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A NUMERICAL PROCEDURE FOR COEFFICIENTS IN GENERALIZED GAUSS-TURÁN QUADRATURES

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ABSTRACT. *A numerical procedure for the coefficients in the generalized Gauss-Turán quadrature formulas is presented. The corresponding nodes as the zeros of s-orthogonal polynomials can be determined by a stable algorithm given in* [10]. *A numerical example is included.*

1. Introduction

We consider the generalized Gauss-Turán quadrature formula (see [17])

$$\int_{\mathbb{R}} f(t)\,d\lambda(t) = \sum_{i=0}^{2s}\sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R_n(f), \tag{1.1}$$

where $d\lambda(t)$ is a nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments

$$\mu_k = \int_{\mathbb{R}} t^k d\lambda(t), \quad k = 0, 1, \ldots ,$$

exist and are finite, and $\mu_0 > 0$. The formula (1.1) is exact for all polynomials of degree at most $2(s+1)n-1$, i.e.,

$$R_n(f) = 0 \qquad \text{for} \quad f \in \mathcal{P}_{2(s+1)n-1}.$$

The knots τ_ν ($\nu = 1, \ldots, n$) in (1.1) are the zeros of the monic polynomial $\pi_n^s(t)$, which minimizes the following integral

$$\int_{\mathbb{R}} \pi_n(t)^{2s+2}\, d\lambda(t),$$

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where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. This polynomial π_n^s is known as s-orthogonal (or s-self associated) polynomial with respect to the measure $d\lambda(t)$ (for some details see [2–7], [11–13]). For $s = 0$, we have the standard case of orthogonal polynomials, and (1.1) then becomes well-known Gauss-Christoffel formula.

In [10] one of us gave a stable method for numerical constructing s-orthogonal polynomials and obtaining the nodes of the generalized Gauss-Turán quadrature formula (1.1). It was an iterative method with quadratic convergence based on a discretized Stieltjes procedure and the Newton-Kantorovič method.

In this paper, in Section 2, we give a numerical procedure for finding the coefficients $A_{i,\nu}$ in (1.1). An alternative method was given by Stroud and Stancu [16] (see also [15]). A numerical example is given in Section 3.

2. The Coefficients in the Generalized Gauss-Turán Quadrature

Let $\tau_\nu = \tau_\nu(s,n)$, $\nu = 1, \ldots, n$, be the zeros of the s-orthogonal polynomial $\pi_n(t)\,(\equiv p_n^s(t))$. If we define ω_ν by

$$\omega_\nu(t) = \left(\frac{\pi_n(t)}{t-\tau_\nu}\right)^{2s+1}, \qquad \nu = 1, \ldots, n,$$

then the coefficients $A_{i,\nu}$ in the generalized Gauss-Turán quadrature (1.1) can be expressed in the form (see [15])

$$A_{i,\nu} = \frac{1}{i!(2s-i)!}\left[D^{2s-i}\frac{1}{\omega_\nu(t)}\int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1} - \pi_n(t)^{2s+1}}{x-t}\, d\lambda(x)\right]_{t=\tau_\nu},$$

where D is the standard differentiation operator. Especially, for $i = 2s$, we have

$$A_{2s,\nu} = \frac{1}{(2s)!(\pi_n{}'(\tau_\nu))^{2s+1}}\int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1}}{t-\tau_\nu}\, d\lambda(x),$$

i.e.,

$$A_{2s,\nu} = \frac{B_\nu{}^{(s)}}{(2s)!(\pi_n'(\tau_\nu))^{2s}}, \qquad \nu = 1, \ldots, n,$$

where $B_\nu{}^{(s)}$ are the Christoffel numbers of the following Gaussian quadrature (with respect to the measure $d\mu(t) = \pi_n^{2s}(t)d\lambda(t)$),

$$\int_{\mathbb{R}} g(t)\, d\mu(t) = \sum_{\nu=1}^{n} B_\nu^{(s)} g(\tau_\nu) + R_n(g), \qquad R_n(\mathcal{P}_{2n-1}) = 0.$$

