CONSTRUCTION OF s - ORTHOGONAL POLYNOMIALS AND TURÁN QUADRATURE FORMULAE

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ABSTRACT: *A connection between Turán quadratures and s-orthogonal polynomials with respect to a nonnegative measure on the real line* $\mathbb{R}$ *is given. Using a discretized Stieltjes procedure and the Newton-Kantorowič method, an iterative method with quadratic convergence for the construction of s-orthogonal polynomials is formulated. Some numerical examples are included. Finally, some considerations about Turán quadrature formulae with Chebyshev measure are given.*

1. INTRODUCTION

In 1950 P. Turán investigated numerical quadratures of the type

$$\int_{-1}^{1} f(t)\,dt = \sum_{\nu=1}^{n} \sum_{i=0}^{k-1} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,k}(f), \tag{1.1}$$

where

$$A_{i,\nu} = \int_{-1}^{1} \ell_{\nu,i}(t)\,dt \qquad (\nu=1,\ldots,n;\ i=0,1,\ldots,k-1)$$

and $\ell_{\nu,i}(t)$ are the fundamental functions of Hermite interpolation. The $A_{i,\nu}$ are Cotes numbers of higher order. The formula (1.1) is exact if f is a polynomial of degree at most kn-1 and the points $-1 \leq \tau_1 < \tau_2 < \ldots < \tau_n \leq 1$ are arbitrary.

For k=1 the formula (1.1), i.e.,

$$\int_{-1}^{1} f(t)\,dt = \sum_{\nu=1}^{n} A_{0,\nu} f(\tau_\nu) + R_{n,1}(f),$$

can be exact for all polynomials of degree $\leq 2n-1$ if the nodes τ_ν are the zeros of the Legendre polynomial P_n. That is the well-known Gauss-Legendre quadrature.

Because of the theorem of Gauss it is natural to ask whether knots τ_ν can be chosen so that the quadrature formula (1.1) will be exact for polynomials of degree not exceeding $(k+1)n-1$. P. Turán [17] showed that the answer is negative for $k=2$, and for $k=3$ it is positive. He proved that the knots τ_ν should be chosen as the zeros of the monic polynomial $\pi_n^*(t)=t^n+\ldots$ which minimizes the following integral

$$\int_{-1}^{1} \pi_n(t)^4 dt,$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$.

More generally, the answer is negative for even, and positive for odd k, and then τ_ν are the zeros of the polynomial minimizing

$$\int_{-1}^{1} \pi_n(t)^{k+1} dt. \tag{1.2}$$

For $k=1$, π_n is the monic Legendre polynomial P_n.

Because of the above, we put $k=2s+1$. Instead of (1.1), it is also interesting to investigate the analogous formula with a weight function $t \to p(t)$,

$$\int_{-1}^{1} f(t)p(t)dt = \sum_{i=0}^{2s} \sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f),$$

or more generally, with some nonnegative measure $d\lambda(t)$ on the real line $\mathbb{R}$,

$$\int_{\mathbb{R}} f(t)d\lambda(t) = \sum_{i=0}^{2s} \sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f). \tag{1.3}$$

This paper is organized as follows. In Section 2 we give a connection between Turán quadratures and s-orthogonal polynomials, which were studied extensively by several Italian mathematicians [12], [7], [13], [14]. Also, in this section we mention a recent method of Vincenti [20] for the computation of the coefficients of s-orthogonal polynomials with respect to an even function. In Section 3 we develop a new method for the numerical construction of s-orthogonal polynomials with respect to an arbitrary weight function. Numerical examples are given in Section 4. Section 5 deals with Turán quadratures with Chebyshev measure.

2. TURAN QUADRATURES AND s-ORTHOGONAL POLYNOMIALS

We consider the Turán quadrature formula (1.3), where $d\lambda(t)$ is a nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k d\lambda(t)$, $k=0,1,\ldots$, exist and are finite, and $\mu_0 > 0$. The formula (1.3) must be exact for all polynomials of degree at most $2(s+1)n-1$. The role of the integral (1.2) is taken over by

$$F = \int_{\mathbb{R}} \pi_n(t)^{2s+2} d\lambda(t),$$

where $F \equiv F(a_0,\ldots,a_{n-1})$, $\pi_n(t) = \sum_{k=0}^{n} a_k t^k$, $a_n = 1$. In order to minimize F we must have

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k d\lambda(t) = 0, \qquad k=0,1,\ldots,n-1. \tag{2.1}$$

Usually, instead of $\pi_n(t)$ we write $P_{s,n}(t)$.

