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INVITED LECTURE

SOME NONSTANDARD TYPES OF ORTHOGONALITY (A SURVEY)

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ABSTRACT. This survey is devoted to some nonstandard types of orthogonal polynomials in the complex plane. Under suitable integrability conditions on w, we consider polynomials orthogonal on a circular arc with respect to a non-Hermitian complex inner product as well as Geronimus' version of orthogonality on a contour in the complex plane. Also, we introduce a class of polynomials orthogonal on some selected radial rays in the complex plane. In both of cases we investigate their existence and uniqueness, recurrence relations, representations and connections with standard polynomials orthogonal on the real line. We also give an introduction to the general theory of orthogonality on the real line and the unit circle. Zero distributions of nonstandard types of orthogonal polynomials are considered.

1. Introduction

The orthogonal systems play an important role in many branches of mathematics, physics and other applied and computational sciences. Especially, orthogonal polynomial systems appear in the Gaussian quadrature processes, the least square approximation of functions, differential and difference equations, Fourier series, etc.

In this survey we mainly consider some classes of nonstandard orthogonal polynomials. The paper is organized as follows. In Section 2 we discuss two standard types of orthogonal polynomials – *polynomials orthogonal on the real line* and *polynomials orthogonal on the unit circle*. The most important properties of such polynomials are presented. Under suitable integrability conditions on a weight function, in Section 3 we consider polynomials orthogonal on the semicircle with respect to a complex-valued inner product.

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A generalization of such nonstandard orthogonal polynomials on a circular arc in the complex plane is treated in Section 4. Geronimus' version of orthogonality on a contour in the complex plane for polynomials orthogonal on the semicircle or on a circular arc is considered in Section 5. Sections 6 and 7 are devoted to a new class of orthogonal polynomials on some selected radial rays in the complex plane. We investigate the existence and uniqueness, recurrence relations, representations and the connection with standard polynomials orthogonal on the real line. Also, the distribution of zeros of such polynomials is included.

2. Standard types of orthogonal polynomials

A standard type of orthogonality is one on the real line with respect to a given non-negative measure $d\lambda(t)$. Namely, let $\lambda\colon \mathbb{R} \to \mathbb{R}$ be a fixed non-decreasing function with infinitely many points of increase for which all moments $\mu_k = \int_{\mathbb{R}} t^k \, d\lambda(t)$, $k = 0, 1, \ldots$, exist and are finite. Then the improper Stieltjes integral $\int_{\mathbb{R}} P(t) \, d\lambda(t)$ exists for every polynomial P. By the application of the Lebesgue-Stieltjes integral $\int_{\mathbb{R}} f(t) \, d\lambda(t)$ to characteristic functions of sets, the function λ engenders a Lebesgue-Stieltjes measure $d\lambda(t)$, which is known also as *m-distribution* (cf. Freud [10]). Moreover, if $t \mapsto \lambda(t)$ is an absolutely continuous function, then we say that $\lambda'(t) = w(t)$ is a *weight function*. In that case, the measure $d\lambda$ can be express as $d\lambda(t) = w(t)\, dt$, where the weight function $t \mapsto w(t)$ is a non-negative and measurable in Lebesgue's sense for which all moments exists and $\mu_0 = \int_{\mathbb{R}} w(t)\, dt > 0$.

In the general case the function λ can be written in the form $\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{s}} + \lambda_{\mathrm{j}}$, where λ_{ac} is absolutely continuous, λ_{s} is singular, and λ_{j} is a jump function.

The set of points of increase of $t \mapsto \lambda(t)$, so-called the *support of the measure*, we denote by $\operatorname{supp}(d\lambda)$. It is always an infinite and closed set. If $\operatorname{supp}(d\lambda)$ is bounded, then the smallest closed interval containing $\operatorname{supp}(d\lambda)$ we will denote by $\Delta(d\lambda)$.

Using the measure $d\lambda(t)$ we can define the inner product (f, g), by

$$(2.1) \qquad (f,g) = \int_{\mathbb{R}} f(t)\overline{g(t)}\, d\lambda(t) \qquad \big(f, g \in L^2(\mathbb{R}) \equiv L^2(\mathbb{R}; d\lambda)\big),$$

and consider a system of (monic) orthogonal polynomials $\{p_k(t)\}$ such that

$$p_k(t) = t^k + \text{terms of lower degree} \quad (k = 0, 1, \ldots),$$
$$(p_k, p_n) = 0, \quad k \neq n, \quad (p_n, p_n) = \|p_n\|^2 > 0.$$

For any m-distribution $d\lambda(t)$ there exists a unique system of polynomials $\{p_k(t)\}$.