Since $B_\nu^{(s)} > 0$, we conclude that $A_{2s,\nu} > 0$. The expressions for the other coefficients ($i < 2s$) become very complicated. For the numerical calculation we can use a tringular system of linear equations obtained from the formula (1.1) by replacing f with the Newton polynomials: $1, t-\tau_1, \ldots, (t-\tau_1)^{2s+1}$, $(t-\tau_1)^{2s+1}(t-\tau_2), \ldots, (t-\tau_1)^{2s+1}(t-\tau_2)^{2s+1}\cdots(t-\tau_n)^{2s}$.

Here, we give a method for the numerical calculation of coefficients of the generalized Gauss-Turán quadrature formula (1.1), starting from the Hermite interpolation problem

$$H_m^{(i)}(\tau_\nu) = f^{(i)}(\tau_\nu),$$

where $\nu = 1, \ldots, n;\ i = 0, 1, \ldots, \alpha_\nu - 1,\ \alpha_1 + \cdots + \alpha_n = m + 1$.

Taking $\alpha_i = 2s + 1$, $i = 1, \ldots, n$ and integrating $f(t) - H_m(t) = r(f;t)$, we obtain

$$\int_{\mathbb{R}} f(t)d\lambda(t) = \sum_{i=0}^{2s}\sum_{\nu=1}^{n} f^{(i)}(\tau_\nu)\int_{\mathbb{R}} l_{\nu,i}(t)d\lambda(t) + R_n(f), \tag{2.1}$$

where

$$l_{\nu,i}(t) = \frac{1}{i\,!}\sum_{k=0}^{2s-i}\frac{1}{k\,!}\left[\frac{(t-\tau_\nu)^{2s+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu}\frac{\Omega(t)}{(t-\tau_\nu)^{2s-i-k+1}},$$

$$\Omega(t) = \left[(t-\tau_1)(t-\tau_2)\cdots(t-\tau_n)\right]^{2s+1},$$

and $R_n(f) = \int_{\mathbb{R}} r(f;t)d\lambda(t)$ is the corresponding remainder term.

Hence, (2.1) becomes the generalized Gauss-Turán quadrature formula (1.1), where $A_{i,\nu}$ are the Cotes numbers of higher order and given by

$$A_{i,\nu} = \int_{\mathbb{R}} l_{\nu,i}(t)d\lambda(t),$$

i.e.,

$$A_{i,\nu} = \frac{1}{i\,!}\sum_{k=0}^{2s-i}\frac{1}{k\,!}\left[\frac{(t-\tau_\nu)^{2s+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu}\int_{\mathbb{R}}\Omega_{\nu,i+k}(t)d\lambda(t),$$

where $i = 0, 1, \ldots, 2s$; $\nu = 1, \ldots, n$ and

$$\Omega_{\nu,j}(t) = \frac{\Omega(t)}{(t-\tau_\nu)^{2s-j+1}} = (t-\tau_\nu)^j\prod_{j\neq\nu}(t-\tau_j)^{2s+1}. \tag{2.2}$$

For $i+k \le 2s$, we can see that $\Omega_{\nu,i+k}(t)$ is a polynomial of degree at most

$$(n-1)(2s+1)+2s = (2s+1)n-1 \le 2(s+1)n-1 = 2N-1,$$

where $N=(s+1)n$.

Hence, the problem of determining of the coefficients of the Gauss-Turán quadrature formula (1.1) is reduced to determination of integrals in (2.2). All of the above integrals in can be found exactly, except for rounding errors, by using a Gauss-Christoffel quadrature formula with respect to the measure $d\lambda(t)$,

$$\int_{\mathbb{R}} g(t)\,d\lambda(t) = \sum_{k=1}^{N} A_k^{(N)} g(\tau_k^{(N)}) + R_N(g), \tag{2.3}$$

taking $N=(s+1)n$ knots. This formula is exact for all polynomials of degree at most $2N-1 = 2(s+1)n-1$.

