CONSTRUCTION OF $s$ - ORTHOGONAL POLYNOMIALS
AND TURÁN QUADRATURE FORMULAE

GRADIMIR V. MILOVANOVIĆ

ABSTRACT: A connection between Turán quadratures and $s$-orthogonal polynomials with respect to a nonnegative measure on the real line $\mathbb{R}$ is given. Using a discretised Stieltjes procedure and the Newton-Kantorovič method, an iterative method with quadratic convergence for the construction of $s$-orthogonal polynomials is formulated. Some numerical examples are included. Finally, some considerations about Turán quadrature formulae with Chebyshev measure are given.

1. INTRODUCTION

In 1950 P. Turán investigated numerical quadratures of the type

$$
\int_{-1}^{1} f(t) \, dt = \sum_{\nu=1}^{n} \sum_{i=0}^{k-1} A_{i,\nu} f^{(i)}(\tau_{\nu}) + R_{n,k}(f),
$$

where

$$
A_{i,\nu} = \int_{-1}^{1} \ell_{\nu,i}(t) \, dt \quad (\nu=1, \ldots, n; \ i=0,1,\ldots,k-1)
$$

and $\ell_{\nu,i}(t)$ are the fundamental functions of Hermite interpolation. The $A_{i,\nu}$ are Cotes numbers of higher order. The formula (1.1) is exact if $f$ is a polynomial of degree at most $kn-1$ and the points $-1 < \tau_1 < \tau_2 < \ldots < \tau_n < 1$ are arbitrary.

For $k=1$ the formula (1.1), i.e.,

$$
\int_{-1}^{1} f(t) \, dt = \sum_{\nu=1}^{n} A_{0,\nu} f(\tau_{\nu}) + R_{n,1}(f),
$$

is exact if $f$ is a polynomial of degree at most $n-1$. These quadratures are called Turán quadratures.
can be exact for all polynomials of degree \( \leq 2n-1 \) if the nodes \( \tau_v \) are the zeros of the Legendre polynomial \( P_n \). That is the well-known Gauss-Legendre quadrature.

Because of the theorem of Gauss it is natural to ask whether knots \( \tau_v \) can be chosen so that the quadrature formula (1.1) will be exact for polynomials of degree not exceeding \((k+1)n-1\).

P. Turán [17] showed that the answer is negative for \( k=2 \), and for \( k=3 \) it is positive. He proved that the knots \( \tau_v \) should be chosen as the zeros of the monic polynomial \( \pi_n^*(t) = t^n + ... \) which minimizes the following integral

\[
\left( \int_{-1}^{1} \pi_n(t)^4 \, dt \right)^{-1/2}
\]

where \( \pi_n(t) = t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 \).

More generally, the answer is negative for even, and positive for odd \( k \), and then \( \tau_v \) are the zeros of the polynomial minimizing

\[
\left( \int_{-1}^{1} \pi_n(t)^{k+1} \, dt \right)^{-1/2}
\]

For \( k=1 \), \( \pi_n \) is the monic Legendre polynomial \( P_n \).

Because of the above, we put \( k=2s+1 \). Instead of (1.1), it is also interesting to investigate the analogous formula with a weight function \( t+p(t) \),

\[
\left( \int_{-1}^{1} f(t)p(t) \, dt \right) = \sum_{i=0}^{2s} \sum_{v=1}^{n} A_{i,v} f^{(i)}(\tau_v) + R(f),
\]

or more generally, with some nonnegative measure \( d\lambda(t) \) on the real line \( \mathbb{R} \),

\[
\left( \int_{\mathbb{R}} f(t) d\lambda(t) \right) = \sum_{i=0}^{2s} \sum_{v=1}^{n} A_{i,v} f^{(i)}(\tau_v) + R(f).
\]
This paper is organized as follows. In Section 2 we give a connection between Turán quadratures and s-orthogonal polynomials, which were studied extensively by several Italian mathematicians [12], [7], [13], [14]. Also, in this section we mention a recent method of Vincenti [20] for the computation of the coefficients of s-orthogonal polynomials with respect to an even function. In Section 3 we develop a new method for the numerical construction of s-orthogonal polynomials with respect to an arbitrary weight function. Numerical examples are given in Section 4. Section 5 deals with Turán quadratures with Chebyshev measure.

