Constr. Approx. (1987) 3: 389–404

CONSTRUCTIVE APPROXIMATION
© 1987 Springer-Verlag New York Inc.

Polynomials Orthogonal on the Semicircle, II

Walter Gautschi, Henry J. Landau, and Gradimir V. Milovanović

Abstract. Generalizing previous work [2], we study complex polynomials $\{\pi_k\}$, $\pi_k(z)=z^k+\cdots$, orthogonal with respect to a complex-valued inner product $(f,g)=\int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta$. Under suitable assumptions on the "weight function" w, we show that these polynomials exist whenever $\operatorname{Re}\int_0^\pi w(e^{i\theta})\,d\theta\neq 0$, and we express them in terms of the real polynomials orthogonal with respect to the weight function $w(x)$. We also obtain the basic three-term recurrence relation. A detailed study is made of the polynomials $\{\pi_k\}$ in the case of the Jacobi weight function $w(z)=(1-z)^\alpha(1+z)^\beta$, $\alpha>-1$, $\beta>-1$, and its special case $\alpha=\beta=\lambda-\frac{1}{2}$ (Gegenbauer weight). We show, in particular, that for Gegenbauer weights the zeros of π_n are all simple and, if $n\geq 2$, contained in the interior of the upper unit half disc. We strongly suspect that the same holds true for arbitrary Jacobi weights. Finally, for the Gegenbauer weight, we obtain a linear second-order differential equation for $\pi_n(z)$. It has regular singular points at $z=1,-1,\infty$ (like Gegenbauer's equation) and an additional regular singular point on the negative imaginary axis, which depends on n.

1. Introduction

Let w be a weight function which is positive and integrable on the open interval $(-1,1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc $D_+=\{z\in\mathbb{C}: |z|<1, \operatorname{Im} z>0\}$. Consider the following two inner products,

$$(1.1)\qquad [f,g]=\int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx,$$

$$(1.2)\qquad (f,g)=\int_\Gamma f(z)g(z)w(z)(iz)^{-1}\,dz=\int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta,$$

where Γ is the circular part of ∂D_+ and all integrals are assumed to exist (possibly) as appropriately defined improper integrals. The first inner product is positive definite and therefore generates a unique set of real orthogonal polynomials $\{p_k\}$,

$$(1.3)\qquad [p_k,p_l]\begin{cases}=0 & \text{if } k\neq l,\\ >0 & \text{if } k=l,\end{cases}\quad k,l=0,1,2,\ldots,$$

Date received: June 30, 1986. Date revised: September 19, 1986. Communicated by Paul Nevai.
AMS classification: Primary 30C10, 30C15, 33A65; Secondary 34A30.
Key words and phrases: Complex orthogonal polynomials, Recurrence relations, Zeros, Differential equation.

where p_k is assumed monic of degree k. The second inner product, on the other hand, is not Hermitian; we deliberately did not conjugate the second factor g and did not integrate with respect to the measure $|w(e^{i\theta})|\,d\theta$. The existence of corresponding orthogonal polynomials, therefore, is not guaranteed. We call a system of complex polynomials $\{\pi_k\}$ *orthogonal on the semicircle* if

$$(1.4)\qquad (\pi_k, \pi_l)\begin{cases} =0 & \text{if } k \neq l, \\ \neq 0 & \text{if } k = l, \end{cases} \quad k, l = 0, 1, 2, \ldots;$$

we assume π_k monic of degree k.

Our interest here is in the orthogonal polynomials $\{\pi_k\}$, their existence, relationship to the polynomials $\{p_k\}$, difference and differential equations, and zeros. A study of these polynomials was initiated in [2], where we considered $w(z) \equiv 1$ and used moment information to construct the polynomials $\{\pi_k\}$. This necessitated lengthy preliminary computations of moment determinants [2, Section 2]. We now obtain these polynomials more directly, and for more general weight functions, using orthogonality as the principal tool of construction.

The paper is organized as follows. In Section 2 we establish the existence of the orthogonal polynomials $\{\pi_k\}$, assuming only that

$$(1.5)\qquad \operatorname{Re}(1, 1) = \operatorname{Re}\int_0^\pi w(e^{i\theta})\,d\theta \neq 0.$$

We furthermore represent π_n as a linear (complex) combination of p_n and p_{n-1}. This then leads quickly to the basic three-term recurrence relation, as is shown in Section 3. The case of Jacobi weights $w(z) = (1-z)^\alpha(1+z)^\beta$, $\alpha > -1$, $\beta > -1$, is considered in Section 4 and is further specialized to Gegenbauer weights ($\alpha = \beta = \lambda - \frac{1}{2}$) in Section 5. Section 6 is devoted to a study of the zeros of π_n in the case of Jacobi and Gegenbauer weights. After a brief discussion of numerical methods, it is shown that, for Gegenbauer weights with $\lambda > -\frac{1}{2}$, the zeros of π_n, $n \geq 2$, are all contained in the upper unit half disc D_+. Numerical evidence seems to suggest that the same is true for arbitrary Jacobi weights. The simplicity of the zeros is shown in the case of Gegenbauer weights. The polynomial π_n then satisfies a linear second-order differential equation, which is derived in Section 7.

2. Existence and Representation of π_n

We assume that w is a weight function, positive on $(-1, 1)$, holomorphic in $D_+ = \{z \in \mathbf{C}: |z| < 1, \operatorname{Im} z > 0\}$, and such that the integrals in (1.1) and (1.2) exist for smooth f and g (possibly) as improper integrals. We shall also assume (1.5). If C_ε, $\varepsilon > 0$, denotes the boundary of D_+ with small circular parts of radius ε and centers at ± 1 spared out, we have by Cauchy's theorem, for any polynomial g,

$$(2.1)\qquad 0 = \int_{C_\varepsilon} g(z)w(z)\,dz$$

$$= \left(\int_{\Gamma_\varepsilon} + \int_{c_{\varepsilon,-1}} + \int_{c_{\varepsilon,1}}\right) g(z)w(z)\,dz + \int_{-1+\varepsilon}^{1-\varepsilon} g(x)w(x)\,dx, \qquad g \in \mathbf{P},$$

where Γ_ε and $c_{\varepsilon,\pm1}$ are the circular parts of C_ε (with radii 1 and ε, respectively). We assume that w is such that

$$\lim_{\varepsilon\downarrow 0}\int_{c_{\varepsilon,\pm1}} g(z)w(z)\,dz=0, \qquad \text{all} \quad g\in\mathbf{P}. \tag{2.2}$$