In order to calculate the derivatives

$$\left[\frac{(t-\tau_\nu)^{2s+1}}{\Omega(t)}\right]_{t=\tau_\nu}^{(k)} \qquad (k=0,1,\ldots,2s;\ \nu=1,\ldots,n), \tag{2.4}$$

we need the following auxiliary result:

Lemma 2.1. *If $g \in C^{(m)}(E)$, $m \in \mathbb{N}_0$, $E \subset \mathbb{R}$, then*

$$(e^g)^{(0)} = e^g, \quad (e^g)^{(p)} = \sum_{l=1}^{p} \binom{p-1}{l-1} g^{(l)} (e^g)^{(p-l)}, \quad p=1,\ldots,m.$$

Proof. Since $(e^g)' = g'e^g$, applying the Leibnitz's formula for the derivative of the product of functions, we have

$$\begin{aligned}(e^g)^{(p)} = (g'e^g)^{(p-1)} &= \sum_{j=0}^{p-1} \binom{p-1}{j} (g')^{(j)} (e^g)^{(p-j-1)} \\ &= \sum_{l=1}^{p} \binom{p-1}{l-1} g^{(l)} (e^g)^{(p-l)},\end{aligned}$$

where $p=1,\ldots,m$. $\square$

Let $a = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = b$, where $[a,b]$ is the smallest closed interval containing $\mathrm{supp}(d\lambda)$, or $a = -\infty$, $b = +\infty$. For $t \in (\tau_{\nu-1}, \tau_{\nu+1})$ we define u_ν by

$$u_\nu(t) = \prod_{i\neq\nu}(t-\tau_\nu)^{-(2s+1)} = (-1)^{n-\nu}\exp\Big[-(2s+1)\sum_{i\neq\nu}\log|t-\tau_i|\Big],$$

i.e., $u_\nu(t) = (-1)^{n-\nu}e^{h_\nu(t)}$, where

$$h_\nu(t) = -(2s+1)\sum_{i\neq\nu} g_i(t), \qquad g_i(t) = \log|t-\tau_i|.$$

Since $g_i^{(j)}(\tau_\nu) = (-1)^{j-1}(j-1)!(\tau_\nu-\tau_i)^{-j}$, $j \geq 1$, we have

$$h_\nu^{(j)}(\tau_\nu) = -(2s+1)(-1)^{j-1}(j-1)!\sum_{i\neq\nu}(\tau_\nu-\tau_i)^{-j}.$$

It is clear that the derivatives in (2.4) are exactly the derivatives of $u_\nu(t)$ in the point $t = \tau_\nu$. Thus, using Lemma 2.1, we can express they in terms of $h_\nu^{(j)}(\tau_\nu)$.

This numerical method for calculating the coefficients $A_{i,\nu}$ can be summarized in the following form:

Proposition 2.2. *Let τ_ν, $\nu = 1,\ldots,n$, be zeros of the s-orthogonal polynom $\pi_n^s(t)$, with respect to the measure $d\lambda(t)$ on $\mathbf{R}$. Then, coefficients of the generalized Gauss-Turán quadrature formula,*

$$\int_{\mathbf{R}} f(t)\,d\lambda(t) = \sum_{i=0}^{2s}\sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f),$$

can be expressed in the form

$$A_{i,\nu} = \frac{1}{i\,!}(-1)^{n-\nu}\sum_{k=0}^{2s-i}\frac{1}{k\,!}\big[e^{h_\nu(t)}\big]^{(k)}_{t=\tau_\nu}\sum_{j=1}^{N} A_j^{(N)}\Omega_{\nu,i+k}(\tau_j^{(N)}),$$

where $A_j^{(N)}$ and $\tau_j^{(N)}$ are weights and nodes of the Gauss-Christoffel quadrature formula (2.3) in $N = (s+1)n$ points, the polynomial $\Omega_{\nu,j}(t)$ is given by (2.2), and $\big[e^{h_\nu(t)}\big]^{(k)}_{t=\tau_\nu}$ is determined by Lemma 2.1.

