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ON AN APPLICATION OF HERMITE'S INTERPOLATION POLYNOMIAL AND SOME RELATED RESULTS

GRADIMIR V. MILOVANOVIĆ AND JOSIP E. PEČARIĆ

ABSTRACT:
In this paper we gave generalizations and improvements of integral inequalities from [1] *and* [2]. *In the proof we used the well-known result for the error of Hermite's interpolation polynomial. Some similar results are also given.*

O JEDNOJ PRIMENI HERMITEOVOG INTERPOLACIONOG POLINOMA I NEKIM SRODNIM REZULTATIMA. *U radu su date generalizacije i poboljšanja integralnih nejednakosti iz* [1] *i* [2]. *U dokazu je korišćen poznati rezultat za grešku Hermiteovog interpolacionog polinoma. Neki slični rezultati su takodje dati.*

1. INTRODUCTION

In the journal Amer. Math. Monthly the following two problems ([1],[2]) are posed:

1° Suppose $f(x)$ has a continuous $(2m)$-th derivative on $a \leq x \leq b$, that $|f^{(2m)}(x)| \leq M$, and that $f^{(r)}(a) = f^{(r)}(b) = 0$ for $r=0,1,\ldots,m-1$. Show that

$$(1) \qquad \left|\int_a^b f(x)\,dx\right| \leq \frac{(m!)^2 M}{(2m)!(2m+1)!}(b-a)^{2m+1}.$$

2° Let $f:[a,b] \to R$ be a continuous function which is twice differentiable in (a,b) and satisfies $f(a) = f(b) = 0$. Prove that

$$(2) \qquad \int_a^b |f(x)|\,dx \leq \frac{1}{12} M(b-a)^3,$$

where $M = \sup|f''(x)|$ for $x \in (a,b)$.

The solution of first problem is given in [3].

The inequalities (1) and (2) are related to IYENGAR's inequality [4, pp. 297-298][1].

In this paper we shall prove some inequalities which generalize (1) and (2) in many senses.

Let us define the two-parameter clas of polynomials $P_n^{(m,k)}$ $(0 \leq m \leq k < n;\ m,k,n \in N)$ by means of

$$P_n^{(m,k)}(x) \equiv P_n^{(m,k)}(x;a,b) =$$

$$= C_n^{(m,k)}(a,b)\ (x-a)^m \int_b^x (x-a)^{k-m}(x-b)^{n-k-1}dx$$

where a and b are real parametars and

$$C_n^{(m,k)}(a,b) = \frac{(-1)^{n-k}(n-m)!}{m!(k-m)!(n-k-1)!}(b-a)^{m-n}.$$

For this polynomials the following relations hold:

$$\frac{d^i}{dx^i}P_n^{(m,k)}(x)\Big|_{x=a} = \delta_{im} \quad (i=0,1,\ldots,k;\ \delta_{im} \text{ is the CRONECKER symbol}),$$

$$\frac{d^i}{dx^i}P_n^{(m,k)}(x)\Big|_{x=b} = 0 \qquad (i=0,1,\ldots,n-k-1),$$

$$P_n^{(m,k)}(x) = C_n^{(m,k)}(a,b) \sum_{i=0}^{k-m} \frac{(b-a)^i}{n-m-i}\binom{k-m}{i}(x-a)^m(x-b)^{n-m-i}$$

$$\int_a^b P_n^{(m,k)}(x)dx = \frac{(n-m)!}{(n+1)!}\binom{k+1}{m+1}(b-a)^{m+1}.$$

If the values of derivatives of function f in $x=a$ and $x=b$ are known, using polynomials $P_n^{(m,k)}$, HERMITE's interpolation polynomial can be represented in the following form:

$$S_{n,k}(x) = \sum_{m=0}^{k-1} P_{n-1}^{(m,k-1)}(x;a,b)f^{(m)}(a) +$$

$$+ \sum_{m=0}^{n-k-1} P_{n-1}^{(m,n-k-1)}(x;b,a)f^{(m)}(b).$$

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1) On some generalizations IYENGAR's inequality see [5-7].

2. MAIN RESULT

We use the following notation

$$M^{[r]}(f;p) = \left(\frac{\int_a^b p(x)\,|f(x)|^r dx}{\int_a^b p(x)dx} \right)^{1/r}, \quad g(x) = f(x) - S_{n,k}(x).$$

THEOREM 1. Let $x \mapsto f(x)$ ne a n-times differentiable function such that $|f^{(n)}(x)| \leqq M$ $(\forall x \in (a,b))$. If $x \mapsto p(x)$ is an integrable function on (a,b) such that

$$0 < c \leqq p(x) \leqq \lambda c \quad (\lambda \geqq 1,\ x \in [a,b]),$$

the following inequality

$$(3) \quad M^{[r]}(g;p) \leqq \frac{MC(b-a)^n}{n!} \left(\frac{\lambda B(rk+1, r(n-k)+1)}{C^r + (\lambda-1)B(rk+1, r(n-k)+1)} \right)^{1/r} \quad (r>0)$$

holds, where B is beta function and $C = k^k(n-k)^{n-k}/n^n$.