The case $d\lambda(t) = p(t)dt$ on $[a,b]$ has been considered by the Italian mathematicians A.Ossicini [12], A.Ghizzetti and A.Ossicini [7], S.Guerra [8], [9]. It is known that there exists a unique

$P_{s,n}(t) = \prod_{\nu=1}^{n} (t-\tau_\nu)$, whose zeros τ_ν are real, distinct and located in the interior of the interval $[a,b]$. These polynomials are known as s-orthogonal (or s-self associated) polynomials in the interval $[a,b]$ with respect to the weight function p.

For s=0 we have the standard case of orthogonal polynomials. The case when s>0 is very difficult. It requires the use of a method with special numerical treatment.

Recently G.Vincenti [20] has considered an iterative process to compute the coefficients of s-orthogonal polynomials in a special case when the interval $[a,b]$ is symmetric with respect to origin, say, $[-b,b]$, and the weight function p is an even function $p(-t)=p(t)$. Then $P_{s,n}(-t) = (-1)^n P_{s,n}(t)$. He considered two cases: when n is even and when n is odd.

In the first case n=2m, $P_{s,n}(t) = \sum_{i=0}^{m} a_i t^{2m-2i}$, $a_0 = 1$. From (2.1) Vincenti obtained a nonlinear system of equations of the form

$$\sum_{i=0}^{m} C_{m+r-i} a_i = -C_{m+r} \qquad (r=0,1,\ldots,m-1),$$

where

$$C_j^{(0)} = \int_0^b p(t) t^{2j} dt, \qquad C_j^{(h)} = \sum_{p,q=0}^{m} C_{j+2m-p-q}^{(h-1)} a_p a_q,$$

and $C_j^{(s)} \equiv C_j$. Then he has solved this system by some iterative method like Newton's method. For n=2m+1, a similar system of equations was obtained.

Vincenti applied his process to the Legendre case. When n and s increase, the process becomes ill-conditioned. So, the author gave numerical results in the following cases: n=2,3, $1 \leq s \leq 10$; n=4,5, $1 \leq s \leq 5$; n=6,7, $1 \leq s \leq 3$; n=8,9, $1 \leq s \leq 2$; n=10,11, s=1. The results were obtained with 18 correct decimal digits, but using an arithmetic with 36 decimal digits.

From (2.1) we can see that this procedure needs the first $2(s+1)n$ moments of the weight function: $\mu_0, \mu_1, \ldots, \mu_{2(s+1)n-1}$. We see that $C_j^{(0)} = \mu_{2j}/2$. Of course, in this special case, the moments of odd order are zero. Here, we have a nonlinear map $V_{n,s}: \mathbb{R}^{2(s+1)n} \to \mathbb{R}^n$, given by $[\mu_0, \mu_1, \ldots, \mu_{2(s+1)n-1}]^T \to [a_0, a_1, \ldots, a_{n-1}]^T$. The problem itself is highly sensitive to small perturbations in the moments, so that any algorithm which theoretically solves the problem using the moments will be subject to severe growth of errors when executed in an arithmetic of finite precision ([4],[5]). It would be useful to find a numerical condition number of the map $V_{n,s}$, but that will not be our aim here.

3. CONSTRUCTION OF s-ORTHOGONAL POLYNOMIALS

In this section we will give a stable procedure for the numerical construction of s-orthogonal polynomials with respect to $d\lambda(t)$ on $\mathbb{R}$. Namely, we will reduce our problem to the standard theory of orthogonal polynomials, and then we will use the Stieltjes procedure ([3],[5]). The main idea is an interpretation of the "orthogonality conditions" (2.1), i.e.,

$$\int_{\mathbb{R}} \pi_n(t) t^k \pi_n(t)^{2s} d\lambda(t) = 0, \qquad k=0,1,\ldots,n-1.$$

For given n and s, we put $d\mu(t) = d\mu^{s,n}(t) = (\pi_n(t))^{2s} d\lambda(t)$. These conditions can be interpreted as

$$\int_{\mathbb{R}} \pi_k^{s,n}(t) t^{\nu} d\mu(t) = 0, \qquad \nu=0,1,\ldots,k-1,$$

where $(\pi_k^{s,n})$ is a sequence of monic orthogonal polynomials with respect to the new measure $d\mu(t)$. Of course, $P_{s,n}(.) = \pi_n^{s,n}(.)$.

As we can see, the polynomials $\pi_k^{s,n}$, $k=0,1,\ldots$, are implicitly defined, because the measure $d\mu(t)$ depends of $\pi_n^{s,n}(t)$. The general class of such polynomials was introduced by H.Engels (see [2, pp. 214-226]).