To conclude this section we mention a particulary interesting case of the Chebyshev mesure $d\lambda(t) = (1-t^2)^{-1/2}dt$. In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\, dt, \qquad k \geq 0.$$

Thus, the Chebyshev-Turán formula

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt = \sum_{i=0}^{2s} \sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R_n(f), \tag{2.5}$$

with $\tau_\nu = \cos \frac{(2\nu-1)\pi}{2n}$, $\nu = 1, \dots, n$ is exact for all polynomials of degree at most $2(s+1)n-1$. Turán has stated a problem of explicit determination of $A_{i,\nu}$ and its behavior as $n \to +\infty$ (see Problem XXVI in [18]). Some characterizations and solution for $s = 2$ were obtained by Micchelli and Rivlin [9], Riess [14], and Varma [19]. One simple answer to Turán question was given by Kis [9].

3. Numerical Example

In this section we give an example when is preferable to use a formula of Turán type instead of the standard Gaussian formula

$$\int_{\mathbb{R}} f(t)\, d\lambda(t) = \sum_{\nu=1}^{n} A_\nu f(t_\nu) + R_n(f), \tag{3.1}$$

for which $R_n(\mathcal{R}_{2n-1}) = 0$. All computations were done on the MICROVAX 3400 computer in Q-arithmetic (machine precision $\approx 1.93 \times 10^{-34}$).

Consider the following simple numerical example

$$I = \int_{-1}^{1} e^t \sqrt{1-t^2}\, dt = 1.775499689212180946878576537\dots\,.$$

Here we have $f(t) = e^t$ and $d\lambda(t) = \sqrt{1-t^2}\, dt$ on $[-1,1]$ (the Chebyshev measure of the second kind). Notice that $f^{(i)}(t) = f(t)$ for every $i \geq 0$.

The Gaussian formula (3.1) and the corresponding Gauss-Turán formula (1.1) give

$$I \approx I_n^G = \sum_{\nu=1}^{n} A_\nu e^{t_\nu} \tag{3.2}$$

and

$$(3.3) \qquad I \approx I_{n,s}^{T} = \sum_{\nu=1}^{n} C_{\nu}^{(s)} e^{\tau_\nu},$$

respectively, where $C_{\nu}^{(s)} = \sum_{i=0}^{2s} A_{i,\nu}$.

Table 3.1 shows the relative errors $|(I_{n,s}^T - I)/I_{n,s}^T|$ for $n = 1(1)5$ and $s = 0(1)5$. (Numbers in parentheses indicate decimal exponents and m.p. is the machine precision.)

TABLE 3.1
Relative errors in quadrature sums $I_{n,s}^T$

n	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
1	1.15(−1)	4.71(−3)	9.72(−5)	1.21(−6)	1.01(−8)	5.98(−11)
2	2.38(−3)	2.05(−7)	3.06(−12)	1.36(−17)	2.40(−23)	1.88(−29)
3	1.97(−5)	1.15(−12)	4.02(−21)	9.26(−31)	m.p.	m.p.
4	8.76(−8)	1.71(−18)	4.68(−31)	m.p.	m.p.	m.p.
5	2.43(−10)	9.40(−25)	m.p.	m.p.	m.p.	m.p.

For $s = 0$ the quadrature formula (3.3) reduces to (3.2), i.e., $I_{n,0}^T \equiv I_n^G$. Notice that Turán formula (3.3) with n nodes has the same degree of exactness as Gaussian formula with $(s+1)n$ nodes.

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