2. TURÁN QUADRATURES AND S-ORTHOGONAL POLYNOMIALS

We consider the Turán quadrature formula (1.3), where \( d\lambda(t) \) is a nonnegative measure on the real line \( \mathbb{R} \), with compact or infinite support, for which all moments \( \mu_k = \int_{\mathbb{R}} t^k d\lambda(t) \), \( k=0,1,\ldots \), exist and are finite, and \( \mu_0 > 0 \). The formula (1.3) must be exact for all polynomials of degree at most \( 2(s+1)n-1 \). The role of the integral (1.2) is taken over by

\[
F = \int_{\mathbb{R}} \pi_n(t)^{2s+2} d\lambda(t),
\]

where \( F \equiv F(a_0, a_1, \ldots, a_{n-1}) \), \( \pi_n(t) = \sum_{k=0}^{n-1} a_k t^k \), \( a_n = 1 \). In order to minimize \( F \) we must have

\[
(2.1) \quad \int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k d\lambda(t) = 0, \quad k=0,1,\ldots,n-1.
\]

Usually, instead of \( \pi_n(t) \) we write \( P_{s,n}(t) \).

The case \( d\lambda(t) = p(t) dt \) on \([a,b]\) has been considered by the Italian mathematicians A.Ossicini [12], A.Ghizzetti and A.Ossicini [7], S.Guerra [8], [9]. It is known that there exists a unique
\[ P_{s,n}(t) = \prod_{\nu=1}^{n} (t - \tau_{\nu}), \] whose zeros \( \tau_{\nu} \) are real, distinct and located in the interior of the interval \([a,b]\). These polynomials are known as \(s\)-orthogonal (or \(s\)-self associated) polynomials in the interval \([a,b]\) with respect to the weight function \(p\).

For \(s=0\) we have the standard case of orthogonal polynomials. The case when \(s>0\) is very difficult. It requires the use of a method with special numerical treatment.

Recently G.Vincenti [20] has considered an iterative process to compute the coefficients of \(s\)-orthogonal polynomials in a special case when the interval \([a,b]\) is symmetric with respect to origin, say, \([-b,b]\), and the weight function \(p\) is an even function \(p(-t)=p(t)\). Then \( P_{s,n}(-t) = (-1)^n P_{s,n}(t) \). He considered two cases: when \(n\) is even and when \(n\) is odd.

In the first case \(n=2m\), \( P_{s,n}(t) = \sum_{i=0}^{m} a_i t^{2m-2i}, a_0 = 1. \) From (2.1) Vincenti obtained a nonlinear system of equations of the form

\[ \sum_{i=0}^{m} C_{m+r-i} a_i = -C_{m+r} \quad (r=0,1,\ldots,m-1), \]

where

\[ C_j^{(0)} = \int_{0}^{b} p(t) t^{2j} dt, \quad C_j^{(h)} = \sum_{p,q=0}^{m} C_{j+2m-p-q} a_p a_q, \]

and \( C_j^{(s)} = C_j \). Then he has solved this system by some iterative method like Newton's method. For \(n=2m+1\), a similar system of equations was obtained.

Vincenti applied his process to the Legendre case. When \(n\) and \(s\) increase, the process becomes ill-conditioned. So, the author gave numerical results in the following cases: \(n=2,3, 1 \leq s \leq 10; n=4,5, 1 \leq s \leq 5; n=6,7, 1 \leq s \leq 3; n=8,9, 1 \leq s \leq 2; n=10,11, s=1. \) The results were obtained with 18 correct decimal digits,
but using an arithmetic with 36 decimal digits.

From (2.1) we can see that this procedure needs the first $2(s+1)n$ moments of the weight function: $\mu_0, \mu_1, \ldots, \mu_{2(s+1)n-1}$.

