FILOMAT (Niš) 9:1 (1995), 101–111

ON PROPERTIES OF SOME NONCLASSICAL ORTHOGONAL POLYNOMIALS

G. V. Milovanović, P. M. Rajković and Z. M. Marjanović

ABSTRACT. *In this paper we consider some sequences of nonclassical orthogonal polynomials which were studied in the papers* [3], [4] *and* [6]. *We find some new relations which they satisfy and discuss their zeros.*

1. Polynomials of the Laguerre type

We consider the generalized Laguerre functional

$$I^{(s)}(P) = \int_0^{+\infty} P(x)x^s e^{-x}\,dx, \quad s \in \mathbf{N}_0, \quad I^{(0)} = I,$$

and the monic generalized Laguerre polynomials $\{\widehat{L}_n^{(s)}(x)\}$, which satisfy the following three-term recurrence relation

$$\begin{aligned}(1.1) \qquad &\widehat{L}_{n+1}^{(s)}(x) = (x - 2n - s - 1)\widehat{L}_n^{(s)}(x) - n(n+s)\widehat{L}_{n-1}^{(s)}(x),\\ &\widehat{L}_{-1}^{(s)}(x) = 0, \quad \widehat{L}_0^{(s)}(x) = 1.\end{aligned}$$

These polynomials can be expressed in the form

$$(1.2) \qquad \widehat{L}_n^{(s)}(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}(k+s+1)_{n-k}x^k,$$

where

$$(m)_s = m(m+1)\cdots(m+s-1), \qquad (m)_0 = 1.$$

We introduce the functional $\mathbf{\Delta}_a$ by

$$\mathbf{\Delta}_a(f) = f(a),$$

Received 22.07.1995
1991 *Mathematics Subject Classification*: 33C45
Supported by Grant 0401A of RFNS through Math. Inst. SANU

and define the functional I_1 by

$$I_1 = I + c\Delta_0, \quad c \in \mathbb{R}.$$

By $\{L_n(x;c)\}$ we denote the corresponding sequence of orthogonal polynomials of the *Laguerre type*. Such polynomials were expressed as a linear combination of $L_n^{(0)}(x)$ and $xL_n^{(2)}(x)$ (see [6]). In the same paper, it was proved that the n-th polynomial of this sequence can be expressed in the form

$$L_n(x;c) = (-1)^n n! \sum_{k=0}^{n} (-1)^k \frac{1 + k(1 + c(n+1))}{(k+1)!} \binom{n}{k} x^k. \tag{1.3}$$

In this section we prove some properties of these polynomials.

Denoting the monic polynomials of the Laguerre type by $\widehat{L}_n(x;c)$, we yield

$$L_n(x;c) = (1 + nc)\widehat{L}_n(x;c), \quad c \neq -\frac{1}{n}, \quad n \in \mathbb{N}. \tag{1.3'}$$

From (1.3) we see that

$$\deg L_n(x;-1/n) = n - 1 \qquad \text{and} \qquad L_n(x;-1/n) = -nL_{n-1}(x;-1/n).$$

Hence, for each $n \in \mathbb{N}$, the sequences $\{\widehat{L}_\nu(x;-1/n)\}_{\nu=0}^{\infty}$ are *quasi-orthogonal* of the order one with respect to I_1, with $c = -1/n$, and the next reccurence relations are not valid for $L_n(x;-1/n)$ and its neighbours.

Theorem 1.1. *The polynomial $\widehat{L}_n(x;c)$ can be expressed in the form*

$$\widehat{L}_n(x;c) = \widehat{L}_n^{(1)}(x) + \lambda_n \widehat{L}_{n-1}^{(1)}(x), \tag{1.4}$$

where

$$\lambda_n = n\, \frac{1 + (n+1)c}{1 + nc},$$

i.e., $\{\widehat{L}_n(x;c)\}$ is quasi-orthogonal of the order one with respect to the functional $I^{(1)}$.