Then, letting $\varepsilon\downarrow 0$ in (2.1), we obtain

$$0=\int_C g(z)w(z)\,dz=\int_\Gamma g(z)w(z)\,dz+\int_{-1}^{1} g(x)w(x)\,dx, \qquad g\in\mathbf{P}. \tag{2.3}$$

The (monic, real) polynomials $\{p_k(z)\}$, orthogonal with respect to the inner product (1.1), as well as the associated polynomials of the second kind,

$$q_k(z)=\int_{-1}^{1}\frac{p_k(z)-p_k(x)}{z-x}\,w(x)\,dx, \qquad k=0,1,2,\ldots, \tag{2.4}$$

are known to satisfy a three-term recurrence relation of the form

$$y_{k+1}=(z-a_k)y_k-b_ky_{k-1}, \qquad k=0,1,2,\ldots, \tag{2.5}$$

where

$$\begin{aligned} y_{-1}&=0, \qquad y_0=1 \qquad \text{for } \{p_k\},\\ y_{-1}&=-1, \qquad y_0=0 \qquad \text{for } \{q_k\}. \end{aligned} \tag{2.6}$$

We denote by m_k and μ_k the moments associated with the inner products (1.1) and (1.2), respectively,

$$m_k=[x^k,1], \qquad \mu_k=(z^k,1), \qquad k\ge 0. \tag{2.7}$$

It is assumed, in (2.5), that

$$b_0=m_0. \tag{2.8}$$

Theorem 2.1. *Let w be a weight function, positive on $(-1,1)$, holomorphic in $D_+=\{z\in\mathbf{C}: |z|<1, \operatorname{Im} z>0\}$, and such that* (2.2) *is satisfied and the integrals in* (2.3) *exist* (*possibly*) *as improper integrals. Assume in addition that* (1.5) *holds. Then there exists a unique system of* (*monic, complex*) *orthogonal polynomials $\{\pi_k\}$ relative to the inner product* (1.2). *Denoting by $\{p_k\}$ the* (*monic, real*) *orthogonal polynomials relative to the inner product* (1.1), *we have*

$$\pi_n(z)=p_n(z)-i\theta_{n-1}p_{n-1}(z), \qquad n=0,1,2,\ldots, \tag{2.9}$$

where

$$\theta_{n-1}=\frac{\mu_0p_n(0)+iq_n(0)}{i\mu_0p_{n-1}(0)-q_{n-1}(0)}, \qquad n=0,1,2,\ldots. \tag{2.10}$$

Alternatively,

$$\theta_n=ia_n+\frac{b_n}{\theta_{n-1}}, \quad n=0,1,2,\ldots; \qquad \theta_{-1}=\mu_0, \tag{2.11}$$

where a_k, b_k *are the recursion coefficients in* (2.5) *and* $\mu_0 = (1, 1)$. *In particular, all* θ_n *are real (in fact, positive) if* $a_n = 0$ *for all* $n \ge 0$. *Finally,*

(2.12) $\quad (\pi_n, \pi_n) = \theta_{n-1}[p_{n-1}, p_{n-1}] \neq 0, \qquad n = 1, 2, 3, \ldots, \qquad (\pi_0, \pi_0) = \mu_0.$

Proof. Assume first that the orthogonal polynomials $\{\pi_k\}$ exist. Putting $g(z) = (1/i)\pi_n(z)z^{k-1}$, $1 \le k < n$, in (2.3), we find

$$0 = \int_\Gamma \pi_n(z) z^k (iz)^{-1} w(z)\,dz - i\int_{-1}^{1} \pi_n(x) x^{k-1} w(x)\,dx$$
$$= (\pi_n, z^k) - i[\pi_n, x^{k-1}] = -i[\pi_n, x^{k-1}], \qquad 1 \le k < n,$$

hence, upon expanding π_n in the polynomials $\{p_k\}$,

(2.13) $\qquad \pi_n(z) = p_n(z) - i\theta_{n-1}p_{n-1}(z), \qquad n = 0, 1, 2, \ldots,$

for some constants θ_{n-1}. To determine these constants, put

$$g(z) = [\pi_n(z) - \pi_n(0)](iz)^{-1} = \frac{1}{i}\left\{\frac{p_n(z) - p_n(0)}{z} - i\theta_{n-1}\frac{p_{n-1}(z) - p_{n-1}(0)}{z}\right\}$$

in (2.3) and use the first expression for g to evaluate the first integral, and the second to evaluate the second integral in (2.3). This gives

(2.14) $\qquad 0 = (\pi_n, 1) - \pi_n(0)(1, 1) + \frac{1}{i}[q_n(0) - i\theta_{n-1}q_{n-1}(0)], \qquad n \ge 1.$

Since $(\pi_n, 1) = 0$, $(1, 1) = \mu_0$, and using (2.13) with $z = 0$, we get (2.10) for $n \ge 1$. Note that the denominator in (2.10) (and the numerator, for that matter) does not vanish, since $\operatorname{Re}\mu_0 \neq 0$ by (1.5) and $p_k(0)$, $q_k(0)$ cannot vanish simultaneously, $\{p_k\}$ and $\{q_k\}$ being linearly independent solutions of (2.5). For $n = 0$, (2.10) yields, by virtue of (2.6), $\theta_{-1} = \mu_0$, which is the definition given in (2.11).

To show the first relation in (2.11), replace n by $n+1$ in (2.10), and use (2.5) for $z = 0$, to obtain

$$\theta_n = \frac{\mu_0 p_{n+1}(0) + iq_{n+1}(0)}{i\mu_0 p_n(0) - q_n(0)}$$
$$= \frac{\mu_0[-a_n p_n(0) - b_n p_{n-1}(0)] + i[-a_n q_n(0) - b_n q_{n-1}(0)]}{i\mu_0 p_n(0) - q_n(0)}$$
$$= \frac{ia_n[i\mu_0 p_n(0) - q_n(0)] - b_n[\mu_0 p_{n-1}(0) + iq_{n-1}(0)]}{i\mu_0 p_n(0) - q_n(0)}$$
$$= ia_n + \frac{b_n}{\theta_{n-1}}, \qquad n \ge 1.$$

Using (2.10) with $n = 1$, (2.5) with $k = 0$, and (2.6), yields

$$\theta_0 = \frac{\mu_0(-a_0) + ib_0}{i\mu_0} = ia_0 + \frac{m_0}{\mu_0},$$

since $b_0 = m_0$ [cf. (2.8)]. Therefore, (2.11) also holds for $n = 0$.