We see that $C_j^{(0)} = \mu_{2j}/2$. Of course, in this special case, the moments of odd order are zero. Here, we have a nonlinear map $V_{n,s} : \mathbb{R}^{2(s+1)n} \rightarrow \mathbb{R}^n$, given by $[\mu_0, \mu_1, \ldots, \mu_{2(s+1)n-1}]^T \mapsto [a_0, a_1, \ldots, a_{n-1}]^T$. The problem itself is highly sensitive to small perturbations in the moments, so that any algorithm which theoretically solves the problem using the moments will be subject to severe growth of errors when executed in an arithmetic of finite precision ([4], [5]). It would be useful to find a numerical condition number of the map $V_{n,s}$, but that will not be our aim here.

3. CONSTRUCTION OF $s$-ORTHOGONAL POLYNOMIALS

In this section we will give a stable procedure for the numerical construction of $s$-orthogonal polynomials with respect to $d\lambda(t)$ on $\mathbb{R}$. Namely, we will reduce our problem to the standard theory of orthogonal polynomials, and then we will use the Stieltjes procedure ([3], [5]). The main idea is an interpretation of the "orthogonality conditions" (2.1), i.e.,

$$\int_{\mathbb{R}} \pi_n(t)^k \pi_n(t)^{2s} d\lambda(t) = 0, \quad k=0,1,\ldots,n-1.$$ 

For given $n$ and $s$, we put $d\mu(t) = d\mu_{s,n}(t) = (\pi_n(t))^{2s} d\lambda(t)$. These conditions can be interpreted as

$$\int_{\mathbb{R}} \pi_k^{s,n}(t)^\nu d\mu(t) = 0, \quad \nu=0,1,\ldots,k-1,$$

where $\pi_k^{s,n}$ is a sequence of monic orthogonal polynomials with respect to the new measure $d\mu(t)$. Of course, $P_{s,n}(.) = \pi_n^{s,n}(.)$. 
As we can see, the polynomials $\pi_k^{s,n}$, $k=0,1,\ldots$, are implicitly defined, because the measure $du(t)$ depends of $\pi_n^{s,n}(t)$. The general class of such polynomials was introduced by H.Engels (see [2, pp. 214-226]).

We will write only $\pi_k(.)$ instead of $\pi_k^{s,n}(.)$. These polynomials satisfy a three-term recurrence relation

\begin{equation}
\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k=0,1,\ldots,
\end{equation}

\begin{align*}
\pi_{-1}(t) &= 0, \quad \pi_0(t) = 1,
\end{align*}

where, because of orthogonality,

\begin{align*}
\alpha_k &= \alpha_k(s,n) = \frac{\langle \pi_k', \pi_k \rangle}{\langle \pi_k, \pi_k \rangle} = \frac{\int \pi_k^2(t) du(t)}{\int \pi_k(t) du(t)}, \\
\beta_k &= \beta_k(s,n) = \frac{\langle \pi_k', \pi_k \rangle}{\langle \pi_{k-1}', \pi_{k-1} \rangle} = \frac{\int \pi_{k-1}^2(t) du(t)}{\int \pi_{k-1}(t) du(t)}.
\end{align*}

and, for example, $\beta_0 = \int_R du(t)$.

The coefficients $\alpha_k$ and $\beta_k$ are the fundamental quantities in the constructive theory of orthogonal polynomials. They provide a compact way of representing orthogonal polynomials, requiring only a linear array of parameters. The coefficients of orthogonal polynomials, or their zeros, in contrast need two-dimensional arrays.

Finding the coefficients $\alpha_k$, $\beta_k$ ($k=0,1,\ldots,n-1$) gives us access to the first $n+1$ orthogonal polynomials $\pi_0, \pi_1', \ldots, \pi_n$. Of course, for a given $n$, we are interested only in the last of them i.e., $\pi_n^{s,n}(.)$. So, for $n=0,1,\ldots$, the diagonal (boxed) elements
In the following table are our \( s \)-orthogonal polynomials \( s_n \).