Proof. Suppose that λ_n exists such that (1.4) is true. Using (1.2) and (1.3)–(1.3′), for the coefficient of the term x^k we yield

$$\frac{1 + k(1 + c(n+1))}{1 + nc} \cdot \frac{n}{n-k} = \frac{n(n+1)}{n-k} - \lambda_n.$$

After some computation we find λ_n which does not depend on k. □

Using the previous relations we can prove the three-term recurrence relation for polynomials $\widehat{L}_n(x;c)$:

Theorem 1.2. *The sequence* $\{\widehat{L}_n(x;c)\}$ *satisfies the three-term recurrence relation*

$$\widehat{L}_{n+1}(x;c) = (x-\alpha_n)\widehat{L}_n(x;c) - \beta_n \widehat{L}_{n-1}(x;c),$$

where

$$\alpha_n = \frac{2n+1+4n(n+1)c+n(n+1)(2n+1)c^2}{(1+nc)(1+(n+1)c))},$$
$$\beta_n = n^2\,\frac{(1+(n-1)c)(1+(n+1)c)}{(1+nc)^2}.$$

The norm of $\widehat{L}_n(x;c)$ is given by

$$\|\widehat{L}_n(x;c)\|^2 = \beta_0\beta_1\cdots\beta_n = (n!)^2\,\frac{1+(n+1)c}{1+nc}.$$

Theorem 1.3. *The zeros of* $\widehat{L}_n(x;c)$ *are real, simple and positive, except for* $c<-1/n$ *when one of them is negative. Denoting these zeros by*

$$\zeta_{n1} < \zeta_{n2} < \cdots < \zeta_{nn},$$

we have, for $c<-2/(n+1)$, *that the lowest zero* ζ_{n1} *satisfies*

$$-\frac{\zeta_{nn}}{n-1} < \zeta_{n1} < 0 \quad \wedge \quad |\zeta_{n1}| < |\zeta_{n2}|. \tag{1.5}$$

Proof. Let ζ_{nj}, $j=k,\dots,n$, be positive zeros of odd multiplicity. Defining $q(x) = \prod_{j=k}^{n}(x-\zeta_{nj})$, we see that the polynomial $xq(x)\widehat{L}_n(x;c)$ does not change sign for $x>0$. Hence

$$I_1(xq(x)\widehat{L}_n(x;c)) = \int_0^{+\infty} xq(x)\widehat{L}_n(x;c)e^{-x}\,dx \neq 0.$$

Since $\{\widehat{L}_n(x;c)\}$ is quasi-orthogonal of the order one, with respect to the functional $I^{(1)}$, it follows $\deg q(x) \geq n-1$.

From (1.3) we have that $\widehat{L}_n(0;c) = (-1)^n n!/(1+nc)$. Hence

$$\operatorname{sign}\widehat{L}_n(0;c) = \begin{cases} (-1)^n, & \text{for } c>-1/n,\\ (-1)^{n+1}, & \text{for } c<-1/n. \end{cases}$$

Since $\widehat{L}_n(x;c) = \prod_{i=1}^{n}(x - \zeta_{ni})$ and $\widehat{L}_n(0;c) = (-1)^n \prod_{i=1}^{n} \zeta_{ni}$, we conclude that all zeros are positive for $c > -1/n$, and only one of them is negative for $c < -1/n$. Also, differentiating (1.3) with respect to x, we find

$$\widehat{L}_n'(0;c) = (-1)^{n+1}\frac{n!n}{2}\cdot\frac{2+c(n+1)}{1+nc}.$$

Thus, for $c < -2/(n=1)$, it is $\widehat{L}_n'(0;c)/\widehat{L}_n(0;c) > 0$. Because of

$$\frac{\widehat{L}_n'(x;c)}{\widehat{L}_n(x;c)} = \sum_{i=1}^{n}\frac{1}{x-\zeta_{ni}},$$

we have $\sum_{i=1}^{n}(-1/\zeta_{ni}) > 0$. Then, from $-1/\zeta_{n1} > \sum_{i=2}^{n}(1/\zeta_{ni})$, we yield