A general property of the inner product (2.1) that $(tf, g) = (f, tg)$ provides the three-term recurrence relation for the (monic) orthogonal polynomials $p_k(t)$,

$$(2.2)\qquad \begin{aligned} &p_{k+1}(t) = (t - a_k)p_k(t) - b_k p_{k-1}(t), \qquad k = 0, 1, 2, \dots,\\ &p_0(t) = 1, \quad p_{-1}(t) = 0. \end{aligned}$$

The recursion coefficients can be expressed in terms of inner product (cf. Milovanović, Mitrinović, Rassias [31, p. 33])

$$a_k = \frac{(tp_k, p_k)}{(p_k, p_k)} \quad (k \geq 0), \quad b_k = \frac{(p_k, p_k)}{(p_{k-1}, p_{k-1})} \quad (k \geq 1).$$

The coefficient b_0, which multiplies $p_{-1}(t) = 0$ in three-term recurrence relation may be arbitrary. Sometimes, it is convenient to define it by $b_0 = \mu_0 = \int_{\mathbb{R}} d\lambda(t)$. Then the norm of p_n can be express in the form

$$\|p_n\| = \sqrt{(p_n, p_n)} = \sqrt{b_0 b_1 \cdots b_n}\,.$$

An interesting and very important property of polynomials $p_n(t)$, $n \geq 1$, is the distribution of zeros. Namely, all zeros of $p_n(t)$ are real and distinct and are located in the interior of the interval $\Delta(d\lambda)$. Also, the zeros of $p_n(t)$ and $p_{n+1}(t)$ interlace, i.e.,

$$\tau_k^{(n+1)} < \tau_k^{(n)} < \tau_{k+1}^{(n+1)} \qquad (k = 1, \dots, n;\ n \in \mathbf{N}),$$

where $\tau_k^{(n)}$, $k = 1, \dots, n$, denote the zeros of $p_n(t)$ in an increasing order

$$\tau_1^{(n)} < \tau_2^{(n)} < \cdots < \tau_n^{(n)}.$$

It is easy to prove that the zeros $\tau_k^{(n)}$ of $p_n(t)$ are the same as the eigenvalues of the following tridiagonal matrix

$$J_n = J_n(d\lambda) = \begin{bmatrix} a_0 & \sqrt{b_1} & & & O \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \sqrt{b_2} & a_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{b_{n-1}} \\ O & & & \sqrt{b_{n-1}} & a_{n-1} \end{bmatrix},$$

which is known as the *Jacobi matrix*. Also, the monic polynomial $p_n(t)$ can be expressed in the following determinant form

$$\hat{\pi}_n(t) = \det(tI_n - J_n),$$

where I_n is the identity matrix of the order n.

Suppose that $\Delta(d\lambda) = [a, b]$. Since every interval (a, b) can be transformed by a linear transformation to one of following intervals: $(-1, 1)$, $(0, +\infty)$, $(-\infty, +\infty)$, we can restrict the consideration (without loss of generality) only to these three intervals. A very important class of orthogonal polynomials on an interval of orthogonality $(a, b) \in \mathbb{R}$ is constituted by so-called the *classical orthogonal polynomials*. Their weight functions $w(t)$ satisfy the differential equation

$$\frac{d}{dt}(A(t)w(t)) = B(t)w(t),$$

where

$$A(t) = \begin{cases} 1 - t^2, & \text{if } (a, b) = (-1, 1), \\ t, & \text{if } (a, b) = (0, +\infty), \\ 1, & \text{if } (a, b) = (-\infty, +\infty), \end{cases}$$

and $B(t)$ is a polynomial of the first degree.

The classical orthogonal polynomials $\{Q_k\}$ on (a, b) can be specificated as the *Jacobi polynomials* $P_k^{(\alpha,\beta)}(t)$ $(\alpha, \beta > -1)$ on $(-1, 1)$, the *generalized Laguerre polynomials* $L_k^s(t)$ $(s > -1)$ on $(0, +\infty)$, and finally as the *Hermite polynomials* $H_k(t)$ on $(-\infty, +\infty)$, with the weight functions

$$t \mapsto (1 - t)^\alpha (1 + t)^\beta, \qquad t \mapsto t^s e^{-t}, \qquad t \mapsto e^{-t^2} \qquad (\alpha, \beta, s > -1),$$

respectively. These polynomials have many nice particular properties (cf. [9], [25], [29], [31], [40], [43]). Some characterizations of the classical polynomials were given in [2–3], [5–6], [9], [20], [23].