Proof. Since $|f^{(n)}(x)| \leqq M$, the inequality

$$|f(x) - S_{n,k}(x)| \leqq \frac{M}{n!} |(x-a)^k (x-b)^{n-k}|$$

is valid, wherefrom (for $r > 0$)

$$(4) \qquad M^{[r]}(g;p) = \frac{M}{n!} \left(\frac{\int_a^b p(x)(x-a)^{rk}(b-x)^{r(n-k)}dx}{\int_a^b p(x)dx} \right)^{1/r}.$$

According to J. KARAMATA's inequality [8] (see also [5]) we have

$$\frac{\int_a^b p(x)(x-a)^{rk}(b-x)^{r(n-k)}dx}{\int_a^b p(x)dx} \leqq \frac{\lambda N \mu}{N + (\lambda-1)\mu},$$

where

$$N = C^r(b-a)^{nr} \text{ and } \mu = (b-a)^{nr} B(rk+1, r(n-k)+1),$$

which combined with (4) gives (3).

From Theorem 1, we directly get the following result:

COROLLARY 1. Let $x \mapsto f(x)$ be a n-times differentiable function such that $|f^{(n)}(x)| \leq M$ $(\forall x \in (a,b))$ and let $f^{(i)}(a) = 0$ $(i = 0,1,\ldots,k-1)$ and $f^{(i)}(b) = 0$ $(i = 0,1,\ldots,n-k-1)$. Then

$$\left(\frac{1}{b-a}\int_a^b |f(x)|^r dx\right)^{1/r} \leq \frac{M(b-a)^n}{n!} B(rk+1, r(n-k)+1)^{1/r} \quad (r>0). \tag{5}$$

For $n = 2m$, $k = m$, $r = 1$, inequality (5) reduces to

$$\int_a^b |f(x)| dx \leq \frac{M(b-a)^{2m+1}(m!)^2}{(2m)!(2m+1)!} \tag{6}$$

which generalize (2), and which is evidently stronger than the inequality (1).

COROLLARY 2. Let function $x \mapsto f(x)$ satisfy the conditions as in Corollary 1. If $x \mapsto p(x)$ is arbitrary nonnegative function, then

$$M^{[r]}(f;p) \leq \frac{Mk^k(n-k)^{n-k}}{n!\,n^n}(b-a)^n \quad (r > 0). \tag{7}$$

REMARK 1. Corollary 2 can be formally obtained from Theorem 1 putting $\lambda \to +\infty$. Using N. ÅSLUND's result ([9]), the inequality (7) can be substituted by a somewhat simpler but weaker inequality

$$M^{[r]}(f;p) \leq \frac{M}{n!}\binom{n}{k}^{-1}(b-a)^n \quad (r > 0).$$

3. SOME SIMILAR RESULTS

According to the results from the previous section and the inequality $\left|\int_a^b h(x)dx\right| \leq \int_a^b |h(x)| dx$, we obtain the following inequality

$$\left|\int_a^b f(x)dx - \sum_{k=1}^{m} \frac{(2m-k)!}{(2m)!}\binom{m}{k}(b-a)^k\left(f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b)\right)\right| \leq \frac{M(m!)^2(b-a)^{2m+1}}{(2m)!(2m+1)!}. \tag{8}$$

REMARK 2. If $f^{(k-1)}(a) = (-1)^k f^{(k-1)}(b)$ $(k=1,\ldots,m)$, inequality (8) reduces to (1).

THEOREM 2. Let $I_n = \{0,1,\ldots,n\}$ and let $\{P_k\}_{k\in I_n}$ be a harmonic sequence of polynomials on $[0,1]$ $(P_n'(x) = P_{n-1}(x))$. If $x\mapsto f(x)$ is n-times differentiable function such that $|f^{(n)}(x)| \leq M$ $(\forall x\in(a,b))$, then

$$\left|P_0\int_a^b f(x)dx - \sum_{k=1}^{n}(-1)^k(b-a)^k(P_k(0)f^{(k-1)}(a)-P_k(1)f^{(k-1)}(b))\right| \leq M(b-a)^{n+1}\int_0^1|P_n(t)|dt. \tag{9}$$

Proof. If $h(t) = f(a+t(b-a))$ we have $\int_a^b f(x)dx = (b-a)\int_0^1 h(t)dt$, wherefrom, applying integration by parts on the last integral, we obtain

$$\int_0^1 h(t)dt = h(1) - \int_0^1 th'(t)dt. \tag{10}$$

Since $P_1(t) = P_0 t + P_1(0)$ $(P_0(t) = P_0)$, equality (10) may be represented in the form

$$P_0\int_0^1 h(t)dt = P_1(1)h(1)-P_1(0)h(0) - \int_0^1 P_2'(t)h'(t)dt.$$

By succesive integration by parts of $\int_0^1 P_2'(t)h'(t)dt$ $(n-1)$-times, we obtain

$$P_0\int_0^1 h(t)dt = \sum_{k=1}^{n}(-1)^k(P_k(0)h^{(k-1)}(0)-P_k(1)h^{(k-1)}(1)) +(-1)^n\int_0^1 P_n(t)h^{(n)}(t)dt,$$

from where (9) follows.

COROLLARY 3. Let function $x\mapsto f(x)$ satisfy the conditions as in Theorem 2 and let $f^{(k)}(b) = (-1)^{k-1}f^{(k)}(a)$ $(k=0,\ldots,n-1)$. Then

$$\left|\int_a^b f(x)dx\right| \leq \frac{M(b-a)^{n+1}}{2^n(n+1)!}. \tag{11}$$

To prove this, take $P_n(t) = \frac{1}{n!}(t-1/2)^n$, in Theorem 2.

REMARK 3. The inequality (11) is obtained in [6] with somewhat stricter conditions for f.

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