**TABLE 3.1**

<table>
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<th>( n )</th>
<th>( d_n s_n(t) )</th>
<th>Orthogonal Polynomials</th>
</tr>
</thead>
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<td>0</td>
<td>((\pi_0 s^{0}(t), 2s , d\lambda(t))</td>
<td>( s_0 )</td>
</tr>
<tr>
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<td>((\pi_1 s^{1}(t), 2s , d\lambda(t))</td>
<td>( s_1 )</td>
</tr>
<tr>
<td>2</td>
<td>((\pi_2 s^{2}(t), 2s , d\lambda(t))</td>
<td>( s_2 )</td>
</tr>
<tr>
<td>3</td>
<td>((\pi_3 s^{3}(t), 2s , d\lambda(t))</td>
<td>( s_3 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

A stable procedure for finding the coefficients \( a_k, \beta_k \) is the discretized Stieltjes procedure, especially for infinite intervals of orthogonality (see Gautschi [5], and Gautschi, Milovanović [6]). Unfortunately, in our case this procedure cannot be used directly, because the measure \( d_\mu(t) \) involves an unknown polynomial \( s_n \). Consequently, we consider the system of nonlinear equations

\[
f_0 = \beta_0 - \int_{\mathbb{R}}^{2s} \pi_n(t) \, d\lambda(t) = 0,
\]

(3.3) \[
f_{2k+1} = \int_{\mathbb{R}}^{2s} \left( a_k - t \right)^{2s} \pi_n(t) \, d\lambda(t) = 0, \quad k=0,1,\ldots,n-1,
\]

\[
f_{2k} = \int_{\mathbb{R}}^{2s} \left( \beta_k \pi_{k-1}(t) - \pi_k(t) \right)^{2s} \pi_n(t) \, d\lambda(t) = 0, \quad k=1,\ldots,n-1,
\]

which follows from (3.2).
Let \( x \) be a \((2n)\)-dimensional column vector with components \( a_0, \beta_0, \ldots, a_{n-1}, \beta_{n-1} \) and \( f(x) \) a \((2n)\)-dimensional vector with components \( f_0, f_1, \ldots, f_{2n-1} \) given by (3.3). If \( W = W(x) \) is the corresponding Jacobi matrix of \( f(x) \), then we can apply Newton-Kantorovič's method

\[
(3.4) \quad x^{[v+1]} = x^{[v]} - W^{-1}(x^{[v]}) f(x^{[v]}), \quad v = 0, 1, \ldots,
\]

for determining the coefficients of the recurrence relation (3.1).

Starting with a reasonable good approximation \( x^{[0]} \), the convergence of the method (3.4) is quadratic.

It is interesting that the elements of Jacobi matrix can be easily computed in the following way:

First, we have to determine the partial derivatives \( \frac{\partial f_k}{\partial \alpha_i} \) and \( \frac{\partial f_k}{\partial \beta_i} \). Differentiating the recurrence relation (3.1) with respect to \( \alpha_i \) and \( \beta_i \) we obtain

\[
a_{k+1, i} = (t - a_k) a_{k, i} - \beta_i a_{k-1, i},
\]

and

\[
b_{k+1, i} = (t - a_k) b_{k, i} - \beta_i b_{k-1, i},
\]

where

\[
a_{k, i} = 0, \quad b_{k, i} = 0, \quad k \leq i,
\]

\[
a_{i+1, i} = -\pi_i(t), \quad b_{i+1, i} = -\pi_{i-1}(t).
\]

These relations are the same as those for \( \pi_k \), but with other initial values. The elements of the Jacobi matrix are

\[
\frac{\partial f_{2k+1}}{\partial \alpha_i} = 2 \int_n^{2s-1} (t) [\left( a_{k,t} \right) p_{k, i}(t) + \frac{1}{2} \pi_i^{2}(t) \pi_{i-1}(t)] d\lambda(t),
\]

\[
\frac{\partial f_{2k}}{\partial \beta_i} = 2 \int_n^{2s-1} (t) [\left( a_{k,t} \right) p_{k, i}(t) + \frac{1}{2} \pi_i^{2}(t) \pi_{i-1}(t)] d\lambda(t),
\]

\[
\frac{\partial f_{2k}}{\partial \beta_i} = 2 \int_n^{2s-1} (t) [\left( a_{k,t} \right) p_{k, i}(t) + \frac{1}{2} \pi_i^{2}(t) \pi_{i-1}(t)] d\lambda(t).
\]
\[
\frac{\partial f_{2k+1}}{\partial \beta_i} = 2 \int \pi_n^{2s-1}(t)(\alpha_k - t)q_{k,i}(t) \, d\lambda(t),
\]

