

LOBATTO QUADRATURE FORMULAS FOR GENERALIZED GEGENBAUER WEIGHT

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ABSTRACT:

The construction of two families of Lobatto type quadrature formulas on $[-1, 1]$ with weight function $p(x) = |x|^\mu (1-x^2)^\alpha$, $\mu, \alpha > -1$, is presented in this paper. This construction is based on the usage of Gauss-Christoffel quadrature formulas and QR algorithm.

LOBATTO-OVE KVADRATURNE FORMULE ZA GENERALISANU GEGENBAUEROVU TEŽINU U radu se daje konstrukcija dve familije kvadraturnih formula Lobatto-ovog tipa na $[-1, 1]$ sa težinskom funkcijom $p(x) = |x|^\mu (1-x^2)^\alpha$, $\mu, \alpha > -1$. Ova konstrukcija se zasniva na korišćenju Gauss-Christoffelovih kvadratura i QR algoritma.

1. GENERALIZED GEGENBAUER POLYNOMIALS

Let $w_k^{(\alpha, \beta)}$ be the sequence of monic orthogonal polynomials with respect to weight function $p(x) = |x|^\mu (1-x^2)^\alpha$ ($\mu, \alpha > -1$, $\beta = (\mu-1)/2$) on $[-1, 1]$. These polynomials were introduced by Lascenov in 1953 ([4], see [1, pp. 155-6]). It is interesting that J. Radecki "rediscovered" the orthogonal polynomials with this weight in 1980 ([6]). Gegenbauer polynomials C_k^λ are a special case of these polynomials ($\mu = 0$, $\alpha = \lambda - 1/2$).

The three term recurrence relation for the polynomials $w_k^{(\alpha, \beta)}$ is

$$(1.1) \quad w_{k+1}^{(\alpha, \beta)}(x) = x w_k^{(\alpha, \beta)}(x) - \Lambda_k w_{k-1}^{(\alpha, \beta)}(x), \quad k = 0, 1, \dots,$$

where

$$\Lambda_{2m} = \frac{m(m+\alpha)}{(2m+\alpha+\beta)(2m+\alpha+\beta+1)}, \quad \Lambda_{2m-1} = \frac{(m+\beta)(m+\alpha+\beta)}{(2m+\alpha+\beta-1)(2m+\alpha+\beta)}$$

for $m = 1, 2, \dots$, except for $\alpha+\beta = -1$ when $\Lambda_1 = (\beta+1)/(\alpha+\beta+2)$.

The norm of generalized Gegenbauer polynomials is given by

$$\|w_{2k}^{(\alpha, \beta)}\|^2 = \frac{k!}{(k+\alpha+\beta+1)_k} B(k+\alpha+1, k+\beta+1) \quad \text{and} \quad \|w_{2k+1}^{(\alpha, \beta)}\|^2 = \|w_{2k}^{(\alpha, \beta+1)}\|^2.$$

2. CONSTRUCTION OF LOBATTO TYPE FORMULAS

Let us construct the Lobatto type quadrature formula

$$(2.1) \quad \int_{-1}^1 p(x)f(x) dx = Af(-1) + Bf(1) + \sum_{k=1}^n C_k f(x_k) + R_{n+2}(f),$$

where $p(x) = |x|^M(1-x^2)^d$, $M, d > -1$. In that sense let us observe the following quadrature problem

$$(2.2) \quad \int_{-1}^1 p(x)F(x) dx = \sum_{k=1}^n C_k F(x_k) + R_{n+2}(F),$$

where $F(x) = (1-x^2)g(x)$ and $R(F) = 0$ for each $g \in P_{2n-1}$ (P_{2n-1} is a set of all algebraic polynomials of degree not higher than $2n-1$). In other words, the parameters C_k and x_k ($k=1, \dots, n$) should be determined in (2.2) such that

$$(2.3) \quad \int_{-1}^1 (1-x^2)p(x)g(x) dx = \sum_{k=1}^n C_k (1-x_k^2)g(x_k) \quad (\forall g \in P_{2n-1}).$$

The formula (2.2) can be interpreted as Gauss-Christoffel quadrature formula with weight $p_1(x) = (1-x^2)p(x) = |x|^M(1-x^2)^{d+1}$, i.e.,

$$(2.4) \quad \int_{-1}^1 p_1(x)g(x) dx = \sum_{k=1}^n A_k g(x_k) + R_n(g),$$

where x_k ($k=1, \dots, n$) are the zeros of the polynomial $W_n^{(M, d)}$, and A_k ($k=1, \dots, n$) are the corresponding Christoffel's numbers. The remainder $R_n(g)$ in the Gauss-Christoffel quadrature formula (2.4) is given by

$$(2.5) \quad R_n(g) = \frac{\|W_n^{(d+1, d)}\|^2}{(2n)!} g^{(2n)}(\xi) \quad (-1 < \xi < 1),$$

assuming $g \in C^{2n}[-1, 1]$.