$$\frac{1}{-\zeta_{n1}} > \frac{n-1}{\zeta_{nn}} \qquad \text{and} \qquad \frac{1}{-\zeta_{n1}} > \frac{1}{\zeta_{n2}},$$

from which follows (1.5). □

EXAMPLE 1.1. The polynomial $\widehat{L}_3(x;1) = x^3 - \frac{11}{4}x^2 + \frac{27}{2}x - \frac{3}{2}$, has positive zeros: $x_1 \approx 0.119747$, $x_2 \approx 2.06541$, $x_3 \approx 6.06483$, but the polynomial $\widehat{L}_3(x;-2) = x^3 - \frac{39}{5}x^2 + \frac{54}{5}x + \frac{6}{5}$, has one negative zero: $x_1 \approx -0.103301$, $x_2 \approx 1.95187$, $x_3 \approx 5.95143$.

Theorem 1.4. *The zeros of $\widehat{L}_n^{(1)}(x)$ and $\widehat{L}_{n-1}^{(1)}(x)$ interlace the zeros of the polynomial $\widehat{L}_n(x;c)$.*

Proof. Let $x_{m,i}^{(1)}$, $i = 1,\dots,m$, be the zeros of $\widehat{L}_m^{(1)}(x)$. Then from (1.4), we have $\widehat{L}_n(x_{n,i}^{(1)};c) = \lambda_n \widehat{L}_{n-1}^{(1)}(x_{n,i}^{(1)})$ and $\widehat{L}_n(x_{n,i+1}^{(1)};c) = \lambda_n \widehat{L}_{n-1}^{(1)}(x_{n,i+1}^{(1)})$. Since between $x_{n,i}^{(1)}$ and $x_{n,i+1}^{(1)}$ there exists an unique zero of $\widehat{L}_{n-1}^{(1)}(x)$, we conclude that $\widehat{L}_{n-1}^{(1)}(x_{n,i}^{(1)})$ and $\widehat{L}_{n-1}^{(1)}(x_{n,i+1}^{(1)})$ must have the opposite signs. Hence, a zero of $\widehat{L}_n(x;c)$ exists in the interval $(x_{n,i}^{(1)}, x_{n,i+1}^{(1)})$ for $i = 1,\dots,n-1$.

In the same way, we can conclude that $\widehat{L}_n(x;c)$ has a zero in the interval $(x_{n-1,i}^{(1)}, x_{n-1,i+1}^{(1)})$ $(i = 1,\dots,n-2)$. □

Remark. In [8] it was proved that $\widehat{L}_n(x)$ and $\widehat{L}_{n-1}(x)$ interlace the zeros of $\widehat{L}_n(x;c)$.

2. Polynomials of the Jacobi type

In this section we consider the functional

$$I^{(\beta,\alpha)}(P) = \int_0^1 x^\beta (1-x)^\alpha P(x)\,dx, \qquad \beta,\alpha > -1,$$

and the monic Jacobi polynomials $\{\widehat{Q}_n^{(\beta,\alpha)}(x)\}$ orthogonal with respect to the functional $I^{(\beta,\alpha)}$. Such polynomials can be expressed by the sum

$$\widehat{Q}_n^{(\beta,\alpha)}(x) = \frac{(-1)^n n!}{(n+\alpha+\beta+1)_n} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}(n+\alpha+\beta+1)_k \frac{(\beta+1)_n}{(\beta+1)_k} x^k,$$

and they satisfy the three-term recurrence relation

$$\widehat{Q}_{n+1}^{(\beta,\alpha)}(x) = (x-a_n)\widehat{Q}_n^{(\beta,\alpha)}(x) - b_n \widehat{Q}_{n-1}^{(\beta,\alpha)}(x), \;\; \widehat{Q}_{-1}^{(\beta,\alpha)}(x) = 0, \;\; \widehat{Q}_0^{(\beta,\alpha)}(x) = 1,$$

where

$$a_n = \gamma_{n+1} - \gamma_n, \qquad \gamma_n = \frac{n(n+\beta)}{2n+\alpha+\beta},$$

$$b_n = \left\{ \frac{(n-1)(\beta+n-1)}{2(2n+\alpha+\beta-1)} - a_n \right\} \gamma_n - \frac{n(\beta+n)}{2(2n+\alpha+\beta+1)} \gamma_{n+1}.$$

