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ON THE LEAST SQUARES APPROXIMATION WITH CONSTRAINTS

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ABSTRACT:
In this paper a family of functions F, defined by

$$F = \{ f: f(-x) = f(x),\ f(1) = 0,\ f \in L^2[-1,1] \}, \qquad (*)$$

is studied. A least squares approximation method (with Chebyshev weight and additional constraints) of the above mentioned functions is given. The method is illustrated by example of polynomial approximation of the function $f(x) = \cos(\pi x/2)$ *on* $[-1,1]$ *by* $\phi(x)$, *with the constraints* $\phi(1) = f(1)$ *and* $\phi(0) = f(0)$.

O SREDNJE KVADRATNOJ APROKSIMACIJI SA OGRANIČENJIMA. *U radu je proučavana familija funkcija F, definisana pomoću* (*). *Dat je jedan metod srednje kvadratne aproksimacije ovih funkcija korišćenjem Čebiševljeve težinske funkcije i dodatnih ograničenja. Metod je ilustrovan na primeru aproksimacije funkcije* $f(x) = \cos(\pi x/2)$ *na* $[-1,1]$ *pomoću* $\phi(x)$, *sa ograničenjima* $\phi(1) = f(1)$ *i* $\phi(0) = f(0)$.

We introduce a family of real functions

$$F = \{ f : f(-x) = f(x),\ f(1) = 0,\ f \in L^2[-1,1] \}$$

and the inner product of two functions f and g by

$$(f,g) = \int_{-1}^{1} f(x)g(x)w(x)\,dx \qquad (f,g \in L^2[-1,1]), \qquad (1)$$

where $w(x) = (1-x^2)^{\lambda - 1/2}$ $(\lambda > -1/2)$.

Let further Π_{2n} be the set of all real polynomials with a degree not higher that 2n and such that the polynomials belong to the set F.

In paper [1] we considered a general method for the lea t squares approximation for the function $f \in F$ in the class $\P_{2n}$, with respect to the norm $\|f\| = \sqrt{(f,f)}$, where the inner product is defined by (1). Hence the approximation ϕ is defined by (see, also [3])

$$\min_{\phi \in \P_{2n}} \| f - \phi \|, \quad \text{when } f \in F.$$

Also, we considered in [1] some other approximations with general constraints. So, for example, when $f \in F$, we could study the problem

$$\min_{\phi_{2n} \in \P_{2n}} \| f - \phi_{2n} \| \quad \text{subject to} \quad \phi_{2n}(a) = f(a),$$

where $a \in [0,1)$. In order to find the minimum of the "distance" $\| f - \phi_{2n} \|$ under the constraints $\phi_{2n}(1) = 0$ and $\phi_{2n}(a) = f(a)$, we represent ϕ_{2n} as a linear combination of the Gegenbauer polynomials $C_{k,\lambda}(x)$, i.e.,

$$\phi_{2n}(x) = \sum_{k=0}^{n} d_k C_{2k,\lambda}(x),$$

and define D to be

$$D = \int_{-1}^{1} (f(x) - \phi_{2n}(x))^2 w(x)\,dx + \alpha\phi_{2n}(1) + \beta(\phi_{2n}(a) - f(a)),$$

where α and β are Lagrange multipliers, whose values we will determine.

We must have

$$\frac{\partial D}{\partial d_k} = -2(f, C_{2i,\lambda}) + 2h_{2i}d_i + \alpha C_{2i,\lambda}(1) + \beta C_{2i,\lambda}(a) = 0,\ i=0,1,\dots,n,$$

where $h_{2i} = \| C_{2i,\lambda} \|^2 = \frac{\lambda}{2i+\lambda} \cdot \frac{C_{2i,\lambda}(1)}{\Lambda(\lambda)}$ and $\Lambda(\lambda) = \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\ \Gamma(\lambda+\frac{1}{2})}$ (Γ is the gamma function).