\[
3.5) \quad \frac{\partial f_{2k}}{\partial \alpha_i} = 2 \int \pi_n^{2s-1}(t)(\beta_k p_{k-1,i}(t) - p_{k,i}(t)) \, d\lambda(t),
\]

\[
\frac{\partial f_{2k}}{\partial \beta_i} = 2 \int \pi_n^{2s-1}(t)\left\{q_k q_{k-1,i}(t) - q_{k,i}(t) + \frac{1}{2}\delta_{ki}\pi_k^{2s-1}(t)\pi_n(t)\right\} \, d\lambda(t),
\]

where \( p_{k,i}(t) = \pi_k(t)(\alpha_k, i t) + s a_{n,i} \pi_k(t) \) and \( q_{k,i}(t) = \kappa(t)(b_k, i n(t) + s b_{n,i} \pi_k(t)) \), and \( \delta_{ki} \) is Kronecker's delta.

All of the above integrals in (3.3) and (3.5) can be found exactly, except for rounding errors, by using a Gauss-Christoffel quadrature formula with respect to the measure \( d\lambda(t) \),

\[
3.6) \quad \int g(t) \, d\lambda(t) = \sum_{k=1}^{N} a_k^{(N)} g(\tau_k^{(N)}) + R_N(g),
\]

taking \( N = (s+1)n \) knots. This formula is exact for all polynomials of degree at most \( 2N-1 = 2(s+1)n - 1 = 2(n-1) + 2ns + 1 \).

Thus, for all calculations we use only the fundamental three-term recurrence relation and the Gauss-Christoffel quadrature 3.6). As initial values \( a_k^{[0]} = a_k(s,n) \) and \( \beta_k^{[0]} = \beta_k(s,n) \) we take the values obtained for \( n-1 \), i.e. \( a_k^{[0]} = a_k(s,n-1) \), \( \beta_k^{[0]} = \beta_k(s,n-1) \), \( \kappa n-2 \). For \( a_{n-1} \) and \( \beta_{n-1} \) we use the corresponding extrapolated values.

In the case \( n=1 \) we solve the equation

\[
\phi(\alpha_0) = \phi(\alpha_0(s,1)) = \int (t-\alpha_0)^{2s+1} \, d\lambda(t) = 0,
\]

and then determine

\[
\beta_0 = \beta_0(s,1) = \int (t-\alpha_0)^{2s} \, d\lambda(t).
\]
4. NUMERICAL EXAMPLES

We will consider two examples, involving Laguerre and Legendre measures.

Example 4.1. \( d\lambda(t) = e^{-t} \, dt \) on \((0, \infty)\).

Using the presented method, we determined the recursion coefficients \( a_k(s,n) \) and \( \beta_k(s,n) \), \( k=0,1,\ldots,n-1 \), for \( s=1(1)5 \) and \( n=1(1)10 \). These coefficients and zeros of \( \tau_n^{s,n}(t) \), \( \varpi_k(s,n) \), \( k=1,\ldots,n \), for some selected values of \( s \) and \( n \), are given in Table 4.1. Numbers in parentheses denote decimal exponents. The zeros \( \varpi_k(s,n) \), \( k=1,\ldots,n \), were obtained as eigenvalues of the (symmetric tridiagonal) Jacobi matrix

\[
J_n = \begin{bmatrix}
    a_0(s,n) & \sqrt{\beta_1(s,n)} & 0 \\
    \sqrt{\beta_1(s,n)} & a_1(s,n) & \sqrt{\beta_2(s,n)} \\
    \vdots & \ddots & \ddots \\
    & & \sqrt{\beta_{n-1}(s,n)} \\
    0 & & a_{n-1}(s,n)
\end{bmatrix}
\]

using the QR algorithm.