We will write only $\pi_k(.)$ instead of $\pi_k^{s,n}(.)$. These polynomials satisfy a three-term recurrence relation

$$\text{(3.1)} \qquad \pi_{k+1} = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k=0,1,\ldots,$$

$$\pi_{-1}(t) = 0, \quad \pi_0(t) = 1,$$

where, because of orthogonality,

$$\alpha_k = \alpha_k(s,n) = \frac{\langle t\pi_k, \pi_k\rangle}{\langle \pi_k, \pi_k\rangle} = \frac{\int_{\mathbb{R}} t\pi_k^2(t)\,d\mu(t)}{\int_{\mathbb{R}} \pi_k^2(t)\,d\mu(t)},$$

(3.2)

$$\beta_k = \beta_k(s,n) = \frac{\langle \pi_k, \pi_k\rangle}{\langle \pi_{k-1}, \pi_{k-1}\rangle} = \frac{\int_{\mathbb{R}} \pi_k^2(t)\,d\mu(t)}{\int_{\mathbb{R}} \pi_{k-1}^2(t)\,d\mu(t)},$$

and, for example, $\beta_0 = \int_{\mathbb{R}} d\mu(t)$.

The coefficients α_k and β_k are the fundamental quantities in the constructive theory of orthogonal polynomials. They provide a compact way of representing orthogonal polynomials, requiring only a linear array of parameters. The coefficients of orthogonal polynomials, or their zeros, in contrast need two-dimensional arrays.

Finding the coefficients α_k, β_k $(k=0,1,\ldots,n-1)$ gives us access to the first n+1 orthogonal polynomials $\pi_0, \pi_1, \ldots, \pi_n$. Of course, for a given n, we are interested only in the last of them i.e., π_n $(\equiv \pi_n^{s,n})$. So, for $n=0,1,\ldots$, the diagonal (boxed) elements

In the following table are our s-orthogonal polynomials $\pi_n^{s,n}$.

TABLE 3.1

n	$d\mu^{s,n}(t)$	Orthogonal Polynomials			
0	$(\pi_0^{s,0}(t))^{2s}d\lambda(t)$	$\boxed{\pi_0^{s,0}}$			
1	$(\pi_1^{s,1}(t))^{2s}d\lambda(t)$	$\pi_0^{s,1}$	$\boxed{\pi_1^{s,1}}$		
2	$(\pi_2^{s,2}(t))^{2s}d\lambda(t)$	$\pi_0^{s,2}$	$\pi_1^{s,2}$	$\boxed{\pi_2^{s,2}}$	
3	$(\pi_3^{s,3}(t))^{2s}d\lambda(t)$	$\pi_0^{s,3}$	$\pi_1^{s,3}$	$\pi_2^{s,3}$	$\boxed{\pi_3^{s,3}}$
⋮					

A stable procedure for finding the coefficients α_k, β_k is the discretized Stieltjes procedure, especially for infinite intervals of orthogonality (see Gautschi [5], and Gautschi, Milovanović [6]). Unfortunately, in our case this procedure cannot be used directly, because the measure $d\mu(t)$ involves an unknown polynomial $\pi_n^{s,n}$. Consequently, we consider the system of nonlinear equations

$$f_0 \equiv \beta_0 - \int_{\mathbb{R}} \pi_n^{2s}(t)\,d\lambda(t) = 0,$$

$$(3.3) \qquad f_{2k+1} \equiv \int_{\mathbb{R}} (\alpha_k - t)\pi_k^2(t)\pi_n^{2s}(t)\,d\lambda(t) = 0, \quad k=0,1,\ldots,n-1,$$

$$f_{2k} \equiv \int_{\mathbb{R}} (\beta_k\pi_{k-1}^2(t) - \pi_k^2(t))\pi_n^{2s}(t)\,d\lambda(t) = 0, \quad k=1,\ldots,n-1,$$

which follows from (3.2).

Let x be a $(2n)$-dimensional column vector with components $\alpha_0, \beta_0, \ldots, \alpha_{n-1}, \beta_{n-1}$ and $f(x)$ a $(2n)$-dimensional vector with components $f_0, f_1, \ldots, f_{2n-1}$, given by (3.3). If $W = W(x)$ is the corresponding Jacobi matrix of $f(x)$, then we can apply Newton-Kantorovič's method

$$(3.4) \qquad x^{[\nu+1]} = x^{[\nu]} - W^{-1}(x^{[\nu]})\, f(x^{[\nu]}), \qquad \nu = 0,1,\ldots,$$

for determining the coefficients of the recurrence relation (3.1) Starting with a reasonable good approximation $x^{[0]}$, the convergence of the method (3.4) is quadratic.