If all $a_n = 0$, then w is symmetric and the reality of the θ_n follows from (2.11) and (5.2) below.

Conversely, defining π_n by (2.9) and (2.10), it follows readily, for $n \geq 2$, from (2.3) that

$$(\pi_n, z^k) = \frac{1}{i}\int_\Gamma \pi_n(z)z^{k-1}w(z)\,dz = i\int_{-1}^{1} \pi_n(x)x^{k-1}w(x)\,dx = 0, \qquad 1 \leq k < n,$$

and from (2.14), (2.10), and (2.9) for $z = 0$, that $(\pi_n, 1) = 0$, $n \geq 1$. Furthermore,

$$\begin{aligned}(\pi_n, \pi_n) &= \int_\Gamma \pi_n(z)z^n w(z)(iz)^{-1}\,dz \\ &= \frac{1}{i}\int_\Gamma \pi_n(z)z^{n-1}w(z)\,dz = i\int_{-1}^{1} \pi_n(x)x^{n-1}w(x)\,dx \\ &= i\int_{-1}^{1} [p_n(x) - i\theta_{n-1}p_{n-1}(x)]x^{n-1}w(x)\,dx \\ &= \theta_{n-1}\int_{-1}^{1} p_{n-1}^2(x)w(x)\,dx,\end{aligned}$$

proving (2.12). ∎

We note from (2.9) that

$$(2.15) \qquad (p_n, 1) = i\theta_{n-1}(p_{n-1}, 1) = \cdots = \prod_{\nu=1}^{n} (i\theta_{\nu-1})(1, 1),$$

and, similarly,

$$(p_n, \pi_k) = i\theta_{n-1}(p_{n-1}, \pi_k), \qquad 1 \leq k < n,$$

which, applied repeatedly, gives

$$\begin{aligned}(2.16) \qquad (p_n, \pi_k) &= \left(\prod_{\nu=k+1}^{n} i\theta_{\nu-1}\right)(p_k, \pi_k) \\ &= i^{-1}\left(\prod_{\nu=k}^{n} i\theta_{\nu-1}\right)[p_{k-1}, p_{k-1}], \qquad 1 \leq k \leq n.\end{aligned}$$

Here, (2.12) has been used in the last step. From (2.15) and (2.16) there follows

$$(2.17) \qquad p_n(z) = \sum_{k=0}^{n}\left(\prod_{\nu=k+1}^{n} i\theta_{\nu-1}\right)\pi_k(z),$$

the inversion of (2.9). (When $k = n$, the empty product in (2.17) is to be interpreted as 1.)

Example 2.1. $w(z) = 1 + z$.

Here, $\mu_0 = (1, 1) = \pi + 2i$, Re $\mu_0 \neq 0$, so that the orthogonal polynomials $\{\pi_k\}$ exist. Furthermore, $b_0 = m_0 = 2$, $a_n = (2n+1)^{-1}(2n+3)^{-1}$ for $n \geq 0$, $b_n = n(n+1)(2n+1)^{-2}$ for $n \geq 1$, so that by (2.11),

$$\theta_0 = \frac{\pi - 4i}{3(2 - i\pi)}, \quad \theta_1 = \frac{3\pi + 8i}{5(4 + i\pi)}, \quad \ldots,$$

by (2.5) and (2.6),

$$p_0(z)=1, \quad p_1(z)=z-\tfrac{1}{3}, \quad p_2(z)=z^2-\tfrac{2}{5}z-\tfrac{1}{5}, \quad \ldots,$$

and by (2.9),

$$\pi_0(z)=1, \quad \pi_1(z)=z-\frac{2}{2-i\pi}, \quad \pi_2(z)=z^2-\frac{i\pi}{4+i\pi}z-\frac{4}{3(4+i\pi)}, \quad \ldots .$$

Example 2.2. $w(z)=z^2$.

Here, $\mu_0=\int_0^\pi e^{2i\theta}\,d\theta=0$, so that (1.5) is violated and thus the polynomials $\{\pi_k\}$ do not exist, even though $w(x)\geq 0$ on $[-1,1]$ and the polynomials $\{p_k\}$ do exist. It is easily seen, in fact, that $q_k(0)=0$ when k is even, so that θ_{n-1} in (2.10) is zero for n even, and undefined for n odd. For an explanation of Example 2.2, see Theorem 5.1.

3. Recurrence Relation

We assume (1.5), so that the orthogonal polynomials $\{\pi_k\}$ exist. Since $(zf,g)=(f,zg)$, it is known that they must satisfy a three-term recurrence relation. In analogy to Section 3 of [2] we write it in the form

$$\begin{gathered}\pi_{k+1}(z)=(z-i\alpha_k)\pi_k(z)-\beta_k\pi_{k-1}(z), \qquad k=0,1,2,\ldots, \\ \pi_{-1}(z)=0, \qquad \pi_0(z)=1.\end{gathered} \tag{3.1}$$

Using (2.9) in (3.1), we get, for $k\geq 1$,

$$p_{k+1}(z)-i\theta_k p_k(z)=(z-i\alpha_k)[p_k(z)-i\theta_{k-1}p_{k-1}(z)]-\beta_k[p_{k-1}(z)-i\theta_{k-2}p_{k-2}(z)],$$

and substituting here for $zp_k(z)$ and $zp_{k-1}(z)$ the expressions obtained from the basic recurrence relation (2.5) yields

$$\begin{aligned}[a_k+i(\theta_k-\theta_{k-1}-\alpha_k)]p_k(z)+[b_k-\beta_k-\theta_{k-1}(\alpha_k+ia_{k-1})]p_{k-1}(z)& \\ +i[\beta_k\theta_{k-2}-b_{k-1}\theta_{k-1}]p_{k-2}(z)\equiv 0, &\qquad k\geq 1.\end{aligned}$$

By the linear independence of the polynomials $\{p_r\}$ we conclude that

$$\begin{aligned} a_k+i(\theta_k-\theta_{k-1}-\alpha_k)&=0, \qquad k\geq 1,\\ b_k-\beta_k-\theta_{k-1}(\alpha_k+ia_{k-1})&=0, \qquad k\geq 1,\\ \beta_k\theta_{k-2}-b_{k-1}\theta_{k-1}&=0, \qquad k\geq 2.\end{aligned} \tag{3.2}$$