Let $I_2 = I^{(0,\alpha)} + c\Delta_0$, $c \in \mathbf{R}$, be a new functional and let $\{P_n^{(0,\alpha)}(x; c\,|\,0)\}$ be the corresponding monic orthogonal polynomials of the *Jacobi type*. In [6] it was proved that the n-th polynomial can be expressed by

$$P_n^{(0,\alpha)}(x; c\,|\,0) = (-1)^n n! \sum_{k=0}^{n} (-1)^k \frac{1 + k(1 + c(n+1)(n+\alpha))}{(k+1)!(n+\alpha+k+1)_{n-k}} \binom{n}{k} x^k.$$

Denoting the monic polynomials of the Jacobi type by $\{\widehat{P}_n^{(0,\alpha)}(x; c\,|\,0)\}$, we have

$$P_n^{(0,\alpha)}(x; c\,|\,0) = (1 + n(n+\alpha)c)\widehat{P}_n^{(0,\alpha)}(x; c\,|\,0), \quad c \neq -\frac{1}{n(n+\alpha)}, \quad n \in \mathbf{N}.$$

Hence it is $\deg P_n^{(0,\alpha)}(x; -\frac{1}{n(n+\alpha)}\,|\,0) = n-1$ and the next reccurence relations are not valid for that polynomial and its neighbours.

Like in Section 1 we can prove the following results:

Theorem 2.1. *The polynomial* $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$ *can be expressed in the form*

$$\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)=\widehat{Q}_n^{(1,\alpha)}(x)+\lambda_n\widehat{Q}_{n-1}^{(1,\alpha)}(x),$$

where

$$\lambda_n=\frac{n(n+\alpha)}{(2n+\alpha)(2n+\alpha+1)}\cdot\frac{1+(n+1)(n+\alpha+1)c}{1+n(n+\alpha)c}.$$

Theorem 2.2. *The sequence* $\{\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)\}$ *satisfies the three-term recurrence relation*

$$\widehat{P}_{n+1}^{(0,\alpha)}(x;c\,|\,0)=(x-\alpha_n)\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)-\beta_n\widehat{P}_{n-1}^{(0,\alpha)}(x;c\,|\,0),$$

where

$$\alpha_n=a_n+\lambda_{n+1}-\lambda_n,\quad \beta_n=b_{n-1}\frac{\lambda_n}{\lambda_{n-1}}.$$

Theorem 2.3. *All zeros of* $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$ *are real, simple and positive, with exception one of them for* $c<-1/(n(n+\alpha))$. *Furthermore, for* $c<-2/((n+1)(n+\alpha))$ *the inequalities* (1.5) *hold.*

EXAMPLE 2.1. The polynomial $\widehat{P}_3^{(0,0)}(x;1\,|\,0)=x^3-\frac{27}{20}x^2+\frac{21}{50}x-\frac{1}{200}$, has all zeros in $(0,1)$: $x_1\approx 0.0123940$, $x_2\approx 0.459337$, $x_3\approx 0.878268$, but the polynomial $\widehat{P}_3^{(0,0)}(x;-2\,|\,0)=x^3-\frac{45}{34}x^2+\frac{33}{85}x+\frac{1}{340}$, has one negative zero: $x_1\approx -0.103301$, $x_2\approx 1.95187$, $x_3\approx 5.95143$.

Theorem 2.4. *The zeros of* $\widehat{Q}_n^{(1,\alpha)}(x)$ *and* $\widehat{Q}_{n-1}^{(1,\alpha)}(x)$ *interlace the zeros ofthe polynomial* $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$.