There are several classes of orthogonal polynomials which are in certain sense close to the classical orthogonal polynomials, so-called *semi-classical* polynomials.

In many applications of orthogonal polynomials it is very important to know the recursion coefficients a_k and b_k. If $d\lambda(t)$ is one of the classical measures, then a_k and b_k are known explicitly. Furthermore, there are certain non-classical cases when we know also these coefficients. For example, we mention here the *generalized Gegenbauer weight* $w(t) = |t|^\mu (1 - t^2)^\alpha$, $\mu, \alpha > -1$, on $[-1, 1]$ (see Lascenov [22] and Chihara [9, pp. 155–156]), the *hyperbolic weight* $w(t) = 1/\cosh t$ on $(-\infty, +\infty)$ ([9, pp. 191–193]), and the *logistic weight* $w(t) = e^{-t}/(1+e^{-t})^2$ on $(-\infty, +\infty)$.

A system of orthogonal polynomials for which the recursion coefficients are not known explicitly will be said to be *strong non-classical* orthogonal polynomials. In such cases there are a few known approaches to compute the first n coefficients a_k, b_k, $k = 0, 1, \ldots, n-1$. Furthermore, for such a purpose there is the package ORTHPOL developed by Gautschi [12]. These coefficients then allow us to compute all orthogonal polynomials of degree $\leq n$ by a straightforward application of the three-term recurrence relation (2.2).

Another type of orthogonality is *orthogonality on the unit circle*. The polynomials orthogonal on the unit circle with respect to a given weight function have been introduced and studied by Szegő [41–43] and Smirnov [37–38]. A more general case was considered by Achieser and Kreĭn [1], Geronimus [16–17], P. Nevai [35–36], Alfaro and Marcellán [4], Marcellán and Sansigre [24], etc. These polynomials are linked with many questions in the theory of time series, digital filters, statistics, image processing, scattering theory, control theory and so on.

The inner product is defined by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\mu(\theta),$$

where $d\mu(\theta)$ is a finite positive measure on the interval $[0, 2\pi]$ whose support is an infinite set. In that case there is a unique system of (monic) orthogonal polynomials $\{\phi_k\}_{k \in \mathbb{N}_0}$. If $\theta \mapsto \mu(\theta)$ is an absolutely continuous function on $[0, 2\pi]$, then we say that $\mu'(\theta) = w(\theta)$ is a *weight function.*

The monic orthogonal polynomials $\{\phi_k\}$ on the unit circle $|z| = 1$ satisfy the recurrence relations

$$\phi_{k+1}(z) = z\phi_k(z) + \phi_{k+1}(0)\phi_k^*(z), \quad \phi_{k+1}^*(z) = \phi_k^*(z) + \overline{\phi_{k+1}(0)} z \phi_k^*(z),$$

for $k = 0, 1, \ldots$, where $\phi_k^*(z) = z^k \bar{\phi}_k(1/z)$.

As we can see these recurrence relations are not three-term relations like (2.2). The values $\phi_k(0)$ are called reflection parameters or Szegő parameters. Defining a sequence of parameters $\{a_k\}$ by $a_k = -\overline{\phi_{k+1}(0)}$, $k = 0, 1, \ldots$, Geronimus [18, Chapter VIII] derived the following three-term recurrence relations:

$$\bar{a}_{k-1}\phi_{k+1}(z) = (\bar{a}_{k-1}z + \bar{a}_k)\phi_k(z) - \bar{a}_k z(1 - |a_{k-1}|^2)\phi_{k-1}(z),$$
$$a_{k-1}\phi_{k+1}^*(z) = (a_{k-1}z + a_k)\phi_k^*(z) - a_k z(1 - |a_{k-1}|^2)\phi_{k-1}^*(z),$$

where $k \in \mathbf{N}$ and $\phi_0(z) = 1$, $\phi_1(z) = z - \bar{a}_0$.