From the last relation in (3.2), and (2.11), we get

$$\beta_k=\frac{\theta_{k-1}}{\theta_{k-2}}b_{k-1}=\theta_{k-1}(\theta_{k-1}-ia_{k-1}) \tag{3.3}$$

for $k\geq 2$. The first relation (3.2) gives

$$\alpha_k=\theta_k-\theta_{k-1}-ia_k, \qquad k\geq 1. \tag{3.4}$$

To verify that (3.3) also holds for $k=1$, it suffices to apply the second relation (3.2) for $k=1$, in combination with (2.11) and (3.4) for $k=1$. With α_k, β_k thus determined, the second relations in (3.2) are automatically satisfied, as follows easily from (2.11). Finally, from $\pi_1(z)=z-i\alpha_0=p_1(z)-i\theta_0=z-a_0-i\theta_0$, one finds

$$\alpha_0=\theta_0-ia_0. \tag{3.4$_0$}$$

Alternatively, by (2.11), we may write (3.4) and (3.4$_0$) as

$$\begin{aligned} \alpha_k &= -\theta_{k-1}+\frac{b_k}{\theta_{k-1}}, \qquad k\geq 1, \\ \alpha_0 &= \frac{b_0}{\theta_{-1}}=\frac{m_0}{\mu_0}. \end{aligned} \tag{3.5}$$

We have proved:

Theorem 3.1. *Under the assumption* (1.5), *the (monic, complex) polynomials* $\{\pi_k\}$ *orthogonal with respect to the inner product* (1.2) *satisfy the recurrence relation*

$$\begin{gathered} \pi_{k+1}(z)=(z-i\alpha_k)\pi_k(z)-\beta_k\pi_{k-1}(z), \qquad k=0,1,2,\ldots, \\ \pi_{-1}(z)=0, \qquad \pi_0(z)=1, \end{gathered} \tag{3.6}$$

where the coefficients α_k, β_k *are given by* (3.4) [*or* (3.5)] *and* (3.3), *respectively, with the* θ_n *defined in* (2.10) [*or* (2.11)]. (The coefficient β_0 in (3.6) is arbitrary; the definition $\beta_0=\mu_0$, however, is sometimes convenient.)

By comparing the coefficient of z^k on the left and right of (3.6), one obtains from (3.4$_0$), (3.4) that

$$\pi_n(z)=z^n-\left(i\theta_{n-1}+\sum_{m=0}^{n-1}a_m\right)z^{n-1}+\cdots, \qquad n\geq 1. \tag{3.7}$$

4. Jacobi Weight

We consider now the case of the Jacobi weight function

$$w(z)=w^{(\alpha,\beta)}(z)=(1-z)^\alpha(1+z)^\beta, \qquad \alpha>-1, \quad \beta>1, \tag{4.1}$$

where fractional powers are understood in terms of their principal branches. We first establish the existence of the corresponding orthogonal polynomials $\{\pi_k\}$.

Theorem 4.1. *We have*

$$\mu_0=\mu_0^{(\alpha,\beta)}=\int_0^\pi w^{(\alpha,\beta)}(e^{i\theta})\,d\theta=\pi+i\,⨍_{-1}^{1}\frac{w^{(\alpha,\beta)}(x)}{x}\,dx, \tag{4.2}$$

where the integral on the right is a Cauchy principal value integral, hence Re $\mu_0\neq 0$.

Proof. Let C_ε, $\varepsilon > 0$, be the contour formed by ∂D_+, with a small semicircle of radius ε about the origin spared out. Then, by Cauchy's theorem,

$$0 = \int_\Gamma \frac{w(z)}{iz}\,dz + \left(\int_{-1}^{-\varepsilon} + \int_\varepsilon^1\right)\frac{w(x)}{ix}\,dx + \int_{c_\varepsilon} \frac{w(z)}{iz}\,dz, \tag{4.3}$$

where Γ and c_ε are the circular parts of C_ε (with radii 1 and ε, respectively). Letting $\varepsilon \downarrow 0$ in (4.3) yields

$$0 = \mu_0 - i\,⨍_{-1}^{1} \frac{w(x)}{x}\,dx - \pi w(0),$$

which proves (4.2). ■

By Theorem 2.1 we therefore have

$$\pi_n(z) = \pi_n^{(\alpha,\beta)}(z) = \hat{P}_n^{(\alpha,\beta)}(z) - i\theta_{n-1}^{(\alpha,\beta)}\hat{P}_{n-1}^{(\alpha,\beta)}(z), \tag{4.4}$$

where $\hat{P}_k^{(\alpha,\beta)}$ are the monic Jacobi polynomials and $\theta_{n-1}^{(\alpha,\beta)}$ is given by (2.10) with the identifications

$$p_k(z) = \hat{P}_k^{(\alpha,\beta)}(z), \qquad q_k(z) \text{ as in } (2.4).$$

Theorem 4.2. *We have*

$$\pi_n^{(\beta,\alpha)}(z) = (-1)^n \bar{\pi}_n^{(\alpha,\beta)}(-z), \tag{4.5}$$

where $\bar{\pi}_n$ denotes the polynomial π_n with all coefficients conjugated, $\bar{\pi}_n(z) = \overline{\pi_n(\bar{z})}$.

Proof. As is well known, $\hat{P}_k^{(\beta,\alpha)}(z) = (-1)^k \hat{P}_k^{(\alpha,\beta)}(-z)$. Since $w^{(\beta,\alpha)}(z) = w^{(\alpha,\beta)}(-z)$, there follows from (4.2) that

$$\mu_0^{(\beta,\alpha)} = \overline{\mu_0^{(\alpha,\beta)}}$$

and from (2.4) that

$$q_k^{(\beta,\alpha)}(z) = (-1)^{k+1} q_k^{(\alpha,\beta)}(-z).$$

Consequently, by (2.10),

$$\theta_{n-1}^{(\beta,\alpha)} = \overline{\theta_{n-1}^{(\alpha,\beta)}},$$

so that, finally,

$$\pi_n^{(\beta,\alpha)}(z) = (-1)^n [\hat{P}_n^{(\alpha,\beta)}(-z) + \overline{i\theta_{n-1}^{(\alpha,\beta)}}\hat{P}_{n-1}^{(\alpha,\beta)}(-z)].$$

This, in view of (4.4), is equivalent to (4.5). ■

The quantity $\mu_0^{(\alpha,\beta)}$ is needed to generate the $\theta_{n-1}^{(\alpha,\beta)}$ by (2.11), and the recursion coefficients α_k, β_k by (3.5) and (3.3). It is of some interest, therefore, to discuss its numerical evaluation. In principle, μ_0 can be computed from the second expression in (4.2), since the principal value integral appearing there can be expressed in terms of the Jacobi function $Q_0^{(\alpha,\beta)}(\xi)$ on the cut, evaluated at $\xi = 0$. This evaluation, however, is not easy, particularly near integer values of α, where it is plagued by cancellation phenomena (see [3]).