Remark. It was proved the interlacing property for $\widehat{Q}_n^{(0,\alpha)}(x)$ and $\widehat{Q}_{n-1}^{(0,\alpha)}(x)$ with respect to $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$ (see [8]).

3. Polynomials of the Legendre type

The sequence of Legendre polynomials $\{P_n(x)\}$ is orthogonal with respect to the functional

$$L(P)=\int_{-1}^{1}P(x)\,dx.$$

This sequence satisfies the three-term recurrence relation

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x),$$

with initial values $P_{-1}(x)=0$ and $P_0(x)=1$. The polynomial $P_n(x)$ can be expressed in the form

$$P_n(x)=\frac{1}{2^n}\sum_{i=0}^{[n/2]}(-1)^i\binom{n}{i}\binom{2n-2i}{n}x^{n-2i}.$$

Let

$$\boldsymbol{I}_3=\boldsymbol{L}+c(\boldsymbol{\Delta}_{-1}+\boldsymbol{\Delta}_1),\qquad c\in\mathbb{R},$$

be a new functional, and $\{P_n(x;c\,|-1,1)\}$ the corresponding orthogonal polynomials of the *Legendre type*. Then the moments $\tilde{\mu}_n=I_3(x^n)$ are given by

$$\tilde{\mu}_n=\begin{cases}0 & \text{for odd } n,\\ \dfrac{2}{n+1}+2c & \text{for even } n.\end{cases}$$

The polynomial $P_n(x;c\,|-1,1)$ can be expressed in a determinant form,

$$P_n(x;c\,|-1,1)=\det\begin{vmatrix}\tilde{\mu}_0 & \tilde{\mu}_1 & \cdots & \tilde{\mu}_{n-1}\\ \tilde{\mu}_1 & \tilde{\mu}_2 & & \tilde{\mu}_n\\ \vdots & & & \\ \tilde{\mu}_{n-1} & \tilde{\mu}_n & & \tilde{\mu}_{2n-1}\\ 1 & x & & x^n\end{vmatrix}.$$

Using a method as in Gautschi and Milovanović [5], we determine

$$\tilde{\Delta}_n=\det[\tilde{\mu}_{i+j}]_{i,j=0,\dots,n-1}.$$

It is known that for determinants

$$H_n=\det\left[\tfrac{1}{2i+2j-3}\right]_{i,j=1}^n,\quad H_0=1,$$

holds

$$H_n=\frac{(2n-2)!!^2}{(2n-1)^2\cdots(4n-5)^2(4n-3)}H_{n-1},\quad n=3,4,\dots.$$

Introducing

$$H_n(c)=\left[\tfrac{1}{2i+2j-3}+c\right]_{i,j=1}^n,$$

we obtain

$$H_n(c)=\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ -c & 1 & \frac{1}{3} & & \frac{1}{2n-1}\\ -c & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1}\\ \vdots & & & & \\ -c & \frac{1}{2n-1} & \frac{1}{2n+1} & & \frac{1}{4n-3}\end{bmatrix}.$$

Lemma 3.1. *We have*

$$H_n(c) = \left(1 + c\binom{2n}{2}\right) H_n.$$

Proof. For $H_n(c)$ we have

$$H_n(c) = H_n + cD_n,$$

where

$$D_n = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ -1 & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -1 & \frac{1}{2n-1} & \frac{1}{2n+1} & & \frac{1}{4n-3} \end{bmatrix}.$$

We can prove that $D_n = \binom{2n}{2} H_n$, $n \in \mathbf{N}$. Subtracting the last row in D_n from all others, except the first one, we obtain

$$D_n = \frac{(2n-2)!!}{(2n-1)\cdots(4n-3)} \begin{bmatrix} 0 & 2n-1 & 2n+1 & \cdots & 4n-3 \\ 0 & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ 0 & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -1 & 1 & 1 & & 1 \end{bmatrix}.$$