Then

$$d_i = \frac{1}{h_{2i}} \{ (f, C_{2i,\lambda}) - \frac{\alpha}{2} C_{2i,\lambda}(1) - \frac{\beta}{2} C_{2i,\lambda}(a) \} \ (i=0,1,\dots,n). \tag{2}$$

The constraints $\phi_{2n}(1) = 0$ and $\phi_{2n}(a) = f(a)$ yield the system of equations

$$A(1,1)\,\frac{\alpha}{2} + A(a,1)\,\frac{\beta}{2} = (f, A(1,x)),$$
$$A(1,a)\,\frac{\alpha}{2} + A(a,a)\,\frac{\beta}{2} = (f, A(a,x)) - f(a), \tag{3}$$

where

$$A(x,t) = \sum_{k=0}^{n} \frac{C_{2k,\lambda}(x)\,C_{2k,\lambda}(t)}{h_{2k}} .$$

We note that $A(x,t) = A(t,x)$.

The explicite solving of equation system (3) in general case is very complicated. In this paper we will consider the case when $\lambda = 0$, i.e. when we have approximation with Chebyshev weight $w(x) = (1-x^2)^{-1/2}$. The method will be illustrated by example of approximation of the function $f(x) = \cos(\pi x/2)$ on $[-1,1]$ with the constraint $\phi_{2n}(0) = f(0) = 1$. By T_k Chebyshev polynomials of the first kind are denoted.

Firstly, we have

(a) $\Lambda(0) = 1/\pi$, $C_{0,0}(x) = T_0(x) = 1$, $h_0 = \|T_0\|^2 = \pi$;

(b) For $i = 1,2,\ldots$

$$T_{2i}(1) = 1, \quad T_{2i}(0) = (-1)^i,$$

$$\lim_{\lambda \to 0} \frac{C_{2i,\lambda}(x)}{\lambda} = \frac{1}{i}\, T_{2i}(x) \qquad \text{(see, e.g. [2,p.81])},$$

$$\lim_{\lambda \to 0} \frac{h_{2i}}{\lambda^2} = \frac{\pi}{2i^2}\, T_{2i}(1) = \frac{\pi}{2i^2} ;$$

(c) $$\lim_{\lambda \to 0} A(x,t) = \frac{2}{\pi} \sum_{i=0}^{n}{}' \frac{T_{2i}(x)\,T_{2i}(t)}{T_{2i}(1)} ,$$

where $\sum_{i=0}^{n}{}' a_i = \frac{1}{2} a_0 + a_1 + \ldots + a_n$.

In this case we have

$$A(1,1) = A(0,0) = (2n+1)/\pi , \quad A(1,0) = A(0,1) = (-1)^n/\pi ,$$

$$A(1,x)=\frac{2}{\pi}\sum_{i=0}^{n}{}' T_{2i}(x),\quad A(0,x)=\frac{2}{\pi}\sum_{i=0}^{n}{}'(-1)^{i}T_{2i}(x).$$

Let $f(x)=\cos(\pi x/2)$. Since $(T_0,f) = J_0(\pi/2)$ and $(T_{2i},f)= =(-1)^{i}J_{2i}(\pi/2)$ $(i=1,2,\ldots)$, where J_m is the Bessel function of the first kind, we have

$$(f,A(1,x))=\frac{2}{\pi}\sum_{i=0}^{n}{}'(T_{2i},f) = J_0(\pi/2)+2\sum_{i=0}^{n}(-1)^{i}J_{2i}(\pi/2).$$

Similarly, we find

$$(f,A(0,x))=J_0(\pi/2)+2\sum_{i=1}^{n}J_{2i}(\pi/2).$$

Now, from (3) we obtain

$$\frac{\alpha}{2}+\frac{\beta}{2}=\pi A_n \qquad\text{and}\qquad \frac{\alpha}{2}-\frac{\beta}{2}=\pi B_n,$$

where

$$A_n=\frac{2n+1-(-1)^n}{4n(n+1)}(2J_0(\pi/2)+4S_n-1),\quad B_n=\frac{2n+1+(-1)^n}{4n(n+1)}(1-4U_n),$$

$$S_1=0,\quad S_n=\sum_{i=1}^{m}J_{2i}(\pi/2),\quad U_n=\sum_{i=1}^{m'}J_{4i-2}(\pi/2),\quad m=\left[\frac{n}{2}\right],\quad m'=\left[\frac{n+1}{2}\right].$$

Let $\phi_{2n}(x) = \sum_{k=0}^{n}d_k C_{2k,\lambda}(x) = d_0C_{0,\lambda}(x)+\sum_{k=1}^{n}(\lambda d_k)\frac{C_{2k,\lambda}(x)}{\lambda}$,

where $\lambda\neq 0$. If $\lambda\to 0$, we have

$$\phi_{2n}(x) = d_0T_0(x)+\sum_{k=1}^{n}a_k^{(n)}T_{2k}(x),$$

where $a_k^{(n)}=\frac{2}{2k}\lim_{\lambda\to 0}(\lambda d_k)$. Also, we put $a_0^{(0)}=d_0$.