It appears much more convenient to use the first expression in (4.2), rewritten in the form

$$\mu_0^{(\alpha,\beta)}=2^{\alpha+\beta}e^{-i\alpha\pi/2}\int_0^\pi e^{i(\alpha+\beta)\theta/2}\sin^\alpha\frac{\theta}{2}\cos^\beta\frac{\theta}{2}\,d\theta, \tag{4.6}$$

and to note that the integrand has a singularity of type θ^α at $\theta=0$ and of type $(\pi-\theta)^\beta$ at $\theta=\pi$. This suggests the use of Gauss-Jacobi quadrature with parameters (β,α) (note the reversal of parameters!). Changing variables, $\theta=(t+1)\pi/2$, indeed yields

$$\begin{aligned}\mu_0^{(\alpha,\beta)}&=2^{\alpha+\beta-1}\pi e^{i(\beta-\alpha)\pi/4}\\&\quad\times\int_{-1}^1 e^{i(\alpha+\beta)t\pi}\left[\frac{\sin((t+1)\pi/4)}{t+1}\right]^\alpha\left[\frac{\cos((t+1)\pi/4)}{1-t}\right]^\beta w^{(\beta,\alpha)}(t)\,dt,\end{aligned} \tag{4.7}$$

where the integrand (except for the weight function $w^{(\beta,\alpha)}$) is now regular on $[-1,1]$.

5. Symmetric Weights and Gegenbauer Weight

We begin by establishing (1.5) for arbitrary symmetric weights not vanishing at the origin.

Theorem 5.1. *If the weight function w, in addition to the assumptions stated at the beginning of Section 2, satisfies*

$$w(-z)=w(z)\quad\textit{and}\quad w(0)>0, \tag{5.1}$$

then

$$\mu_0=(1,1)=\pi w(0), \tag{5.2}$$

and the system of orthogonal polynomials $\{\pi_k\}$ exists uniquely.

Proof. Proceeding as in the proof of Theorem 4.1, we find

$$0=\mu_0-i\,⨍_{-1}^1\frac{w(x)}{x}\,dx-\pi w(0).$$

Here, the Cauchy principal value integral on the right vanishes because of symmetry of w. This proves (5.2). ∎

Under the assumption (5.1) we have $a_k=0$, all $k\ge0$, in (2.5), hence, by (3.4_0), (3.4), and (3.3),

$$\alpha_0=\theta_0,\qquad \alpha_k=\theta_k-\theta_{k-1},\qquad \beta_k=\theta_{k-1}^2,\qquad k\ge1. \tag{5.3}$$

Furthermore, by (2.11),

$$\theta_0 = \frac{m_0}{\mu_0}, \qquad \theta_1 = \frac{b_1}{\theta_0},$$

$$(5.4) \qquad \left.\begin{aligned} \theta_{2m} &= \theta_0 \frac{b_2 b_4 \cdots b_{2m}}{b_1 b_3 \cdots b_{2m-1}} \\ \theta_{2m+1} &= \frac{1}{\theta_0} \frac{b_1 b_3 \cdots b_{2m+1}}{b_2 b_4 \cdots b_{2m}} \end{aligned}\right\} \qquad m = 1, 2, 3, \ldots .$$

In particular, for the Gegenbauer weight

$$(5.5) \qquad w(z) = (1 - z^2)^{\lambda - 1/2}, \qquad \lambda > -\tfrac{1}{2},$$

we have $p_k(z) = \hat{C}_k^\lambda(z)$—the monic Gegenbauer polynomials—for which, as is well known,

$$(5.6) \qquad \begin{aligned} b_0 &= m_0 = \sqrt{\pi}\, \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)}, \\ b_k &= \frac{k(k + 2\lambda - 1)}{4(k + \lambda)(k + \lambda - 1)}, \qquad k \geq 1. \end{aligned}$$

Therefore, by (2.11),

$$\theta_n = \frac{n(n + 2\lambda - 1)}{4(n + \lambda)(n + \lambda - 1)} \frac{1}{\theta_{n-1}}, \quad n = 1, 2, \ldots, \qquad \theta_0 = \frac{\Gamma(\lambda + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\lambda + 1)},$$

and thus, by induction,

$$(5.7) \qquad \theta_0 = \frac{\Gamma(\lambda + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\lambda + 1)}, \qquad \theta_k = \frac{1}{\lambda + k} \frac{\Gamma((k+2)/2)\Gamma(\lambda + (k+1)/2)}{\Gamma((k+1)/2)\Gamma(\lambda + (k/2))}, \qquad k = 1, 2, 3, \ldots .$$

For $\lambda = \frac{1}{2}$ (i.e., $w(z) \equiv 1$), this reduces to equation (3.3) of [2].

6. The Zeros of $\pi_n(z)$

6.1. Computation of the Zeros

It follows from (3.1) that the zeros of $\pi_n(z)$ are the eigenvalues of the (complex, tridiagonal) matrix

$$(6.1) \qquad J_n = \begin{bmatrix} i\alpha_0 & 1 & & & 0 \\ \beta_1 & i\alpha_1 & 1 & & \\ & \beta_2 & i\alpha_2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \beta_{n-1} & i\alpha_{n-1} \end{bmatrix}.$$

The elements of J_n are easily computed from (3.4_0), (3.4), (3.3), and (2.11), once the recursion coefficients a_k, b_k for the orthogonal polynomials $\{p_k\}$ are known.

The value of μ_0 required in this computation is best obtained from the definition (2.7) by numerical integration, as is described, e.g., in the case of the Jacobi weight at the end of Section 4. To compute the eigenvalues of (6.1) in the general case, we recommend the EISPACK routine COMQR (see, e.g., p. 277 of [4]).