Also, subtracting the last column from all others, except the first one, we obtain

$$D_n = \frac{(2n-2)!!^2}{(2n-1)^2\cdots(4n-5)^2(4n-3)} \left\{(4n-3)H_{n-1} + D_{n-1}\right\},$$

from which, by induction, we finish the proof. □

Lemma 3.2. *If*

$$H_n'(c) = \det\left[\tfrac{1}{2i+2j-1} + c\right]_{i,j=1}^{n}, \qquad H_n' = H_n'(0),$$

then

$$H_n'(c) = \left[1 + c\binom{2n+1}{2}\right] H_n', \quad n = 1, 2, \dots .$$

Thus,

$$\tilde{\Delta}_n = \Delta_n \left[1 + \binom{n}{2}c\right] \left[1 + \binom{n+1}{2}c\right],$$

where

$$\Delta_n = \det[L(x^{i+j})]_{i,j=0,\dots,n-1}.$$

Theorem 3.3. *The polynomials $P_n(x;c\,|-1,1)$ satisfy the following three-term recurrence relation*

$$(n+1)\left(1+\tbinom{n}{2}c\right)P_{n+1}(x;c\,|-1,1) = (2n+1)\left(1+\tbinom{n+1}{2}c\right)xP_n(x;c\,|-1,1) - n\left(1+\tbinom{n+2}{2}c\right)P_{n-1}(x;c\,|-1,1).$$

Proof. By $\{\widehat{P}_n(x;c\,|-1,1)\}$ we denote the sequence of monic polynomials of the Legendre type. According to the property $(xf,g)=(f,xg)$ of the inner product defined by the functional $\boldsymbol{I}_3$: $(f,g)=\boldsymbol{I}_3(fg)$, we conclude (cf. [2] and [9]) that this sequence satisfies a three-term reccurence relation of the form

$$\widehat{P}_{n+1}(x;c\,|-1,1) = x\widehat{P}_n(x;c|-1,1) - \beta_n\widehat{P}_{n-1}(x;c\,|-1,1),$$

where (see [3])

$$\beta_n = \frac{\tilde{\Delta}_{n-1}\tilde{\Delta}_{n+1}}{\tilde{\Delta}_n^2}.$$

Knowing a relation for the monic Legendre polynomials and the determinant Δ_n, we yield

$$\widehat{P}_{n+1}(x;c\,|-1,1) = x\widehat{P}_n(x;c\,|-1,1) - \frac{n^2}{4n^2-1}\frac{(1+\binom{n-1}{2}c)(1+\binom{n+2}{2}c)}{(1+\binom{n}{2}c)(1+\binom{n+1}{2}c)}\widehat{P}_{n-1}(x;c\,|-1,1).$$

Putting

$$P_n(x;c\,|-1,1) = \frac{(2n)!}{2^n n!^2}\widehat{P}_n(x;c|-1,1),$$

we finish the proof. □

By induction and using the three-term recurrence relation, we obtain:

Theorem 3.4. *The polynomial $P_n(x;c\,|-1,1)$ can be expressed in the form*

$$P_n(x;c\,|-1,1) = \frac{1}{2^n}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}\left\{1+\left(\binom{n}{2}+2k\right)c\right\}x^{n-2k},$$

where $n=2,3,\ldots$.

Remark. This formula was derived in [6] with a mistake.

The last formula shows that the sequence $\{P_n(x;c\,|-1,1)\}$ has a degenerated property in the sence that is

$$\widehat{P}_n(x;-\tbinom{n}{2}^{-1}\,|-1,1) = \widehat{P}_{n-2}(x;-\tbinom{n}{2}^{-1}\,|-1,1).$$

Theorem 3.5. *The polynomials $P_n(x;c\,|-1,1)$ are quasi-orthogonal of the second order with respect to the functional*

$$J^{(1,1)}(P)=\int_{-1}^{1}P(x)(1-x^2)\,dx.$$

The polynomial $P_n(x;c\,|-1,1)$ has at least $n-2$ different zeros with odd multiplicity in $(-1,1)$.