Example 4.2. \( d\lambda(t) = dt \) on \((-1,1)\). In this (Legendre) case, the coefficients \( a_k(s,n) \) are equal to zero, so the computation can be simplified. The system of equations (3.3) becomes

\[
g_0 = f_0 = \frac{1}{2n} \int_{-1}^{1} 2s(t) \, dt = 0,
\]

\[
g_k = f_{2k} = \frac{1}{2n} \int_{-1}^{1} (\varpi_k(t) - \varpi_k(t) \, \varpi_n(t) \, dt = 0, \quad k=1,\ldots,n
\]
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<td>2.83768214565785(0)</td>
<td>1.2080680183997(2)</td>
<td>1.9621882995226(1)</td>
</tr>
<tr>
<td></td>
<td>3</td>
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<td>2.99532971959448(2)</td>
<td>4.094261010120(1)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5.36078046528124(1)</td>
<td>5.63228814211952(2)</td>
<td>7.4649521550663(1)</td>
</tr>
</tbody>
</table>

Table 4.2 shows the numerical results for s = 1, 3, 5 and n = 3, 5, 10. The corresponding zeros τ_v(s,n), k=1,...,n, are given in Table 4.3.
### TABLE 4.2

<table>
<thead>
<tr>
<th>n</th>
<th>ν</th>
<th>θ_v(1, n)</th>
<th>θ_v(3, n)</th>
<th>θ_v(5, n)</th>
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<td>0.26465724288258 (-7)</td>
<td>0.30161811315945 (-13)</td>
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<tr>
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</table>

### TABLE 4.3

<table>
<thead>
<tr>
<th>n</th>
<th>ν</th>
<th>τ_v(1, n)</th>
<th>τ_v(3, n)</th>
<th>τ_v(5, n)</th>
</tr>
</thead>
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<td>±0.8453661637477</td>
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<td>0.81443918557776</td>
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<td>0.8453661637477</td>
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<tr>
<td>5</td>
<td>1,5</td>
<td>±0.92711786960989</td>
<td>±0.93810619284349</td>
<td>±0.94197468869998</td>
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<td>±0.57330378590709</td>
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<td>3</td>
<td>0.56086741916164</td>
<td>0.57330378590709</td>
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</tr>
<tr>
<td>10</td>
<td>1,10</td>
<td>±0.98066259593659</td>
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<td></td>
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<td>±0.88618806147381</td>
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<td></td>
<td>3,8</td>
<td>±0.69262442514005</td>
<td>±0.69957700233546</td>
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<td>4,7</td>
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<td>±0.44838741280314</td>
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<td>±0.15247058767942</td>
<td>±0.15437687188524</td>
<td>±0.15503560566469</td>
</tr>
</tbody>
</table>
i. Turán Quadratures with Chebyshev Weight

Now, we will consider again the quadrature formula of Turán (1.3). If we define \( \omega_v \), by

\[
\omega_v(t) = \left( \frac{\pi_n(t)}{t - \tau_v} \right)^{2s+1}, \quad v = 1, \ldots, n,
\]

where \( \pi_n(t) = \pi_{s,n}(t) \) and \( \tau_v = \tau_{v}(s,n) \), then the coefficients \( A_{1,v} \) in Turán quadrature (1.3) can be expressed in the form [16]

\[
A_{1,v} = \frac{1}{i!(2s-i)!} \left[ D^{2s-i} \omega_v(t) \int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1} - \pi_n(t)^{2s+1}}{x - t} d\lambda(x) \right]_{t = \tau_v},
\]

where \( D \) is the standard differentiation operator. Especially, for \( i = 2s \), we have

\[
A_{2s,v} = \frac{1}{(2s)!(\pi'_{n}(\tau_v))^{2s+1}} \int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1}}{t - \tau_v} d\lambda(x),
\]

i.e.,

\[
A_{2s,v} = \frac{B_{v}^{(s)}}{(2s)!(\pi'_{n}(\tau_v))^{2s}}, \quad v = 1, \ldots, n,
\]

(5.1)

where \( B_{v}^{(s)} \) are the Christoffel numbers of the following quadrature (with respect to the measure \( d\mu(t) = \pi_{n}^{2s}(t)d\lambda(t) \))

\[
(5.2) \quad \int_{\mathbb{R}} g(t) d\mu(t) = \sum_{v=1}^{n} B_{v}^{(s)} g(\tau_v) + R_n(g), \quad R_n(\mathbb{P}_{2n-1}) = 0,
\]

So we have \( A_{2s,v} > 0 \).