It is interesting that the elements of Jacobi matrix can be easily computed in the following way:

First, we have to determine the partial derivatives $a_{k,i} = \frac{\partial \pi_k}{\partial \alpha_i}$ and $b_{k,i} = \frac{\partial \pi_k}{\partial \beta_i}$. Differentiating the recurrence relation (3.1) with respect to α_i and β_i we obtain

$$a_{k+1,i} = (t - \alpha_k) a_{k,i} - \beta_k a_{k-1,i},$$

and

$$b_{k+1,i} = (t - \alpha_k) b_{k,i} - \beta_k b_{k-1,i},$$

where

$$a_{k,i} = 0, \quad b_{k,i} = 0, \quad k \leq i,$$

$$a_{i+1,i} = -\pi_i(t), \quad b_{i+1,i} = -\pi_{i-1}(t).$$

These relations are the same as those for π_k, but with other initial values. The elements of the Jacobi matrix are

$$\frac{\partial f_{2k+1}}{\partial \alpha_i} = 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)\left[(\alpha_k - t) p_{k,i}(t) + \frac{1}{2}\delta_{ki}\pi_k^2(t)\pi_n(t)\right] d\lambda(t),$$

$$\frac{\partial f_{2k+1}}{\partial \beta_i} = 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)(\alpha_k - t) q_{k,i}(t) d\lambda(t),$$

$$(3.5) \quad \frac{\partial f_{2k}}{\partial \alpha_i} = 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)(\beta_k p_{k-1,i}(t) - p_{k,i}(t)) d\lambda(t),$$

$$\frac{\partial f_{2k}}{\partial \beta_i} = 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)\{(\beta_k q_{k-1,i}(t) - q_{k,i}(t)) + \frac{1}{2}\delta_{ki}\pi_{k-1}^2(t)\pi_n(t)\} d\lambda(t),$$

here $p_{k,i}(t) = \pi_k(t)(a_{k,i}\pi_n(t) + sa_{n,i}\pi_k(t))$ and $q_{k,i}(t) =$ $_k(t)(b_{k,i}\pi_n(t) + sb_{n,i}\pi_k(t))$, and δ_{ki} is Kronecker's delta.

All of the above integrals in (3.3) and (3.5) can be found xactly, except for rounding errors, by using a Gauss-Christoffel ıadrature formula with respect to the measure $d\lambda(t)$,

$$(3.6) \quad \int_{\mathbb{R}} g(t) d\lambda(t) = \sum_{k=1}^{N} A_k^{(N)} g(\tau_k^{(N)}) + R_N(g),$$

aking $N=(s+1)n$ knots. This formula is exact for all polynomials f degree at most $2N-1 = 2(s+1)n - 1 = 2(n-1) + 2ns + 1$.

Thus, for all calculations we use only the fundamental three-erm recurrence relation and the Gauss-Christoffel quadrature 3.6). As initial values $\alpha_k^{[0]} = \alpha_k^{[0]}(s,n)$ and $\beta_k^{[0]} = \beta_k^{[0]}(s,n)$ we tae the values obtained for $n-1$, i.e. $\alpha_k^{[0]} = \alpha_k(s,n-1)$, $\beta_k^{[0]} = \beta_k(s,n-1)$, $\leq n-2$. For α_{n-1} and β_{n-1} we use the corresponding extrapolated alues.

In the case $n=1$ we solve the equation

$$\Phi(\alpha_0) = \Phi(\alpha_0(s,1)) = \int_{\mathbb{R}} (t-\alpha_0)^{2s+1} d\lambda(t) = 0,$$

and then determine

$$\beta_0 = \beta_0(s,1) = \int_{\mathbb{R}} (t-\alpha_0)^{2s} d\lambda(t).$$

4. NUMERICAL EXAMPLES

We will consider two examples, involving Laguerre and Legendre measures.

Example 4.1. $d\lambda(t) = e^{-t}dt$ on $(0,\infty)$.

Using the presented method, we determined the recursion coefficients $\alpha_k(s,n)$ and $\beta_k(s,n)$, $k=0,1,\dots,n-1$, for $s = 1(1)5$ and $n = 1(1)10$. These coefficients and zeros of $\pi_n^{s,n}$, $\tau_k(s,n)$, $k=1,\dots,n$, for some selected values of s and n, are given in Table 4.1. Numbers in parentheses denote decimal exponents.The zeros $\tau_k(s,n)$, $k=1,\dots,n$, were obtained as eigenvalues of the (symmetric tridiagonal) Jacobi matrix

$$J_n = \begin{bmatrix} \alpha_0(s,n) & \sqrt{\beta_1(s,n)} & & & O \\ \sqrt{\beta_1(s,n)} & \alpha_1(s,n) & \sqrt{\beta_2(s,n)} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}(s,n)} \\ O & & & \sqrt{\beta_{n-1}(s,n)} & \alpha_{n-1}(s,n) \end{bmatrix},$$

using the QR algorithm.