Comparing (2.2) and (2.4) we find $C_k = A_k/(1-x_k^2)$, $k=1, \dots, n$, and $R_{n+2}(F) = R_n(g)$.

Lemma 2.1. Suppose

$$H(x) = \begin{cases} (h(x) - h(x_0))/(x - x_0), & x \neq x_0, \\ h'(x_0), & x = x_0, \end{cases}$$

where $h \in C^{m+1}[-1, 1]$ and $-1 \leq x_0 \leq 1$. Then $H^{(k)}(x) = h^{(k+1)}(\xi_k)/(k+1)$, $k=0, 1, \dots, m$, where ξ_k is between x and x_0 .

Proof. From

$$H(x) = \int_0^1 h'(x_0 + t(x-x_0)) dt,$$

we find

$$H^{(k)}(x) = \int_0^1 t^k h^{(k+1)}(x_0 + t(x-x_0)) dt.$$

Using the mean-value theorem we obtain $H^{(k)}(x) = h^{(k+1)}(\xi_k)/(k+1)$, with ξ_k between x and x_0 . ■

Now, we suppose that function F , defined by $F(x) = (1-x^2)g(x)$, belongs to $C^{2n+2}[-1,1]$. According to the previous lemma, we obtain $g^{(2n)}(\xi) = -F^{(2n+2)}(\eta)/((2n+1)(2n+2))$ ($-1 < \eta < 1$), i.e.,

$$(2.6) \quad R_{n+2}(F) = R_n(g) = - \frac{\|w_n^{(\alpha+1, \beta)}\|^2}{(2n+2)!} F^{(2n+2)}(\eta) \quad (-1 < \eta < 1).$$

The quadrature formula (2.1) can be obtained now from (2.2). Namely, if we take

$$F(x) = f(x) + \frac{1}{2}(x-1)f(-1) - \frac{1}{2}(x+1)f(1),$$

then the formula (2.1) follows from (2.2), where x_k ($k=1, \dots, n$) are the zeros of the $w_n^{(\alpha+1, \beta)}$, A_k are the Christoffel's numbers, $C_k = A_k/(1-x_k^2)$ ($k=1, \dots, n$),

$$(2.7) \quad A = B = \frac{1}{2} \left(\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} - \sum_{k=1}^n \frac{A_k}{1+x_k} \right).$$

The equality $A = B$ holds, because the weight $p_1(x) = |x|^M (1-x^2)^{\alpha+1}$ is an even function.

Thus, we have

$$(2.8) \quad \int_{-1}^1 |x|^M (1-x^2)^\alpha f(x) dx = A(f(-1)+f(1)) + \sum_{k=1}^n C_k f(x_k) + R_{n+2}(f),$$

where x_k, A_k ($k=1, \dots, n$) are the parameters of Gauss-Christoffel formula (2.4), $C_k = A_k/(1-x_k^2)$, and A is defined by (2.7).

Assuming $f \in C^{2n+2}[-1,1]$, we have

$$(2.9) \quad R_{n+2}(f) = R_{n+2}(F) = D_n f^{(2n+2)}(\eta) \quad (-1 < \eta < 1),$$

where $D_n = -\|w_n^{(\alpha+1, \beta)}\|^2/(2n+2)!$. So, we conclude that the Lobatto formula (2.8) has the algebraic degree of precision $2n+1$.

It is easy to prove that (2.8) can be represented in the form

$$(2.10) \int_{-1}^1 |x|^{\mu}(1-x^2)^{\alpha} f(x) dx = A(f(-1)+f(1)) + C_0 f(0) + \\ + \sum_{k=1}^m C_k (f(x_k) + f(-x_k)) + R_{n+2}(f),$$

where $m = [n/2]$, A is defined by (2.7), $R_{n+2}(f)$ by (2.9), and

$$C_k = \frac{A_k}{1-x_k^2}, \quad x_k > 0, \quad k=1, \dots, m,$$

$$C_0 = 0 \quad (n=2m) \quad \text{or} \quad C_0 = A_{m+1} \quad (n=2m+1).$$

Since the three term recurrence relation for the polynomials $w_k^{(\alpha+1, \beta)}(x)$, $k=0, 1, \dots$, is known, it is possible to determine x_k and A_k ($k=1, \dots, n$) in Gauss formula (2.4), i.e., in formula (2.10), using QR algorithm (see [3], [2]). The obtained parameters, for different values of μ and α , are given in Table 2.1. The error constant D_n is given too. Numbers in parentheses denote decimal exponents.