3. Orthogonality on the semicircle

Polynomials orthogonal on the semicircle

$$\Gamma_0 = \{z \in \mathbf{C} \mid z = e^{i\theta},\ 0 \le \theta \le \pi\}$$

have been introduced by Gautschi and Milovanović [14–15]. The inner product is given by

$$(f, g) = \int_\Gamma f(z)g(z)(iz)^{-1}\, dz,$$

where Γ is the semicircle $\Gamma = \{z \in \mathbf{C} \mid z = e^{i\theta}, 0 \le \theta \le \pi\}$. Alternatively,

$$(f, g) = \int_0^\pi f(e^{i\theta})g(e^{i\theta})\, d\theta.$$

This inner product is not Hermitian, but the corresponding (monic) orthogonal polynomials $\{\pi_k\}$ exist uniquely and satisfy a three-term recurrence relation of the form

$$\pi_{k+1}(z) = (z - i\alpha_k)\pi_k(z) - \beta_k\pi_{k-1}(z), \qquad k = 0, 1, 2, \ldots,$$
$$\pi_{-1}(z) = 0, \quad \pi_0(z) = 1.$$

Notice that the inner product possesses the property $(zf, g) = (f, zg)$.

The general case of complex polynomials orthogonal with respect to a *complex weight function* was considered by Gautschi, Landau and Milovanović [13]. Namely, let $w: (-1, 1) \mapsto \mathbf{R}_+$ be a weight function which can be extended to a function $w(z)$ holomorphic in the half disc

$$D_+ = \{z \in \mathbf{C} \mid |z| < 1, \operatorname{Im} z > 0\},$$

and

$$(3.1) \qquad (f, g) = \int_\Gamma f(z)g(z)w(z)(iz)^{-1}\, dz = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\, d\theta.$$

Together with (3.1) consider the inner product

$$(3.2) \qquad [f, g] = \int_{-1}^1 f(x)\overline{g(x)}w(x)\, dx,$$

which is positive definite and therefore generates a unique set of real (monic) orthogonal polynomials $\{p_k\}$:

$$[p_k, p_m] = 0 \quad \text{for} \quad k \ne m \quad \text{and} \quad [p_k, p_m] > 0 \quad \text{for} \quad k = m \quad (k, m \in \mathbf{N}_0).$$

On the other hand, the inner product (3.1) is not Hermitian; the second factor g is not conjugated and the integration is not with respect to the measure $|w(e^{i\theta})|\,d\theta$. The existence of corresponding orthogonal polynomials, therefore, is not guaranteed.

We call a system of complex polynomials $\{\pi_k\}$ *orthogonal on the semicircle* if

$$(\pi_k, \pi_m) = 0 \quad \text{for } k \neq m \quad \text{and} \quad (\pi_k, \pi_m) > 0 \quad \text{for } k = m \quad (k, m \in \mathbf{N}_0).$$

where we assume that π_k is monic of degree k.

The existence of the orthogonal polynomials $\{\pi_k\}$ can be established assuming only that

$$\text{(3.3)} \qquad \operatorname{Re}(1,1) = \operatorname{Re}\int_0^\pi w(e^{i\theta})\,d\theta \neq 0.$$

Assume that the weight function w is positive on $(-1,1)$, holomorphic in D_+ and such that the integrals in (3.1) and (3.2) exist for smooth f and g (possibly) as improper integrals. We also assume that the condition (3.3) is satisfied.

Let C_ε, $\varepsilon > 0$, denote the boundary of D_+ with small circular parts of radius ε and centres at ± 1 spared out and let $\mathcal{P}$ be the set of all algebraic polynomials. Further, let Γ_ε and $C_{\varepsilon,\pm 1}$ be the circular parts of C_ε with radii 1 and ε, respectively.

Then, using Cauchy's theorem and assuming that w is such that for all $g \in \mathcal{P}$,

$$\text{(3.4)} \qquad \lim_{\varepsilon \to 0} \int_{C_{\varepsilon,\pm 1}} g(z)w(z)\,dz = 0,$$

we obtain

$$\text{(3.5)} \qquad 0 = \int_C g(z)w(z)\,dz = \int_\Gamma g(z)w(z)\,dz + \int_{-1}^1 g(x)w(x)\,dx, \quad g \in \mathcal{P}.$$