Example 4.2. $d\lambda(t) = dt$ on $(-1,1)$. In this (Legendre) cas the coefficients $\alpha_k(s,n)$ are equal to zero, so the computatio can be simplified. The system of equations (3.3) becomes

$$g_0 = f_0 = \beta_0 - \int_{-1}^{1} \pi_n^{2s}(t)dt = 0,$$

$$g_k = f_{2k} = \int_{-1}^{1} (\beta_k\pi_{k-1}^2(t) - \pi_k^2(t))\pi_n^{2s}(t)dt = 0, \quad k=1,\dots,n$$

TABLE 4.1

(s,n)	k	$\alpha_k(s,n)$	$\beta_k(s,n)$	$\tau_{k+1}(s,n)$
(1,5)	0	1.53297437454020(0)	1.95429735674308(6)	3.8619211523014(-1)
	1	5.58879530809235(0)	3.09769990936949(0)	2.5326808971664(0)
	2	9.67825960904726(0)	1.44873867444755(1)	6.8055964648137(0)
	3	1.38195768909663(1)	3.44094720328124(1)	1.3770543148954(1)
	4	1.79144743230187(1)	6.31554867162230(1)	2.5039067879500(1)
(1,10)	0	1.51947559720794(0)	1.15245141095965(18)	1.9845989648554(-1)
	1	5.54285984605682(0)	3.04910058102535(0)	1.2852724641604(0)
	2	9.57433648078956(0)	1.42156585447179(1)	3.3633782573586(0)
	3	1.36134600304330(1)	3.35373242373804(1)	6.4866460030154(0)
	4	1.76626196755034(1)	6.10680235778704(1)	1.0743607524688(1)
	5	2.17262963633088(1)	9.68925818155147(1)	1.6274303555431(1)
	6	2.58125737578751(1)	1.41154607302825(2)	2.3303521691882(1)
	7	2.99352863173826(1)	1.94114444641607(2)	3.2216061440735(1)
	8	3.40937486293287(1)	2.56171126328495(2)	4.3764898673766(1)
	9	3.80755964787433(1)	3.26547484073315(2)	5.9920103669108(1)
(2,5)	0	3.06241261660323(0)	1.11900724691562(16)	5.1108081782716(-1)
	1	8.17357215072018(0)	6.27220780166492(0)	3.6504048515689(0)
	2	1.43542025111386(1)	3.14187808183856(1)	1.0011553444478(1)
	3	2.06411614818251(1)	7.61775799352481(1)	2.0452776123775(1)
	4	2.68361238086797(1)	1.41467716850165(2)	3.7441657331318(1)
(3,5)	0	2.58905931144849(0)	5.71776101144993(27)	6.3593164870754(-1)
	1	1.07564139072170(1)	1.05185172722828(1)	4.7669589415140(0)
	2	1.90289971948242(1)	5.47855138833478(1)	1.3215882166030(1)
	3	2.74628973371076(1)	1.34292199752058(2)	2.7133552841620(1)
	4	3.57594914086306(1)	2.50922121763235(2)	4.9844533561357(1)
(4,5)	0	3.11368201971988(0)	6.65045548992180(40)	7.6048752765420(-1)
	1	1.33381242208130(1)	1.58333974393260(1)	5.8827138815968(0)
	2	2.37031258589862(1)	8.45825858503624(1)	1.6419218171525(1)
	3	3.42845650702239(1)	2.08746777684076(2)	3.3813401673707(1)
	4	4.46834996793661(1)	3.91510787488863(2)	6.2247175594627(1)
(5,5)	0	3.63680292296229(0)	8.46508537128994(54)	8.8474548516636(-1)
	1	1.59190911806156(1)	2.22147113900203(1)	6.9978980073417(0)
	2	2.83768214565758(1)	1.20806800183997(2)	1.9621882995226(1)
	3	4.11061379266411(1)	2.99537220959448(2)	4.0492610101210(1)
	4	5.36078046528124(1)	5.63228814211952(2)	7.4649521550663(1)

Table 4.2 shows the numerical results for $s=1,3,5$ and $n=3,5,10$. The corresponding zeros $\tau_\nu(s,n)$, $k=1,\dots,n$, are given in Table 4.3.

TABLE 4.2

n	ν	$\beta_\nu(1,n)$	$\beta_\nu(3,n)$	$\beta_\nu(5,n)$
3	0	0.483864899809040(-1)	0.999799077102820(-4)	0.284169237312933(-6)
	1	0.396390615424778	0.438361519822241	0.455125737914133
	2	0.266920571579793	0.262372968797798	0.259637334393080
5	0	0.313354730979678(-2)	0.264465724288258(-7)	0.301618113315945(-13)
	1	0.397514379556632	0.440125755974452	0.456936553362545
	2	0.266421480435867	0.261489083023563	0.258693332791772
	3	0.256509353896241	0.254475851257394	0.253414689828449
	4	0.253674592138278	0.252629769731300	0.252061944536419
10	0	0.314536690060498(-5)	0.261903853328827(-16)	0.290667534992279(-27)
	1	0.398771414276302	0.442152192689833	0.459032427879297
	2	0.266409589288295	0.261261065487811	0.258382986818575
	3	0.256307280251967	0.254101849534999	0.253013674028616
	4	0.253361155621508	0.252167886595534	0.251600165348871
	5	0.252110174900276	0.251373736923891	0.251025244228691
	6	0.251467087710631	0.250973891152692	0.250737300372080
	7	0.251096334167793	0.250747641830448	0.250575234680807
	8	0.250866757894766	0.250611009696396	0.250478257846294
	9	0.250718964459874	0.250526857803099	0.250419693077896