If the weight w is symmetric, then $\beta_k = \theta_{k-1}^2$ is positive [cf. (5.4)], and (6.1) can be transformed into a real matrix. It follows indeed by a similarity transformation with the diagonal matrix $D_n = \mathrm{diag}(1, i\theta_0, i^2\theta_0\theta_1, i^3\theta_0\theta_1\theta_2, \ldots) \in \mathbf{C}^{n\times n}$ that the eigenvalues of (6.1) are equal to $\zeta_\nu = i\eta_\nu$, where η_ν are the eigenvalues of the real nonsymmetric tridiagonal matrix

$$(6.2) \quad -iD_n^{-1}J_nD_n = \begin{bmatrix} \alpha_0 & \theta_0 & & & 0 \\ -\theta_0 & \alpha_1 & \theta_1 & & \\ & -\theta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \theta_{n-2} \\ 0 & & & -\theta_{n-2} & \alpha_{n-1} \end{bmatrix} \quad (w \text{ symmetric}).$$

These can be computed by the EISPACK routine HQR (p. 330 of [4]).

6.2. *Jacobi Weight*

For the general Jacobi weight (4.1), with parameters $\alpha > -1$, $\beta > -1$, we have only numerical results; they were obtained by the procedure described in the preceding subsection. All indications are, however, that the zeros of $\pi_n^{(\alpha,\beta)}$ for $n \geq 2$ are always contained in the upper unit half disc $D_+ = \{z \in \mathbf{C}: |z| < 1, \mathrm{Im}\, z > 0\}$. This was verified numerically for:

(i) $\alpha = -0.75(0.25)1.0$, $\beta = -0.75(0.25)\alpha$ [when $\beta > \alpha$, the analogous fact follows from (4.5)] and $n = 2(1)13, 16, 17, 24, 25$;
(ii) $\alpha = -0.75(0.25)1.0$, $\beta = -0.9, -0.99, -0.999, -0.9999$ and n as in (i);
(iii) miscellaneous values $\alpha > 1$, $\beta < \alpha$ and n as in (i).

A proof of this remarkable property will be given in the next subsection in the special case $\alpha = \beta > -1$, in which case we also show that all zeros are simple.

6.3. *Symmetric Weights and Gegenbauer Weight*

We first assume that w is any symmetric weight function (subject to the conditions of Theorem 2.1),

$$(6.3) \qquad w(-z) = w(z), \qquad w(0) > 0.$$

Then all $\theta_{n-1} > 0$ [cf. (5.4)].

Exactly as in Section 5 of [2] one proves:

Theorem 6.1. *If $\zeta \in \mathbf{C}$ is a zero of π_n, then so is $-\bar{\zeta}$. The zeros of π_n are thus located symmetrically with respect to the imaginary axis. Moreover, all zeros have positive imaginary parts and, in fact, are contained in the half strip $S_+ = \{z \in \mathbf{C}: \mathrm{Im}\, z > 0, -\xi_n \leq \mathrm{Re}\, z \leq \xi_n\}$, where ξ_n is the largest zero of p_n.*

More detailed information about the zeros is provided by the following theorem.

Theorem 6.2. *All zeros of π_n are contained in $D_+ = \{z \in \mathbb{C}: |z| < 1, \text{ Im } z > 0\}$ (in fact, in $D_+ \cap S_+$; cf. Theorem 6.1), with the possible exception of a single (simple) zero on the positive imaginary axis.*

Proof. Consider first a zero ζ of π_n not on the imaginary axis (hence $n \geq 2$). By Theorem 6.1 it suffices to prove $|\zeta| < 1$. Since, again by Theorem 6.1, there is another zero, $-\bar{\zeta}$, we have

$$\pi_n(x) = p_n(x) - i\theta_{n-1}p_{n-1}(x) = (x - \zeta)(x + \bar{\zeta})r_{n-2}(x), \tag{6.4}$$

where $r_{n-2} \not\equiv 0$ is a polynomial of degree $n-2$. Therefore,

$$0 = \int_{-1}^{1} \pi_n(x)\overline{r_{n-2}(x)}w(x)\,dx = \int_{-1}^{1} (x - \zeta)(x + \bar{\zeta})|r_{n-2}(x)|^2 w(x)\,dx, \tag{6.5}$$

where the first relation in (6.5) follows from the first relation (6.4) and the orthogonality of the p_k. Since

$$(x - \zeta)(x + \bar{\zeta}) = x^2 - 2ix \text{ Im } \zeta - |\zeta|^2,$$

there follows, by taking the real part of (6.5),

$$\int_{-1}^{1} (x^2 - |\zeta|^2)|r_{n-2}(x)|^2 w(x)\,dx = 0,$$

which implies $|\zeta| < 1$.

By the same argument one shows that π_n cannot have two distinct zeros, or a double zero, on the imaginary axis, all with imaginary parts ≥ 1. ∎

We specialize now to Gegenbauer weights,

$$w(z) = (1 - z^2)^{\lambda - 1/2}, \qquad \lambda > -\tfrac{1}{2}, \tag{6.6}$$

and denote the corresponding polynomials π_n by

$$\pi_n^\lambda(z) = \hat{C}_n^\lambda(z) - i\theta_{n-1}^\lambda \hat{C}_{n-1}^\lambda(z), \qquad \theta_{n-1}^\lambda \text{ given by (5.7)}. \tag{6.7}$$

Here, $\hat{C}_k^\lambda$ denote the monic Gegenbauer polynomials. We shall abbreviate, when convenient, $\theta_{n-1} = \theta_{n-1}^\lambda$.

For a more detailed study of the zeros of π_n^λ, we need the following lemmas.

Lemma 6.3. *For $x > 0$, $0 < s < 1$, one has*

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}. \tag{6.8}$$

Proof. See [1].

Lemma 6.4. *The quantities $\theta_{n-1} = \theta_{n-1}^\lambda$ [cf. (5.7)] satisfy*

$$\begin{aligned} &\theta_{n-1}^\lambda < \tfrac{1}{2} \quad \text{if } -\tfrac{1}{2} < \lambda < 0 \text{ or } \lambda > 1, \\ &\tfrac{1}{2} < \theta_{n-1}^\lambda \leq \max(\theta_1^\lambda, \theta_2^\lambda) \quad \text{if } 0 < \lambda < 1, \qquad n = 2, 3, \ldots . \\ &\theta_{n-1}^\lambda = \tfrac{1}{2} \quad \text{if } \lambda = 0 \text{ or } \lambda = 1. \end{aligned} \tag{6.9}$$

Proof. An elementary computation, based on (5.7), shows that

$$\theta_{k+2}^{\lambda} \gtreqless \theta_k^{\lambda} \quad \text{if and only if} \quad \lambda(\lambda-1) \gtreqless 0. \tag{6.10}$$