The (monic, real) polynomials $\{p_k\}$, orthogonal with respect to the inner product (3.2), as well as the associated polynomials of the second kind,

$$q_k(z) = \int_{-1}^1 \frac{p_k(z) - p_k(x)}{z - x} w(x)\,dx \qquad (k = 0, 1, 2, \ldots),$$

are known to satisfy a three-term recurrence relation of the form

$$y_{k+1} = (z - a_k)y_k - b_k y_{k-1} \qquad (k = 0, 1, 2, \dots), \tag{3.6}$$

where

$$y_{-1} = 0,\ y_0 = 1 \ \text{ for } \{p_k\} \qquad \text{and} \qquad y_{-1} = -1,\ y_0 = 0 \ \text{ for } \{q_k\}. \tag{3.7}$$

Denote by m_k and μ_k the moments associated with the inner products (3.1) and (3.2), respectively,

$$\mu_k = (z^k, 1), \qquad m_k = [x^k, 1], \qquad k \geq 0,$$

where, in view of (3.7), $b_0 = m_0$.

Gautschi, Landau and Milovanović [13] proved the following result:

Theorem 3.1. *Let w be a weight function, positive on $(-1, 1)$, holomorphic in $D_+ = \{z \in \mathbb{C} \mid |z| < 1,\ \mathrm{Im}\, z > 0\}$, and such that (3.4) is satisfied and the integrals in (3.5) exist (possibly) as improper integrals. Assume in addition that*

$$\mathrm{Re}(1, 1) = \mathrm{Re} \int_0^\pi w(e^{i\theta})\, d\theta \neq 0.$$

Then there exists a unique system of (monic, complex) orthogonal polynomials $\{\pi_k\}$ relative to the inner product (3.1). Denoting by $\{p_k\}$ the (monic, real) orthogonal polynomials relative to the inner product (3.2), we have

$$\pi_k(z) = p_k(z) - i\theta_{k-1} p_{k-1}(z) \qquad (k = 0, 1, 2, \dots), \tag{3.8}$$

where

$$\theta_{k-1} = \frac{\mu_0 p_k(0) + i q_k(0)}{i\mu_0 p_{k-1}(0) - q_{k-1}(0)} \qquad (k = 0, 1, 2, \dots). \tag{3.9}$$

Alternatively,

$$\theta_k = i a_k + \frac{b_k}{\theta_{k-1}} \quad (k = 0, 1, 2, \dots); \qquad \theta_{-1} = \mu_0, \tag{3.10}$$

where a_k, b_k are the recursion coefficients in (3.6) and $\mu_0 = (1, 1)$. In particular, all θ_k are real (in fact, positive) if $a_k = 0$ for all $k \geq 0$. Finally,

$$(\pi_k, \pi_k) = \theta_{k-1}[p_{k-1}, p_{k-1}] \neq 0 \quad (k = 1, 2, \dots), \qquad (\pi_0, \pi_0) = \mu_0. \tag{3.11}$$

As we can see, relation (3.8), with (3.9), gives a connection between orthogonal polynomials on the semicircle and the standard polynomials orthogonal on $[-1,1]$ with respect to the same weight function w. The norms of these polynomials are in relation (3.11).

In the sequel we assume that condition (3.3) is satisfied, so that the orthogonal polynomials $\{\pi_k\}$ exist. Since $(zf,g)=(f,zg)$, it is known that they must satisfy a three-term recurrence relation

$$
\begin{aligned}
&\pi_{k+1}(z)=(z-i\alpha_k)\pi_k(z)-\beta_k\pi_{k-1}(z), \qquad k=0,1,2,\ldots,\\
&\pi_{-1}(z)=0, \quad \pi_0(z)=1.
\end{aligned} \tag{3.12}
$$

Using the representation (3.8), we can find a connection between the coefficients in (3.12) and the corresponding coefficients in the three-term recurrence relation (3.6) for polynomials $\{p_k\}$ (see [13]):

Theorem 3.2. *Under the assumption* (3.3), *the (monic, complex) polynomials* $\{\pi_k\}$ *orthogonal with respect to the inner product* (3.1) *satisfy the recurrence relation* (3.12), *where the coefficients* α_k, β_k *are given by*

$$
\alpha_k=\theta_k-\theta_{k-1}-ia_k, \qquad \beta_k=\frac{\theta_{k-1}}{\theta_{k-2}}b_{k-1}=\theta_{k-1}(\theta_{k-1}-ia_{k-1}),
$$

for $k\ge 1$ *and* $\alpha_0=\theta_0-ia_0$, *with the* θ_k *defined in Theorem* 3.1.

