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AN EXTREMAL PROBLEM FOR ALGEBRAIC POLYNOMIALS WITH A PRESCRIBED ZERO

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Let π_n^1 be a set of polynomials $P(z)=a_0+a_1z+\ldots+a_nz^n$, where $z=e^{it}$ and $a_0,a_1,\ldots,a_n$ are real numbers, with the condition $P(1)=0$. Let us define $\|P\|_V$ and $\|P\|_L$ by

$$\|P\|_V^2=\frac{1}{2\pi}\int_0^{2\pi}|P(e^{it})|^2dt \quad \text{and} \quad \|P\|_L^2=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{it})}{e^{it}-1}\right|^2dt. \tag{1}$$

In a great number of papers (see, for example, [1], [2],[3],[4] several inequalities involving the norms (1) were given. Also, a gre number of inequalities for polynomials from π_n^1 were proved (see, for example, [1], [5], [6], [7], [8], [9]). These inequalities give the estimation of $|P'(1)|$ in term of $\max|P(z)|$, $\min|P(z)|$, $\|P\|_V$, $\|P\|_L$, etc. The aim of this paper is to estimate $|P^{(m)}(1)|$ $(m\le n)$ for polynomials from π_n^1 in term of the norms defined by (1).

Theorem 1. *Let P be a polynomial from the set π_n^1. Then*

$$\sup_{P\in\pi_n^1}\frac{|P^{(m)}(1)|}{\|P\|_V}=C_{n,m} \qquad (m\le n), \tag{2}$$

where

$$C_{n,m}=m!\left(\sum_{k=0}^{n}\binom{k}{m}^2-\frac{1}{n+1}\binom{n+1}{m+1}^2\right)^{1/2}.$$

The supremum in (2) is attained for $P(z) = (z-1)\sum_{k=1}^{n} x_k z^{k-1}$, $z=e^{it}$, *where*

$$x_k = A\left(\frac{n^{(m)}}{m+1}k - \sum_{i=0}^{k-1} i^{(m)}\right),$$

A *is an arbitrary constant and* $x^{(s)} = x(x-1)\ldots(x-s+1)$.

<u>Proof</u>. Let $P\in\pi_n^1$. Then, we can write it in the form

$$P(z) = (z-1)(x_1+x_2z+\ldots+x_nz^{n-1}) = \sum_{k=0}^{n}(-1)^k\Delta x_k z^k, \quad x_0=x_{n+1}=0,$$

where $\Delta x_k = x_{k+1}-x_k$. Now

$$P^{(m)}(z) = \sum_{k=0}^{n}(-1)^k\Delta x_k k^{(m)} z^{k-m},$$

so we obtain

$$P^{(m)}(1) = \sum_{k=0}^{n}(-1)^k k^{(m)}\Delta x_k \quad \text{and} \quad \|P\|_V^2 = \sum_{k=0}^{n}(\Delta x_k)^2.$$

Let $y_k = (-1)^k\Delta x_k$ for $k=0,1,\ldots,n$. Then (2) can be reduced to the solution of the extremal problem:

$$\text{Minimize } \sum_{k=0}^{n} y_k^2$$

(3) $$\text{subject to } P^{(m)}(1) = \sum_{k=0}^{n} k^{(m)} y_k = C \quad \text{and} \quad \sum_{k=0}^{n} y_k = 0.$$

We consider the associated function F, viz.

$$F = \sum_{k=0}^{n} y_k^2 - \lambda\left(\sum_{k=0}^{n} k^{(m)} y_k - C\right) - \mu\left(\sum_{k=0}^{n} y_k\right),$$

where λ and μ are Lagrange multipliers, whose values we will determine. We must have

$$\frac{\partial F}{\partial y_k} = 2y_k - \lambda k^{(m)} - \mu = 0, \quad k=0,1,\ldots,n,$$

which yields

$$y_k = \frac{\lambda}{2} k^{(m)} + \frac{\mu}{2}, \qquad k=0,1,\ldots,n.$$

Let

$$S_i = \sum_{k=0}^{n} (k^{(m)})^i = (m!)^i \sum_{k=0}^{n} \binom{k}{i}^i .$$

From the constraints (3), we obtain $\mu/2 = D_0/D$ and $\lambda/2 = D_1/D$, where $D = S_0 S_2 - S_1^2$, $D_0 = -S_1 C^2$ and $D_1 = -S_0 C$. Now

$$F = \frac{1}{D^2} \sum_{k=0}^{n} (D_0 + D_1 k^{(m)})^2 = \frac{S_0}{D} C^2 = \frac{S_0}{S_0 S_2 - S_1^2} \, | P^{(m)}(1) |^2,$$

wherefrom we obtain (2).

Since $\Delta x_k = y_k$, we have

$$x_k = - \sum_{i=0}^{k-1} y_i = - \frac{1}{D} \sum_{i=0}^{k-1} (D_0 - D_1 i^{(m)}),$$

i.e.,

$$x_k = A \left(\frac{n^{(m)}}{m+1} k - \sum_{i=0}^{k-1} i^{(m)} \right) \qquad (A = \text{const}),$$

wherefrom we obtain the condition when in (2) the equality holds.

Remark 1. We will give $C_{n,m}^2$ for some actual values of m. So

$$C_{n,1}^2 = \frac{n(n+1)(n+2)}{12},$$

$$C_{n,2}^2 = \frac{n(n^2-1)(n+2)(4n-3)}{45},$$

$$C_{n,3}^2 = \frac{n(n^2-1)(n^2-4)(15n^2-35n+12)}{1680},$$

$$\vdots$$

$$C_{n,n}^2 = \frac{n(n!)^2}{n+1}.$$

Note that the constant $C_{n,1}^2$ is determined in the paper [5].

Theorem 2. *Let the polynomial P be from the set* π_n^1. *Then*

$$(4)\qquad \sup_{P\in\pi_n^1} \frac{|P^{(m)}(1)|}{\|P\|_L} = \left((m-1)!\sum_{k=0}^{m-1} \frac{\binom{m-1}{k}}{k!}\cdot\frac{n^{(m+k)}}{m+k}\right)^{1/2} \qquad (m \le m).$$

The equality in (4) holds for $P(z) = A(z-1)\sum_{k=1}^{n} \frac{1}{k} k^{(m)} z^{k-1}$, $z = e^{it}$.

Proof. Since

$$\|P\|_L^2 = \sum_{k=1}^{n} x_k^2 ,$$

on the basis of Schwarz's inequality we obtain

$$|P^{(m)}(1)|^2 = \Big(\sum_{k=m}^{n} \frac{k^{(m)}}{k} x_k\Big)^2 \le \Big(\sum_{k=m}^{n} \frac{k^{(m)2}}{k^2}\Big)\Big(\sum_{k=m}^{n} x_k^2\Big)$$

$$\le \Big(\sum_{k=1}^{n} \frac{k^{(m)2}}{k^2}\Big)\Big(\sum_{k=1}^{n} x_k^2\Big) ,$$

i.e. (4).

Remark 2. Using the same proof as in Theorem 2, we can get an inequality of the form $|P^{(m)}(1)| \le \overline{C}_{n,m}\|P\|_V$, but the constant $\overline{C}_{n,m}$ is not best possible, i.e. the above inequality is rough.

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