The expressions for the other coefficients (\( i < 2s \)) become very complicated.

For the numerical calculation we can use a triangular system of linear equations obtained from the formula (1.3) by replacing \( f \)
with the Newton polynomials: 1, \( t - \tau_1 \), ..., \( (t - \tau_1)^{2s+1} \),
\( (t - \tau_1)^{2s+1} (t - \tau_2) \), ..., \( (t - \tau_1)^{2s+1} (t - \tau_2)^{2s+1} \)...

Particularly interesting is the case of the Chebyshev weight
\[ p(t) = (1-t^2)^{-1/2}. \]

In 1930, S. Bernstein [1] showed that \( 2^{1-n} T_n(t) \) minimizes all integrals of the form
\[ \frac{1}{k!} \int_{-1}^{1} \frac{|\tau_n(t)|^{k+1}}{\sqrt{1-t^2}} \, dt, \quad k \geq 0. \]

So the Turán-Chebyshev formula
\[
(5.3) \quad \frac{1}{k!} \int_{-1}^{1} (1-t^2)^{-1/2} f(t) \, dt = \sum_{i=0}^{2s} \sum_{v=1}^{n} A_{i,v} f(i) (\tau_v) + R(f),
\]
with \( \tau_v = \cos \left( \frac{2v-1}{2n} \pi \right) \), \( v=1, \ldots, n \), is exact for polynomials of degree not exceeding \( 2(s+1)n-1 \). Turán has stated a problem of explicit determination of \( A_{i,v} \), and its asymptotic behavior as \( n \to \infty \) (Problem XXVI in [18]). In this regard, Micchelli and Rivlin ([11]) have proved the following characterization: If \( f \in \mathcal{P}_{2(s+1)n-1} \) then
\[
\frac{1}{k!} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} \, dt = \frac{n}{2} \left\{ \sum_{v=1}^{n} f(\tau_v) + \sum_{j=1}^{s} \sum_{i=j}^{2j} \alpha_j f[\tau_1^{2j}, \ldots, \tau_n^{2j}] \right\},
\]
where
\[ \alpha_j = (-1)^j \frac{(1/2)_j}{2j 4^{(n-1)j}}, \quad j=1,2,\ldots, \]
and \( g[y_1^r, \ldots, y_m^r] \) designate the divided difference of the function \( g \), where each \( y_j \) is repeated \( r \) times.
For $s=1$, the solution of the Turán problem XXVI is given by

$$A_0, \nu = \frac{\pi}{n}, \quad A_1, \nu = -\frac{\pi \nu}{4n^3}, \quad A_2, \nu = \frac{\pi}{4n^3} (1 - \nu^2).$$

In 1975 R.D. Riess [15], and in 1984 A.K. Varma [19], using very different methods, obtained the explicit solution of the Turán problem for $s=2$:

$$A_0, \nu = \frac{\pi}{n}, \quad A_1, \nu = -\frac{\pi \nu}{64n^5} (20n^2 - 1), \quad A_2, \nu = \frac{\pi}{64n^5} [3 + (20n^2 - 7)(1 - \nu^2)],$$

$$A_3, \nu = -\frac{6\pi \nu}{64n^5} (1 - \nu^2), \quad A_4, \nu = \frac{\pi}{64n^5} (1 - \nu^2)^2.$$ 

Notice that (5.1), for the Chebyshev weight, reduces to

$$A_{2s}, \nu = \frac{\pi}{4^n n^{2s+1} (s!)^2} (1 - \nu^2)^s, \quad \nu = 1, \ldots, n.$$