Alternatively, the coefficients α_k can be expressed in the form

$$
\alpha_k=-\theta_{k-1}+\frac{b_k}{\theta_{k-1}}, \quad k\ge 1, \qquad \alpha_0=\frac{b_0}{\theta_{-1}}=\frac{m_0}{\mu_0}.
$$

It is interesting to consider the zero distribution of polynomials $\pi_n(z)$. From (3.12) it follows that the zeros of $\pi_n(z)$ are the eigenvalues of the (complex, tridiagonal) matrix

$$
J_n=\begin{bmatrix}
i\alpha_0 & 1 & & & \mathrm{O}\\
\beta_1 & i\alpha_1 & 1 & & \\
 & \beta_2 & i\alpha_2 & \ddots & \\
 & & \ddots & \ddots & 1\\
\mathrm{O} & & & \beta_{n-1} & i\alpha_{n-1}
\end{bmatrix}, \tag{3.13}
$$

where α_k and β_k are given in Theorem 3.2.

If the weight w is symmetric, i.e.,

$$
w(-z)=w(z), \qquad w(0)>0, \tag{3.14}
$$

then $\mu_0 = (1,1) = \pi w(0) > 0$, $a_k = 0$, $\theta_k > 0$, for all $k \geq 0$, and

$$\alpha_0 = \theta_0, \quad \alpha_k = \theta_k - \theta_{k-1}, \quad \beta_k = \theta_{k-1}^2, \quad k \geq 1.$$

In that case J_n can be transformed into a real nonsymmetric tridiagonal matrix

$$A_n = -iD_n^{-1}J_nD_n = \begin{bmatrix} \alpha_0 & \theta_0 & & & O \\ -\theta_0 & \alpha_1 & \theta_1 & & \\ & -\theta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \theta_{n-2} \\ O & & & -\theta_{n-2} & \alpha_{n-1} \end{bmatrix},$$

where $D_n = \operatorname{diag}(1, i\theta_0, i^2\theta_0\theta_1, i^3\theta_0\theta_1\theta_2, \dots) \in \mathbb{C}^{n\times n}$. The eigenvalues η_ν, $\nu = 1, \dots, n$, of A_n can be calculated using the EISPACK subroutine HQR (see [39]). Then all the zeros ζ_ν, $\nu = 1, \dots, n$, of $\pi_n(z)$ are given by $\zeta_\nu = i\eta_\nu$, $\nu = 1, \dots, n$.

In [13] we proved the following result for a symmetric weight (3.14):

Theorem 3.3. *All zeros of π_n are located symmetrically with respect to the imaginary axis and contained in $D_+ = \{z \in \mathbb{C} \mid |z| < 1,\ \operatorname{Im} z > 0\}$, with the possible exception of a single (simple) zero on the positive imaginary axis.*

If we define the half strip $S_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0,\ -\xi_n \leq \operatorname{Re} z \leq \xi_n\}$, where ξ_n is the largest zero of the real polynomial p_n, then we can prove that all zeros of π_n are also in S_+ (see [13] and [15]). Thus, all zeros are contained in $D_+ \cap S_+$.

For the Gegenbauer weight $w(z) = (1-z^2)^{\lambda-1/2}$, $\lambda > -1/2$, the exceptional case from Theorem 3.3 can only arise if $n = 1$ and $-1/2 < \lambda \leq 0$. Likewise, no exceptional cases seem to occur for Jacobi weights $w(z) = (1-z)^\alpha(1+z)^\beta$, $\alpha, \beta > -1$, if $n \geq 2$, as was observed by several numerical computations (see [13]). However, in a general case, Gautschi [11] exhibited symmetric functions w for which $\pi_n(\,\cdot\,; w)$, for arbitrary fixed n, has a zero iy with $y \geq 1$.

Some applications of these polynomials in numerical integration and numerical differentiation can be found in [8], [26–28].