TABLE 4.3

n	ν	$\tau_\nu(1,n)$	$\tau_\nu(3,n)$	$\tau_\nu(5,n)$
3	1,3	±0.81443918557776	±0.83709885235857	±0.84543661637477
	2	0.	0.	0.
5	1,5	±0.92711786960989	±0.93810619284349	±0.94197468869998
	2,4	±0.56086741916164	±0.57330378590709	±0.57774579736053
	3	0.	0.	0.
10	1,10	±0.98066259593659	±0.98398991804138	±0.98512298236202
	2,9	±0.87750022098482	±0.88396182054293	±0.88618806147381
	3,8	±0.69262442514005	±0.69957700233546	±0.70197668437523
	4,7	±0.44320099195064	±0.44838741280314	±0.45017897460267
	5,6	±0.15247058767942	±0.15437687188524	±0.15503560566469

5. TURÁN QUADRATURES WITH CHEBYSHEV WEIGHT

Now, we will consider again the quadrature formula of Turán (1.3). If we define ω_ν, by

$$\omega_\nu(t) = \left(\frac{\pi_n(t)}{t-\tau_\nu}\right)^{2s+1}, \quad \nu=1,\dots,n,$$

where $\pi_n(t) = \pi_n^{s,n}(t)$ and $\tau_\nu = \tau_\nu(s,n)$, then the coefficients $A_{i,\nu}$ in Turán quadrature (1.3) can be expressed in the form [16]

$$A_{i,\nu} = \frac{1}{i!\,(2s-i)!}\left[D^{2s-i}\,\frac{1}{\omega_\nu(t)}\int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1} - \pi_n(t)^{2s+1}}{x-t}\,d\lambda(x)\right]_{t=\tau_\nu},$$

where D is the standard differentiation operator. Especially, for i=2s, we have

$$A_{2s,\nu} = \frac{1}{(2s)!\,(\pi_n'(\tau_\nu))^{2s+1}}\int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1}}{t-\tau_\nu}\,d\lambda(x),$$

i.e.,

$$(5.1)\qquad A_{2s,\nu} = \frac{B_\nu^{(s)}}{(2s)!\,(\pi_n'(\tau_\nu))^{2s}}, \qquad \nu=1,\dots,n,$$

where $B_\nu^{(s)}$ are the Christoffel numbers of the following quadrature (with respect to the measure $d\mu(t) = \pi_n^{2s}(t)\,d\lambda(t)$)

$$(5.2)\qquad \int_{\mathbb{R}} g(t)\,d\mu(t) = \sum_{\nu=1}^{n} B_\nu^{(s)} g(\tau_\nu) + R_n(g), \quad R_n(\mathbb{P}_{2n-1}) = 0,$$

So we have $A_{2s,\nu} > 0$.

The expressions for the other coefficients (i<2s) become very complicated.

For the numerical calculation we can use a triangular system of linear equations obtained from the formula (1.3) by replacing f with the Newton polynomials: $1,\ t-\tau_1,\ \ldots,\ (t-\tau_1)^{2s+1}$, $(t-\tau_1)^{2s+1}(t-\tau_2),\ldots,\ (t-\tau_1)^{2s+1}(t-\tau_2)^{2s+1}\ldots(t-\tau_n)^{2s}$.

Particularly interesting is the case of the Chebyshev weight

$$p(t) = (1-t^2)^{-1/2}.$$

In 1930, S. Bernstein [1] showed that $2^{1-n}T_n(t)$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\,dt, \qquad k \geq 0.$$

So the Turån-Chebyshev formula

$$(5.3)\quad \int_{-1}^{1} (1-t^2)^{-1/2} f(t)\,dt = \sum_{i=0}^{2s}\sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f),$$

with $\tau_\nu = \cos\frac{(2\nu-1)\pi}{2n}$, $\nu=1,\ldots,n$, is exact for polynomials of degree not exceeding $2(s+1)n-1$. Turån has stated a problem of explicit determination of $A_{i,\nu}$ and its asymptotic behavior as $n\to\infty$ (Problem XXVI in [18]). In this regard, Micchelli and Rivlin ([11]) have proved the following characterization: If $f\in\mathbb{P}_{2(s+1)n-1}$ then

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\,dt = \frac{\pi}{n}\left\{ \sum_{\nu=1}^{n} f(\tau_\nu) + \sum_{j=1}^{s} \alpha_j f'[\tau_1^{2j},\ldots,\tau_n^{2j}] \right\},$$

where

$$\alpha_j = (-1)^j \frac{\binom{-1/2}{j}}{2j\,4^{(n-1)j}}, \qquad j=1,2,\ldots,$$

and $g[y_1^r,\ldots,y_m^r]$ designate the devided difference of the function g, where each y_j is repeated r times.