Furthermore, by Stirling's formula applied to (5.7),

$$\lim_{k\to\infty} \theta_k^{\lambda} = \tfrac{1}{2}. \tag{6.11}$$

Therefore, if $-\frac{1}{2}<\lambda<0$ or $\lambda>1$, we have $\theta_{k+2}^{\lambda}>\theta_k^{\lambda}$, $k=1,2,3,\ldots$, hence, by (6.11), $\theta_{n-1}^{\lambda}<\frac{1}{2}$, all $n\geq 2$. This proves the first inequality in (6.9). Suppose, next, that $0<\lambda<1$. Then, $\theta_{k+2}^{\lambda}<\theta_k^{\lambda}$, $k=1,2,3,\ldots$, hence, again by (6.11), $\frac{1}{2}<\theta_{n-1}^{\lambda}\leq \max(\theta_1^{\lambda},\theta_2^{\lambda})$ for all $n\geq 4$ (and trivially for $n=2$ and 3), proving the second relation in (6.9). Finally, if $\lambda=0$ or $\lambda=1$, then $\theta_{n-1}^{\lambda}=\frac{1}{2}$, all $n\geq 2$. ∎

Theorem 6.5. *If $\lambda>-\frac{1}{2}$, then all zeros of $\pi_n^{\lambda}(z)$, $n\geq 2$, are contained in $D_+=\{z\in\mathbf{C}: |z|<1, \operatorname{Im} z>0\}$.*

Proof. By virtue of Theorem 6.2, it suffices to show that π_n^{λ}, $n\geq 2$, can have no purely imaginary zero with imaginary part ≥ 1.

Thus, consider a zero $\zeta=iy$ of π_n^{λ}. By (2.9) [or (6.7)],

$$p_n(iy)-i\theta_{n-1}p_{n-1}(iy)=0,$$

where $p_n=\hat{C}_n^{\lambda}$ and $\theta_{n-1}=\theta_{n-1}^{\lambda}$. Therefore,

$$\frac{p_n(iy)}{ip_{n-1}(iy)}=\theta_{n-1}. \tag{6.12}$$

Letting

$$\omega_k(y)=\frac{p_k(iy)}{ip_{k-1}(iy)}, \qquad k=1,2,3,\ldots, \tag{6.13}$$

one finds from the recurrence formula for the p_k [cf. (2.5) and (2.6)] that

$$\omega_1(y)=y, \qquad \omega_k(y)=y+\frac{b_{k-1}}{\omega_{k-1}(y)}, \quad k=2,3,\ldots,$$

hence, since $b_{k-1}>0$,

$$\omega_k(y)\geq 1 \qquad \text{for} \quad y\geq 1.$$

Therefore, the left-hand side of (6.12) is ≥ 1 for $y\geq 1$ and $n\geq 1$. We now show that

$$\theta_{n-1}<1, \qquad n\geq 2, \tag{6.14}$$

so that (6.12) cannot hold for $y\geq 1$, when $n\geq 2$, and thus π_n^{λ}, $n\geq 2$, cannot have a zero iy with $y\geq 1$.

By Lemma 6.4, the inequality (6.14) is certainly true if $-\frac{1}{2}<\lambda\le 0$ or $\lambda\ge 1$, and if $0<\lambda<1$ will follow from $\theta_1<1$, $\theta_2<1$. Now using (5.7) and the upper bound in Lemma 6.3 (with $x=\lambda$, $s=\frac{1}{2}$), one gets

$$\theta_1=\frac{\sqrt{\pi}\Gamma(\lambda+1)}{2(\lambda+1)\Gamma(\lambda+\frac{1}{2})}<\frac{\sqrt{\pi}}{2(\lambda+1)}(\lambda+1)^{1/2}=\frac{\sqrt{\pi}}{2(\lambda+1)^{1/2}}<\frac{\sqrt{\pi}}{2}<1,$$

if $\lambda>0$. Likewise,

$$\theta_2=\frac{2}{\sqrt{\pi}(\lambda+2)}\frac{\Gamma(\lambda+\frac{3}{2})}{\Gamma(\lambda+1)}<\frac{2}{\sqrt{\pi}(\lambda+2)}(\lambda+\tfrac{3}{2})^{1/2}<\sqrt{\frac{5}{2\pi}}<1,\qquad 0<\lambda<1. \quad \blacksquare$$

Theorem 6.5 does not hold for $n=1$, $-\frac{1}{2}<\lambda\le 0$, since the zero $i\alpha_0=i\theta_0$ then has a modulus that increases from 1 to ∞ when λ decreases from 0 to $-\frac{1}{2}$. It does hold, however, for $n=1$, $\lambda>0$, as can be shown.

An alternative proof of Theorem 6.5, valid however only for $\lambda>0$, can be given on the basis of Rouché's theorem, as in [2].

To prove the simplicity of the zeros of π_n^λ, we need

Lemma 6.6. *The quantity $\theta_{n-1}=\theta_{n-1}^\lambda$ satisfies the inequality*

$$4(n+\lambda-1)^2\theta_{n-1}^2<n(n+2\lambda-1),\qquad n\ge 2,\quad \lambda>-\tfrac{1}{2}. \tag{6.15}$$

Proof. The right-hand side of (6.15) is positive, since $n+2\lambda-1>n-2\ge 0$. From (5.7) we have

$$4(n+\lambda-1)^2\theta_{n-1}^2=\left(2\frac{\Gamma((n+1)/2)\Gamma(\lambda+(n/2))}{\Gamma(n/2)\Gamma(\lambda+(n-1)/2)}\right)^2$$
$$=\left[2\cdot\frac{n}{2}\cdot\left(\lambda+\frac{n-1}{2}\right)\right]^2\left(\frac{\Gamma((n+1)/2)\Gamma(\lambda+(n/2))}{\Gamma((n/2)+1)\Gamma(\lambda+(n+1)/2)}\right)^2,$$

so that (6.15) becomes

$$\left(\frac{\Gamma((n/2)+1)\Gamma(\lambda+(n+1)/2)}{\Gamma((n+1)/2)\Gamma(\lambda+(n/2))}\right)^2>\tfrac{1}{4}n(n+2\lambda-1). \tag{6.16}$$

This follows immediately from the lower bound in Lemma 6.3, first applied with $x=\frac{1}{2}n$, $s=\frac{1}{2}$, and then with $x=\lambda+\frac{1}{2}n-\frac{1}{2}$, $s=\frac{1}{2}$. $\blacksquare$

Theorem 6.7. *If $\lambda>-\frac{1}{2}$, all zeros of $\pi_n^\lambda(z)$ are simple.*

Proof. The proof is analogous to the one given in Section 5 of [2] for the case $\lambda=\frac{1}{2}$. It suffices, of course, to assume $n\ge 2$.