From (2) it follows

$$a_0^{(n)}=\frac{(f,T_0)}{h_0}-\frac{1}{h_0}\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=J_0(\pi/2)-A_n,$$

$$a_k^{(n)}=\frac{1}{k}\lim_{\lambda\to 0}\left\{\frac{\lambda^2}{h_{2k}}\left[\left(f,\frac{C_{2k,\lambda}}{\lambda}\right)-\frac{\alpha}{2}\frac{C_{2k,\lambda}(1)}{\lambda}-\frac{\beta}{2}\frac{C_{2k,\lambda}(0)}{\lambda}\right]\right\},$$

i.e.,

$$a_k^{(n)} = \frac{1}{k}\cdot\frac{2k^2}{\pi}\cdot\frac{1}{k}\{(f,T_{2k}) - \frac{\alpha}{2} - (-1)^k \frac{\beta}{2}\}.$$

For k=2i and k=2i-1, we have

$$a_{2i}^{(n)} = 2(J_{4i}(\pi/2) - A_n), \quad a_{2i-1}^{(n)}=-2(J_{4i-2}(\pi/2) - B_n).$$

Now, using the reprezentation of Chebyshev polynomials in terms of hypergeometric functions

$$T_{2k}(x) = {}_2F_1(-k,k,\frac{1}{2};1-x^2) = \sum_{i=0}^{k} (-1)^i \frac{k}{k+i} 4^i \binom{k+i}{2i}(1-x^2)^i,$$

we get

$$\phi_{2n}(x) = \sum_{k=1}^{n} c_{n,k}(1-x^2)^k,$$

where

$$c_{n,k} = (-4)^k \sum_{i=k}^{n} \frac{i}{i+k}\binom{i+k}{2k} a_i^{(n)}.$$

The coefficients $c_{n,k}$ satisfy the constraint $\sum_{k=1}^{n} c_{n,k} = 1.$

Let us notice that the approximation formula

$$\cos(\pi x/2) \approx \sum_{k=1}^{n} c_{n,k}(1-x^2)^k \qquad (x \in [-1,1]) \tag{4}$$

requires n+1 multiplications, n-1 additions and one substraction. Of course, it has been assumed using of Horner scheme.

For determination of coefficients $c_{n,k}$ a FORTRAN program is made. Using this program in Q-arithmetic on a VAX 11/780 computer we determined the coefficients for n=1(1)10. For example, for n=8 we obtained:

$c_{8,1}$ = 0.78539 81633 97447 84132 7376,
$c_{8,2}$ = 0.19634 95408 49376 96189 8180,
$c_{8,3}$ = 0.01742 92582 36237 65674 5118,
$c_{8,4}$ = 0.00080 00973 75076 07291 0125,
$c_{8,5}$ = 0.00002 25060 22557 93209 5682,

$$c_{8,6} = 0.00000\ 04281\ 80281\ 65762\ 0963,$$
$$c_{8,7} = 0.00000\ 00058\ 75704\ 29522\ 5866,$$
$$c_{8,8} = 0.00000\ 00000\ 63317\ 58217\ 6690.$$

In this case, the error in approximation (4) is not large than $4.66 \cdot 10^{-18}$.

In the table the upper bound of error in approximation formula (4) is given for n=1(1)10. Numbers in parentheses indicate decimal exponents.

n	1	2	3	4	5
ε_n	5.60(-2)	9.25(-4)	9.36(-6)	6.12(-8)	2.78(-10)
n	6	7	8	9	10
ε_n	9.23(-13)	2.34(-15)	4.66(-18)	7.50(-21)	9.91(-24)

Here

$$\varepsilon_n = \max_{0 \le x \le 1} \left| \cos(\pi x/2) - \sum_{k=1}^{n} c_{n,k}(1-x^2)^k \right| .$$

R E F E R E N C E S

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