Proof. Because of the orthogonality, we have for any polynomial $p(x)$ of degree k ($k\le n-3$) that

$$I_3(P_n(x;c\,|-1,1)p(x)(1-x^2))=0,$$

i.e.,

$$\int_{-1}^{1}P_n(x;c\,|-1,1)p(x)(1-x^2)\,dx=0.$$

So, we yield quasi-orthogonality of the order 2. Finally, let $x_{n1},x_{n2},\ldots,x_{nk}$ be all distinct zeros of $P_n(x;c\,|-1,1)$ with odd multiplicity and which are in $(-1,1)$. If we introduce the node polynomial $p(x)=(x-x_{n1})(x-x_{n2})\cdots(x-x_{nk})$, then the polynomial $P_n(x;c\,|-1,1)p(x)(1-x^2)$ does not change sign in $(-1,1)$. Therefore,

$$\int_{-1}^{1}P_n(x;c\,|-1,1)p(x)(1-x^2)\,dx\neq 0.$$

Because of that, we conclude that $\deg p(x)\ge n-2$. □

EXAMPLE 3.2. The polynomial

$$P_3(x;c\,|-1,1)=5(1+3c)x^3-3(1+5c)x,$$

has the zeros

$$x_1=0,\quad x_{2,3}=\pm\sqrt{\frac{3(1+5c)}{5(1+3c)}}.$$

All zeros of $P_3(x;c\,|-1,1)$ are in $(-1,1)$ if $c>-1/3$. For $c<-1/3$, two of them are out of $(-1,1)$.

Remark. According to Favard's theorem (see [2]) there are exist the weight distributions $w_i(x)$, $i=1,2,3$, corresponding to the previous functionals J_i:

$$J_i(P)=\int_{-\infty}^{+\infty}P(x)\,dw_i(x),\qquad i=1,2,3.$$

These distributions are (see [3])

$$w_1(x)=e^{-x}+c\delta(x),\ w_2(x)=(1-x)^{\alpha}+c\delta(x),\ w_3(x)=1+c\big(\delta(x-1)+\delta(x+1)\big),$$

where $\delta(x)$ is the Dirac's delta function.

References

[1] G. Szego, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. Vol. 23, Providence, R.I., 1975.

[2] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[3] L. L. Littlejohn and S. D. Shore, *Nonclassical orthogonal polynomials as solutions to second order differential equations*, Canad. Math. Bull. **25** (1982), 291–295.

[4] A. B. Mingarelli and A. M. Krall, *Legendre type polynomials under an indefinite inner product*, SIAM J. Math. Anal. **14** (1983), 399–402.

[5] W. Gautschi and G. V. Milovanović, *Polynomials orthogonal on the semicircle*, J. Approx. Theory **46** (1986), 230–250.

[6] A. L. Kuzmina, *On some nonclassical orthogonal polynomials*, Izv. Vyssh. Uchebn. Zaved. Mat. **1987**, 47–52. (Russian)

[7] M. Alfaro, F. Marcellan, M. L. Rezola and A. Ronveaux, *On orthogonal polynomials of Sobolev type: Algebraic properties and zeros*, SIAM J. Math. Anal. **23** (1992), 737–757.

[8] T. E. Perez and M. A. Pinar, *Global properties of zeros for Sobolev-type orthogonal polynomials*, J. Comput. Appl. Math. **49** (1993), 225–232.

[9] W. D. Evans, L. L. Littlejohn, F. Marcellan, C. Markett and A. Ronveaux, *On recurrence relations for Sobolev orthogonal polynomials*, SIAM J. Math. Anal. **26** (1995), 446–467.

Grdimir V. Milovanović and Zvezdan M. Marjanović: Faculty of Electronic Engineering, Department of Mathematics, University of Niš, 18000 Niš, Yugoslavia

Predrag M. Rajković: Faculty of Mechanical Engineering, Department of Mathematics, University of Niš, 18000 Niš, Yugoslavia