For s=1, the solution of the Turán problem XXVI is given by

$$A_{0,\nu}=\frac{\pi}{n}\;,\; A_{1,\nu}=-\frac{\pi\tau_\nu}{4n^3}\;,\quad A_{2,\nu}=\frac{\pi}{4n^3}\,(1-\tau_\nu^2)\,.$$

In 1975 R.D. Riess [15],and in 1984 A.K. Varma[19] , using very different methods, obtained the explicit solution of the Turán problem for s=2:

$$A_{0,\nu}=\frac{\pi}{n}\;,\quad A_{1,\nu}=-\frac{\pi\tau_\nu}{64n^5}\,(20n^2-1)\,,\; A_{2,\nu}=\frac{\pi}{64n^5}\,[3+(20n^2-7)(1-\tau_\nu^2)],$$

$$A_{3,\nu}=-\frac{6\pi\tau_\nu}{64n^5}\,(1-\tau_\nu^2)\,,\quad A_{4,\nu}=\frac{\pi}{64n^5}\,(1-\tau_\nu^2)^2.$$

Notice that (5.1), for the Chebyshev weight, reduces to

$$A_{2s,\nu}=\frac{\pi}{4^s n^{2s+1}(s!)^2}\,(1-\tau_\nu^2)^s,\qquad \nu=1,\ldots,n.$$

One simple answer to Turán question was given by O. Kis [10]. His result can be stated in the following form: If g is an even trigonometric polynomial of degree at most 2(s+1)n-1, then

$$\int_0^\pi g(\theta)\,d\theta=\frac{\pi}{n(s!)^2}\sum_{j=0}^{s}\frac{S_j}{4^j n^{2j}}\sum_{\nu=1}^{n} g^{(2j)}\Big(\frac{2\nu-1}{2n}\pi\Big),$$

where S_{s-j} (j=0,1,...,s) denotes the symmetric elementary polynomials with respect to the numbers $1^2,\ 2^2,\ \ldots,s^2$, i.e. ,

$$S_s=1,\quad S_{s-1}=1^2+2^2+\ldots+s^2,\quad\ldots,\quad S_0=1^2\cdot 2^2\cdots s^2.$$

Consequently,

$$\int_{-1}^{1}(1-t^2)^{-1/2}f(t)\,dt=\frac{\pi}{n(s!)^2}\sum_{j=0}^{s}\frac{S_j}{4^j n^{2j}}\sum_{\nu=1}^{n}\Big[D^{2j}f(\cos\theta)\Big]_{\theta=\frac{2\nu-1}{2n}\pi}\,.$$

Using the expansion

$$D^{2k} f(\cos\theta) = \sum_{i=1}^{2k} a_{k,i}(t) f^{(i)}(t), \qquad \cos\theta = t, \quad k>0,$$

where the functions $a_{i,j} \equiv a_{i,j}(t)$ are given recursively by

$$a_{k+1,1} = (1-t^2)a''_{k,1} - ta'_{k,1},$$

$$a_{k+1,2} = (1-t^2)a''_{k,2} - ta'_{k,2} + 2(1-t^2)a'_{k,1} - ta_{k,1},$$

$$a_{k+1,i} = (1-t^2)a''_{k,i} - ta'_{k,i} + 2(1-t^2)a'_{k,i-1} - ta_{k,i-1} + (1-t^2)a_{k,i-2}$$

$$(k = 3, \ldots, 2k),$$

$$a_{k+1,2k+1} = 2(1-t^2)a'_{k,2k} - ta_{k,2k} + (1-t^2)a_{k,2k-1},$$

$$a_{k+1,2k+2} = (1-t^2)a_{k,2k},$$

with $a_{1,1} = -t$ and $a_{1,2} = 1-t^2$, we obtain the formula (5.3). For example, when s=3, we have

$$A_{0,\nu} = \frac{\pi}{n}, \quad A_{1,\nu} = \frac{\pi\tau_\nu}{2304n^7}(784n^4+56n^2-1),$$

$$A_{2,\nu} = \frac{\pi}{2304n^7}\{(784n^4-392n^2+31)(1-\tau_\nu^2) + 168n^2-15\},$$

$$A_{3,\nu} = -\frac{\pi\tau_\nu}{2304n^7}\{(336n^2-89)(1-\tau_\nu^2) + 15\},$$

$$A_{4,\nu} = \frac{\pi}{2304n^7}\{(56n^2-65)(1-\tau_\nu^2)^2 + 45(1-\tau_\nu^2)\},$$

$$A_{5,\nu} = \frac{\pi\tau_\nu}{2304n^7}\{674(1-\tau_\nu^2)^2 - 240(1-\tau_\nu^2)\}, \quad A_{6,\nu} = \frac{\pi}{2304n^7}(1-\tau_\nu^2)^3.$$

