 3$. 
The prime on the summation indicates that the first term is halved.

Table 1
The coefficients $\lambda_{i,v}$ from (1.1) when $n = 8, s = 1, \ell = 2$.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$i = 0$</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.5952674160697(−05)</td>
<td>4.59137825945918(−08)</td>
<td>1.78172812946410(−09)</td>
</tr>
<tr>
<td>2</td>
<td>3.5952674160697(−05)</td>
<td>3.8923907523565(−08)</td>
<td>1.4499370787881(−08)</td>
</tr>
<tr>
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<td>3.5952674160697(−05)</td>
<td>2.6081196219861(−08)</td>
<td>3.2364041577609(−08)</td>
</tr>
<tr>
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<td>3.5952674160697(−05)</td>
<td>9.1238709338389(−09)</td>
<td>4.5031650407851(−09)</td>
</tr>
<tr>
<td>5</td>
<td>3.5952674160697(−05)</td>
<td>−9.1238709338389(−09)</td>
<td>4.5031650407851(−09)</td>
</tr>
<tr>
<td>6</td>
<td>3.5952674160697(−05)</td>
<td>−2.6081196219861(−08)</td>
<td>3.2364041577609(−08)</td>
</tr>
<tr>
<td>7</td>
<td>3.5952674160697(−05)</td>
<td>−3.8923907523565(−08)</td>
<td>1.4499370787881(−08)</td>
</tr>
<tr>
<td>8</td>
<td>3.5952674160697(−05)</td>
<td>−4.59137825945918(−08)</td>
<td>1.78172812946410(−09)</td>
</tr>
</tbody>
</table>

Table 2
The coefficients $\lambda_{i,v}$ from (1.1) when $n = 5, s = 4, \ell = 3$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$v = 1$</th>
<th>$v = 2$</th>
<th>$v = 3$</th>
<th>$v = 4$</th>
<th>$v = 5$</th>
</tr>
</thead>
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<td>0</td>
<td>1.25663706143592(−05)</td>
<td>$\lambda_{v,1}$</td>
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<td>$\lambda_{v,1}$</td>
<td>$\lambda_{v,1}$</td>
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<tr>
<td>1</td>
<td>3.7024752993068(−08)</td>
<td>2.08292752398086(−08)</td>
<td>$\lambda_{v,1}$</td>
<td>$\lambda_{v,1}$</td>
<td>$\lambda_{v,1}$</td>
</tr>
<tr>
<td>2</td>
<td>3.488444585458585(−09)</td>
<td>2.31538143400258(−08)</td>
<td>3.53076812496826(−08)</td>
<td>$\lambda_{v,2}$</td>
<td>$\lambda_{v,2}$</td>
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<tr>
<td>3</td>
<td>2.36970846458994(−11)</td>
<td>9.90357058373003(−11)</td>
<td>0</td>
<td>$\lambda_{v,2}$</td>
<td>$\lambda_{v,2}$</td>
</tr>
<tr>
<td>4</td>
<td>4.90753754611256(−13)</td>
<td>1.854604059978200(−11)</td>
<td>4.26753946063638(−11)</td>
<td>$\lambda_{v,2}$</td>
<td>$\lambda_{v,2}$</td>
</tr>
<tr>
<td>5</td>
<td>3.58661614290293(−15)</td>
<td>9.82393244205083(−14)</td>
<td>0</td>
<td>$\lambda_{v,2}$</td>
<td>$\lambda_{v,2}$</td>
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<tr>
<td>6</td>
<td>3.2887129769073(−17)</td>
<td>7.43617507016751(−15)</td>
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<tr>
<td>7</td>
<td>1.44536614395942(−19)</td>
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<td>$\lambda_{v,2}$</td>
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</tr>
<tr>
<td>8</td>
<td>5.18296467575908(−22)</td>
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<td>6.2331875712263(−18)</td>
<td>$\lambda_{v,2}$</td>
<td>$\lambda_{v,2}$</td>
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Table 3
The values of $\varrho_0$ for certain values of $n$ when $s = 1$ and $\ell = 3$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\varrho_0$</th>
<th>$n$</th>
<th>$\varrho_0$</th>
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</thead>
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<td>3</td>
<td>2.24</td>
<td>7</td>
<td>1.38</td>
</tr>
<tr>
<td>4</td>
<td>1.79</td>
<td>8</td>
<td>1.33</td>
</tr>
<tr>
<td>5</td>
<td>1.58</td>
<td>9</td>
<td>1.29</td>
</tr>
<tr>
<td>6</td>
<td>1.46</td>
<td>10</td>
<td>1.26</td>
</tr>
</tbody>
</table>