One simple answer to Turán question was given by O. Kis [10]. His result can be stated in the following form: If $g$ is an even trigonometric polynomial of degree at most $2(s+1)n-1$, then

$$\int_0^{\pi} g(\theta) d\theta = \frac{\pi}{n(s!)^2} \sum_{j=0}^{s} \frac{S_{j/2}}{4^n n^{2j}} \sum_{\nu=1}^{n} g(2j) \left(\frac{2\nu-1}{2n}\right),$$

where $S_{s-j}$ $(j=0, 1, \ldots, s)$ denotes the symmetric elementary polynomials with respect to the numbers $1^2, 2^2, \ldots, s^2$, i.e.,

$$S_s = 1, \quad S_{s-1} = 1^2 + 2^2 + \ldots + s^2, \ldots, \quad S_0 = 1^2 2^2 \ldots s^2.$$ 

Consequently,

$$\int_{-1}^{1} (1-t^2)^{-1/2} f(t) dt = \frac{\pi}{n(s!)^2} \sum_{j=0}^{s} \frac{S_{j/2}}{4^n n^{2j}} \sum_{\nu=1}^{n} \left[ D_{2j}^2 f(\cos \theta) \right]_\theta = \frac{2\nu-1}{2n} \pi.$$
Using the expansion

\[ D^{2k} f(\cos \theta) = \sum_{i=1}^{2k} a_{k,i}(t) f^{(i)}(t), \quad \cos \theta = t, \quad k > 0, \]

where the functions \( a_{i,j} = a_{i,j}(t) \) are given recursively by

\[ a_{k+1,1} = (1-t^2)a_{k,1} - t a'_{k,1} \]
\[ a_{k+1,2} = (1-t^2)a_{k,2} - t a'_{k,2} + 2(1-t^2)a'_{k,1} - t a_{k,1} \]
\[ a_{k+1,i} = (1-t^2)a_{k,i} - t a'_{k,i} + 2(1-t^2)a'_{k,i-1} - t a_{k,i-1} + (1-t^2)a_{k,i-2} \]

\( (k = 3, \ldots, 2k) \),

\[ a_{k+1,2k+2} = (1-t^2)a_{k,2k} - t a_{k,2k} + (1-t^2)a_{k,2k-1} \]
\[ a_{k+1,2k+3} = (1-t^2)a_{k,2k+1} \]

with \( a_{1,1} = -t \) and \( a_{1,2} = 1-t^2 \), we obtain the formula (5.3). For example, when \( s = 3 \), we have

\[ A_{0,1} = \frac{\pi}{n}, \quad A_{1,1} = \frac{\pi \tau}{2304n^2} (784n^4 + 56n^2 - 1), \]
\[ A_{2,1} = \frac{\pi}{2304n^2} \left\{ (784n^4 - 392n^2 + 31)(1-\tau^2) + 168n^2 - 15 \right\}, \]
\[ A_{3,1} = -\frac{\pi \tau}{2304n^2} \left\{ (336n^2 - 89)(1-\tau^2) + 15 \right\}, \]
\[ A_{4,1} = \frac{\pi}{2304n^2} \left\{ (56n^2 - 65)(1-\tau^2)^2 + 45(1-\tau^2) \right\}, \]
\[ A_{5,1} = -\frac{\pi \tau}{2304n^2} \left\{ 674(1-\tau^2)^2 - 240(1-\tau^2) \right\}, \quad A_{6,1} = \frac{\pi}{2304n^2} (1-\tau^2)^3. \]

To conclude, we mention the corresponding formula (5.2) for the Chebyshev weight,

\[ (5.4) \quad \int_{-1}^{1} g(t) \frac{n^{2s}}{\sqrt{1-t^2}} dt = \frac{\pi}{48n} \left\{ \sum_{\nu=1}^{n} g(\tau_\nu) + \mathbb{R}_n(g) \right\}, \]
where \( T_v = \cos(2v-1) \frac{\pi}{2n} \), \( v=1,\ldots,n \). Note that all weights are equal, that is, the formula (5.4) is one of Chebyshev type.

The last formula can be reduced to a "cosinus" formula

\[
\int_{-1}^{1} f(\cos x) \cos^2 (nx) \, dx = \frac{\pi}{n} \frac{2^8}{8} \sum_{v=1}^{n} f(\cos(2v-1) \frac{\pi}{2n}) + R_n (f),
\]

where \( R_n (f) = 0 \) if \( f \in \mathbb{P}_2(s+1)n-1 \).

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REFERENCES