To conclude, we mention the corresponding formula (5.2) for the Chebyshev weight,

$$(5.4) \qquad \int_{-1}^{1} g(t)\frac{\hat{T}_n^{2s}(t)}{\sqrt{1-t^2}}dt = \frac{\pi}{4^{sn}n}\binom{2s}{s}\sum_{\nu=1}^{n} g(\tau_\nu) + R_n(g),$$

where $\tau_\nu = \cos(2\nu-1)\frac{\pi}{2n}$, $\nu=1,\ldots,n$. Note that all weights are equal, that is, the formula (5.4) is one of Chebyshev type.

The last formula can be reduced to a "cosinus" formula

$$\int_0^\pi f(\cos x)\cos^{2s}(nx)\,dx = \frac{\pi}{n4^s}\binom{2s}{s}\sum_{\nu=1}^{n} f\left(\cos(2\nu-1)\frac{\pi}{2n}\right) + R_n(f),$$

where $R_n(f) \equiv 0$ if $f \in \mathbb{P}_{2(s+1)n-1}$.

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REFERENCES

1. S. BERNSTEIN: Sur les polynomes orthogonaux relatifs à un segment fini. J. Math. Pures Appl. (9)9(1930), 127-177.

2. H. ENGELS: Numerical quadrature and cubature. Academic Press, London, 1980.

3. W. GAUTSCHI: Construction of Gauss-Christoffel quadrature formulas. Math. Comp. 22(1968), 251-270.

4. W. GAUTSCHI: A survey of Gauss-Christoffel quadrature formulae. In: E.B. Christoffel - The Influence of his Work on Mathematics and the Physical Sciences (P.L. Butzer and F. Fehér, eds.), Birkhäuser Verlag, Basel, 1981, pp. 72-147.

5. W. GAUTSCHI: On generating orthogonal polynomials. SIAM J. Sci. Statist. Comput. 3(1982), 289-317.

6. W. GAUTSCHI and G.V. MILOVANOVIĆ: Gaussian quadrature involving Einstein and Fermi functions with an application to summation of series. Math. Comp. 44(1985), 177-190.

7. A. GHIZZETTI and A. OSSICINI: Su un nuovo tipo di sviluppo di una funzione in serie di polinomi. Rend. Accad. Naz. Lincei (8)43(1967), 21-29.

8. S. GUERRA: Polinomi generati da successioni peso e teoremi di rappresentazione di una funzione in serie di tali polinomi. Rend. Ist. Mat. Univ. Trieste 8(1976), 172-194.

9. S. GUERRA: Su un determinante collegato ad un sistema di polinomi ortogonali. Rend. Ist. Mat. Univ. Trieste 10(1978), 66-79.

10. O. KIS: Remark on mechanical quadrature *(Russian)*. Acta Math. Acad. Sci. Hungar. 8(1957), 473-476.

11. C.A. MICCHELLI and T.J. RIVLIN: Turàn formulae highest precision quadrature rules for Chebyshev coefficients. IBM J. Res. Develop. 16(1972), 372-379.

12. A. OSSICINI: Costruzione di formule di quadratura di tipo Gaussiano. Ann. Mat. Pura Appl. (4)72(1966), 213-238.

13. A. OSSICINI and F. ROSATI: Funzioni caratteristiche nelle formule di quadratura gaussiane con nodi multipli. Bull. Un. Mat. Ital. (4)11(1975), 224-237.

14. A. OSSICINI and F. ROSATI: Sulla convergenza dei funzionali ipergaussiani. Rend. Mat. (6)11(1978), 97-108.

15. R.D. RIESS: Gauss-Turàn quadratures of Chebyshev type and error formulae. Computing 15(1975), 173-179.

16. D.D. STANCU: Asupra unor formule generale de integrare numerica. Acad. R. P. Romîne. Stud. Cerc. Mat. 9(1958), 209-216.

17. P. TURÁN: On the theory of the mechanical quadrature. Acta Sci. Math. Szeged. 12(1950), 30-37.

18. P. TURÁN: On some open problems of approximation theory. J. Approx. Theory 29(1980), 23-85.

19. A.K. VARMA: On optimal quadrature formulae. Studia Sci. Math. Hungar. 19(1984), 437-446.

20. G. VINCENTI: On the computation of the coefficients of s-orthogonal polynomials. SIAM J. Numer. Anal. 23(1986), 1290-1294.