Table 2.1.

n	$\mu=1, \alpha=1$	D_n
5	$A = 0.00416 \ 6666 \ 6667$ $C_0 = 0.06666 \ 6666 \ 6667$ $x_1 = 0.47596 \ 3149 \ 4780$ $C_1 = 0.13571 \ 2782 \ 5494$ $x_2 = 0.79410 \ 4487 \ 7608$ $C_2 = 0.07678 \ 7217 \ 4506$	-3.56(-13)
6	$A = 0.00250 \ 0000 \ 0000$ $C_0 = 0.$ $x_1 = 0.27017 \ 4062 \ 5470$ $C_1 = 0.08475 \ 4772 \ 4316$ $x_2 = 0.58907 \ 0255 \ 6048$ $C_2 = 0.11198 \ 0943 \ 8813$ $x_3 = 0.83964 \ 4097 \ 1558$ $C_3 = 0.05076 \ 4283 \ 6871$	-4.07(-15)
n	$\mu=1, \alpha=-0.5$	D_n
5	$A = 0.24380 \ 9523 \ 8095$ $C_0 = 0.09523 \ 8095 \ 2381$ $x_1 = 0.55743 \ 0069 \ 1997$ $C_1 = 0.27682 \ 6047 \ 3616$ $x_2 = 0.88327 \ 8443 \ 5619$ $C_2 = 0.43174 \ 5381 \ 2099$	-1.80(-12)
6	$A = 0.20897 \ 9591 \ 8367$ $C_0 = 0.$ $x_1 = 0.31495 \ 1060 \ 8466$ $C_1 = 0.12948 \ 4966 \ 1689$ $x_2 = 0.67091 \ 8400 \ 9874$ $C_2 = 0.27970 \ 5391 \ 4893$ $x_3 = 0.91394 \ 1854 \ 3340$ $C_3 = 0.38183 \ 0050 \ 5051$	-2.13(-15)

We note that the formula (2.10) is very useful when $f(-1) = f(1) = 0$. For example, this case appears in the least-square approximation with constraint ([5]). The use of quadrature formula in this case requires only n evaluations of function f , and the formula (2.10) integrates the first $2n+1$ powers exactly.

3. QUADRATURES WITH DERIVATIVES

Now, we construct the formula

$$(3.1) \quad \int_{-1}^1 p(x) f(x) dx = Af(-1) + Bf(1) + Df'(-1) + Ef'(1) + \sum_{k=1}^n C_k f(x_k) + R_{n+4}(f),$$

where $p(x) = |x|^\mu (1-x^2)^\alpha$, $\mu, \alpha > -1$. Using the same idea, we consider the following quadrature problem

$$(3.2) \quad \int_{-1}^1 p(x) F(x) dx = \sum_{k=1}^n C_k F(x_k) + R_{n+4}(F)$$

in which $R_{n+4}(F) = 0$ for each $g \in P_{2n-1}$, where $F(x) = (1-x^2)^2 g(x)$.

In order to determine x_k and C_k ($k=1, \dots, n$), we consider (3.2) as Gauss-Christoffel formula with the weight function $p_2(x) = (1-x^2)^2 p(x) = |x|^\mu (1-x^2)^{\alpha+2}$, i.e. as

$$(3.3) \quad \int_{-1}^1 p_2(x) g(x) dx = \sum_{k=1}^n A_k g(x_k) + R_n(g),$$

Here x_k ($k=1, \dots, n$) are the zeros of the $w_n^{(\alpha+2, \beta)}(x)$ and A_k ($k=1, \dots, n$) are the corresponding Christoffel's numbers. So, we find $C_k = A_k / (1-x_k^2)^2$, $k=1, \dots, n$, and $R_{n+4}(F) = R_n(g)$. Assuming $F \in C^{2n+4}[-1, 1]$, we have

$$(3.4) \quad R_{n+4}(F) = \frac{\|w_n^{(\alpha+2, \beta)}\|^2}{(2n+4)!} F^{(2n+4)}(\xi) \quad (-1 < \xi < 1).$$

The quadrature formula (3.1) can be obtained from (3.2) now.