and $\varrho_k$ are the coefficients from Fourier–Chebyshev series of the form

$$w_n,\ell(t) = \sqrt{1-t^2} = \sum_{k=0}^{\infty} \varrho_k T_{2kn}(t),$$

where convergence holds with respect to the weighted $L^1$-norm

$$\int_{-1}^{1} |f(t)| \frac{dt}{\sqrt{1-t^2}}.$$

The prime on the summation indicates that the first term is halved.

The values of $\lambda_{i,v}$ corresponding to Figs. 2 and 3 are displayed in Tables 1 and 2.

4. Error bounds of the type (1.6)

In this section we study the quantity $\max_{x \in \mathbb{C}} |K_{n,s}(z; w_{n,\ell})|$, where $|K_{n,s}(z)|$ is given by (2.6). Computation shows that $|K_{n,s}(z; w_{n,\ell})|$, $z \in \mathbb{C}$, attains its maximum on the real axis ($z = \pm(\varrho + \varrho^{-1})/2$) if $\varrho > \varrho_0(n,s,\ell)$. Numerical values of $\varrho_0$ for certain values of $n, s$ and $\ell$ have been determined by MATLAB and are shown in Tables 3 and 4. Displayed values are optimal in the sense that $|K_{n,s}(z; w_{n,\ell})|$ does not attain its maximum at $\varrho = \varrho_0$ when $\varrho = \varrho_0$.

This empirical observation can be verified asymptotically as $\varrho \to \infty$. A lengthy calculation reveals that

$$|K_{n,s}(z; w_{n,\ell})| \sim \frac{\pi}{n^{2k+2}4^{k+\varrho/2}} \frac{1 + 2 \cos \varrho}{\varrho^2}^{1/2}, \quad \varrho \to \infty.$$

Using the facts $|V_{n,k}(\xi)| \leq 4^{k+1}$ and $(\varrho_2 - \cos 2\vartheta)(\varrho_{2n} + \cos 2\vartheta) \leq (\varrho_2 - 1)(\varrho_{2n} + 1)$ we obtain the following crude, but very simple inequality

$$|K_{n,s}(z; w_{n,\ell})| \leq \frac{\pi}{n^{2k}4^{k+1}} \frac{1}{(\varrho - \varrho^{-1})(\varrho^n - \varrho^{-n})^{2k}}.$$

(4.1)
Table 4
The values of $\varrho_0$ for certain values of $\ell$ when $s = 2$ and $n = 5$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\varrho_0$</th>
<th>$\ell$</th>
<th>$\varrho_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.64</td>
<td>5</td>
<td>1.70</td>
</tr>
<tr>
<td>2</td>
<td>1.66</td>
<td>6</td>
<td>1.71</td>
</tr>
<tr>
<td>3</td>
<td>1.68</td>
<td>7</td>
<td>1.71</td>
</tr>
<tr>
<td>4</td>
<td>1.69</td>
<td>8</td>
<td>1.72</td>
</tr>
</tbody>
</table>

We conclude with some remarks about quadrature formulae studied here. In general, the nodes of Gauss–Turán quadrature formulae vary both with $n$ and $s$, whereas in this case they are independent of $s$. This allows one to get higher precision by increasing $s$, without recalculating nodes. The convergence of (1.1) with respect to $w_{n,\ell}(t)$, when $s \to \infty$, immediately follows from (1.6) and (4.1). See also [4, Theorem 4.3].

References