Let ζ be a zero of $\pi_n=\pi_n^\lambda$. By (2.9),

$$p_n(\zeta)=i\theta_{n-1}p_{n-1}(\zeta).$$

Therefore, using again (2.9),

$$\begin{aligned}\pi_n'(\zeta) &= p_n'(\zeta) - i\theta_{n-1}p_{n-1}'(\zeta) \\ &= \frac{1}{p_{n-1}(\zeta)}[p_n'(\zeta)p_{n-1}(\zeta) - p_n(\zeta)p_{n-1}'(\zeta)].\end{aligned} \tag{6.17}$$

Using

$$(1-\zeta^2)p_k'(\zeta) = (k+2\lambda)\zeta p_k(\zeta) - 2(k+\lambda)p_{k+1}(\zeta)$$

to remove the derivatives on the right of (6.17), and employing the recurrence relation

$$p_{k+1}(\zeta) = \zeta p_k(\zeta) - \frac{k(k+2\lambda-1)}{4(k+\lambda-1)(k+\lambda)}p_{k-1}(\zeta),$$

yields, after some computation,

$$\begin{aligned}\pi_n'(\zeta) = \frac{p_{n-1}(\zeta)}{2(1-\zeta^2)(n+\lambda-1)}&[n(n+2\lambda-1) - 4(n+\lambda-1)^2\theta_{n-1}^2 \\ &-2(2n+2\lambda-1)(n+\lambda-1)\zeta i\theta_{n-1}].\end{aligned}$$

If $\zeta = \alpha + i\beta$, the expression in brackets becomes

$$\begin{aligned}n(n+2\lambda-1) - 4(n+\lambda-1)^2\theta_{n-1}^2 + 2(2n+2\lambda-1)(n+\lambda-1)\beta\theta_{n-1} \\ -2(2n+2\lambda-1)(n+\lambda-1)\alpha i\theta_{n-1}\end{aligned}$$

and is clearly nonzero by virtue of Lemma 6.6 and $\beta > 0$. ■

7. Differential Equation

The following theorem generalizes Theorem 6.1 of [2].

Theorem 7.1. *The polynomial $\pi_n^\lambda(z)$ in (6.7) satisfies the differential equation*

$$P(z)y'' + Q(z)y' + R(z)y = 0, \tag{7.1}$$

where

$$\begin{aligned}P(z) &= (1-z^2)[n(n+2\lambda-1) - 4(n+\lambda-1)^2\theta_{n-1}^2 \\ &\qquad -2(2n+2\lambda-1)(n+\lambda-1)zi\theta_{n-1}], \\ Q(z) &= -(2\lambda+1)[n(n+2\lambda-1) - 4(n+\lambda-1)^2\theta_{n-1}^2]z \\ &\qquad +2(2n+2\lambda-1)(n+\lambda-1)(1+2\lambda z^2)i\theta_{n-1}, \\ R(z) &= n^2(n+2\lambda)(n+2\lambda-1) - 4(n-1)(n+2\lambda-1)(n+\lambda-1)^2\theta_{n-1}^2 \\ &\qquad -2n(n+2\lambda-1)(2n+2\lambda-1)(n+\lambda-1)zi\theta_{n-1}.\end{aligned} \tag{7.2}$$

Proof. We only sketch the proof, since it is analogous to the one given in Section 6 of [2]. We put $u = \hat{C}^{\lambda}_{n-1}(z)$, $v = 2(n+\lambda-1)\pi^{\lambda}_{n}(z)$ and define

$$\omega(z) = (z-1)^{(1/2)(n+2\lambda-1)-i(n+\lambda-1)\theta_{n-1}}(z+1)^{(1/2)(n+2\lambda-1)+i(n+\lambda-1)\theta_{n-1}}.$$

Then,

$$(z^2-1)[\omega(z)u]' = \omega(z)v. \tag{7.3}$$

Substituting u, u', u'' from (7.3) into Gegenbauer's differential equation

$$(z^2-1)u'' + (2\lambda+1)zu' - (n-1)(n+2\lambda-1)u = 0$$

gives

$$\frac{1}{b(z)}v' + \frac{a(z)}{b(z)}v + \int \frac{\omega}{z^2-1}v\,dz = 0, \tag{7.4}$$

where

$$a(z) = (z^2-1)^{-1}[-nz + 2(n+\lambda-1)i\theta_{n-1}],$$
$$b(z) = [\omega(z)(z^2-1)]^{-1}[n(n+2\lambda-1) - 4(n+\lambda-1)^2\theta^2_{n-1}$$
$$-2(2n+2\lambda-1)(n+\lambda-1)zi\theta_{n-1}].$$

Now differentiating (7.4) and multiplying the result by $-\omega(z)(z^2-1)^2b^2(z)$ yields (7.1) and (7.2) after some computation. ■

We remark that (7.1) has regular singular points at $1, -1, \infty$, and an additional regular singular point which depends on n and, by Lemma 6.6, is located on the negative imaginary axis.

Acknowledgment. The work of the first author was supported in part by the National Science Foundation under Grant DCR-8320561.

References

1. W. Gautschi (1959): *Some elementary inequalities relating to the gamma and incomplete gamma function.* J. Math. and Phys., **38**:77–81.
2. W. Gautschi, G. V. Milovanović (1986): *Polynomials orthogonal on the semicircle.* J. Approx. Theory, **46**:230–250.
3. W. Gautschi, Jet Wimp (to appear): *Computing the Hilbert transform of a Jacobi weight function.* BIT.
4. B. T. Smith, J. M. Boyle, J. J. Dongarra, B. S. Garbow, Y. Ikebe, V. C. Klema, C. B. Moler (1976): Matrix Eigensystem Routines—EISPACK Guide, 2nd edn. Lecture Notes in Computer Science, vol. 6. New York: Springer-Verlag.

W. Gautschi
Department of Computer Sciences
Purdue University
West Lafayette
Indiana 47907
U.S.A.

H. J. Landau
AT&T Bell Laboratories
600 Mountain Avenue
Murray Hill
New Jersey 07974
U.S.A.

G. V. Milovanović
Faculty of Electronic Engineering
Department of Mathematics
University of Niš
18000 Niš
Yugoslavia