Namely, if we put $F(x) = f(x) - H(x)$, where H is the Hermite's interpolation polynomial, i.e.,

$$H(x) = \frac{1}{4}(x-1)^2((x+2)f(-1) + (x+1)f'(-1)) + \frac{1}{4}(x+1)^2((2-x)f(1) + (x-1)f'(1)),$$

we obtain (3.1), where

$$A = B = \frac{1}{2} \left(\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} - \sum_{k=1}^n \frac{A_k}{(1-x_k^2)^2} \right),$$

$$D = E = \frac{1}{4} \left(\frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)} - \sum_{k=1}^n \frac{A_k}{1-x_k^2} \right),$$

and

$$R_{n+4}(f) = D_n f^{(2n+4)}(\eta) \quad (-1 < \eta < 1),$$

with $D_n = \|w_n^{(\alpha+2, \beta)}\|^2 / (2n+4)!$. The remainder in this form holds when

$$f \in C^{2n+4}[-1, 1].$$

The formula (3.1) can be expressed in the form

$$(3.5) \quad \int_{-1}^1 |x|^\mu (1-x^2)^\alpha f(x) dx = A(f(-1) + f(1)) + D(f'(-1) - f'(1)) + \\ + C_0 f(0) + \sum_{k=1}^m C_k (f(x_k) + f(-x_k)) + R_{n+4}(f),$$

where $m = [n/2]$ and

$$C_k = \frac{A_k}{(1-x_k^2)^2}, \quad x_k > 0, \quad k = 1, \dots, m,$$

$$C_0 = 0 \quad (n = 2m) \quad \text{or} \quad C_0 = A_{m+1} \quad (n = 2m+1).$$

The parameters of obtained formula are given in Table 3.1.

Table 3.1.

n	$\mu = 1, \alpha = 1$	D_n
4	$A = 0.01750 \ 0000 \ 0000$ $D = 0.00083 \ 3333 \ 3333$ $x_1 = 0.31825 \ 5412 \ 0882$ $C_1 = 0.11334 \ 5262 \ 4903$ $x_2 = 0.68566 \ 9063 \ 1092$ $C_2 = 0.11915 \ 4737 \ 5097$	5.92(-13)
5	$A = 0.01125 \ 0000 \ 0000$ $D = 0.00041 \ 6666 \ 6667$ $x_0 = 0.$ $C_0 = 0.05555 \ 5555 \ 5556$ $x_1 = 0.43819 \ 9425 \ 2873$ $C_1 = 0.12242 \ 2043 \ 6271$ $x_2 = 0.74698 \ 1434 \ 6273$ $C_2 = 0.08855 \ 0178 \ 5952$	8.13(-16)
n	$\mu = 1, \alpha = -0.5$	D_n
4	$A = 0.42840 \ 8163 \ 2653$ $D = 0.00870 \ 7482 \ 9932$ $x_1 = 0.36995 \ 8442 \ 6479$ $C_1 = 0.18035 \ 3176 \ 9663$ $x_2 = 0.76837 \ 7171 \ 6978$ $C_2 = 0.39123 \ 8659 \ 7684$	2.10(-12)
5	$A = 0.37616 \ 3265 \ 3061$ $D = 0.00580 \ 4988 \ 6621$ $x_0 = 0.$ $C_0 = 0.07407 \ 4074 \ 0741$ $x_1 = 0.49896 \ 8388 \ 1747$ $C_1 = 0.21981 \ 7276 \ 4594$ $x_2 = 0.82104 \ 0480 \ 5363$ $C_2 = 0.36698 \ 2421 \ 1974$	3.20(-15)

4. EXAMPLE

We consider the integral

$$I = \int_{-1}^1 |x| (1-x^2) \cos \frac{\pi x}{2} dx = \frac{48}{\pi^2} \left(\frac{2}{\pi} - \frac{4}{\pi^2} - \frac{1}{6} \right) = 0.31450924354905\dots,$$

where $p(x) = |x| (1-x^2)$, i.e., $\mu = 1$, $\alpha = 1$, and $f(x) = \cos \frac{\pi x}{2}$.

Table 4.1 represents the relative errors in the value of integral I corresponding to the n -point Gauss-Christoffel rule (GC), the Lobatto formula (2.10), and the generalized Lobatto formula (3.5), in that order.

Since $f(-1) = f(1) = 0$ and $f'(-1) = f'(1) = \pi$, all these quadrature rules require n functional evaluations. However, their algebraic degree precision are $2n-1$, $2n+1$, and $2n+3$, respectively.

Table 4.1.

n	G.C	(2.10)	(3.5)
2	2.04(-2)	7.62(-4)	1.80(-5)
3	5.17(-4)	9.16(-6)	1.20(-7)
4	4.60(-6)	4.83(-8)	4.04(-10)
5	3.64(-8)	2.44(-10)	1.38(-12)
6	1.47(-10)	6.88(-13)	6.62(-16)

The results given in Table 4.1. are in good agreement with the previous theoretical results.

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