4. Orthogonality on a circular arc

A generalization of polynomials orthogonal on the semicircle was given by M.G. de Bruin [7] for the circular arc

$$\Gamma_R = \{z \in \mathbb{C} \mid z = -iR + e^{i\theta}\sqrt{R^2+1},\ \varphi \leq \theta \leq \pi - \varphi,\ \tan\varphi = R\}.$$

He considered the polynomials $\{\pi_k^R\}$ orthogonal on Γ_R with respect to the complex inner product

$$(4.1) \qquad (f,g) = \int_{\varphi}^{\pi-\varphi} f_1(\theta) g_1(\theta) w_1(\theta)\, d\theta,$$

where $\varphi \in (0, \pi/2)$, and for $f(z)$ the function $f_1(\theta)$ is defined by

$$f_1(\theta) = f\big(-iR + e^{i\theta}\sqrt{R^2+1}\big), \qquad R = \tan\varphi.$$

Alternatively, the inner product (4.1) can be expressed in the form

$$(4.2) \qquad (f,g) = \int_{\Gamma_R} f(z) g(z) w(z) (iz - R)^{-1}\, dz.$$

Under suitable integrability conditions on the weight function w, which is positive on $(-1, 1)$ and is holomorphic in the moon-shaped region

$$M_+ = \left\{ z \in \mathbf{C} \;\middle|\; |z + iR| < \sqrt{R^2+1},\ \mathrm{Im}\, z > 0 \right\},$$

where $R > 0$, the polynomials $\{\pi_k^R\}$ orthogonal on the circular arc Γ_R with respect to the complex inner product (4.1) always exist and have similar properties like polynomials orthogonal on the semicircle.

For $R = 0$ the arc Γ_R reduces to the semicircle Γ, and polynomials $\{\pi_k^R\}$ to $\{\pi_k\}$. It is easy to prove that the condition

$$\mathrm{Re} \int_{\Gamma_R} w(z)(iz - R)^{-1}\, dz = \mathrm{Re} \int_{\varphi}^{\pi-\varphi} w_1(\theta)\, d\theta \neq 0$$

is automatically satisfied for $R > 0$ in contrast to the case $R = 0$ (see condition (3.3)).

Quite analogous results to Theorems 3.1–3.4 were proved by de Bruin [7]. For example, for polynomials $\{\pi_k\}$ (the upper index R is omitted) equalities (3.8) and (3.11), as well as the three-term recurrence relation (3.12) hold, where now the θ_k is given by

$$\theta_k = -R + ia_k + \frac{b_k}{\theta_{k-1}} \quad (k = 0, 1, 2, \dots); \qquad \theta_{-1} = \mu_0,$$

instead of (3.10). Also, for the symmetric weight, $w(z) = w(-z)$, all zeros of π_n are contained in M_+ with the possible exception of just one simple zero situated on the positive imaginary axis.

Let $\{\pi_n\}$ be the set of polynomials orthogonal on the circular arc Γ_R, with respect to the inner product (4.1), i.e., (4.2). Milovanović and Rajković [33] introduced the polynomials $\{\pi_n^*\}$ orthogonal on the symmetric down circular arc Γ_R^* with respect to the inner product defined by

$$(f,g)^* = \int_{\Gamma_R^*} f(z)g(z)w(z)(iz+R)^{-1}\,dz, \tag{4.3}$$

where $\Gamma_R^* = \{z \in \mathbb{C} \mid z = iR + e^{-i\theta}\sqrt{R^2+1},\ \varphi \le \theta \le \pi - \varphi,\ \tan\varphi = R\}$. Such polynomials are called *dual orthogonal polynomials* with respect to polynomials $\{\pi_n\}$.

Let M be a lentil-shaped region with the boundary $\partial M = \Gamma_R \cup \Gamma_R^*$, i.e.,

$$M = \left\{z \in \mathbb{C} \mid |z \pm iR| < \sqrt{R^2+1}\right\},$$

where $R > 0$.

We assume that w is a weight function, positive on $(-1,1)$, holomorphic in M, and such that the integrals in (4.2), (4.3), and (3.2) exist for smooth functions f and g (possibly) as improper integrals. Under the same additional conditions on w and f, like previous, we have

$$0 = \int_\Gamma f(z)w(z)\,dz + \int_{-1}^{1} f(x)w(x)\,dx,$$

where $\Gamma = \Gamma_R$ or Γ_R^*. Then both systems of the orthogonal polynomials $\{\pi_n\}$ and $\{\pi_n^*\}$ exist uniquely.

The inner products in (4.2) and (4.3) define the moment functionals

$$\mathcal{L}z^k = \mu_k, \qquad \mu_k = (z^k,1) = \int_{\Gamma_R} z^k w(z)(iz-R)^{-1}\,dz$$

and

$$\mathcal{L}^*z^k = \mu_k^*, \qquad \mu_k^* = (z^k,1)^* = \int_{\Gamma_R^*} z^k w(z)(iz+R)^{-1}\,dz,$$

respectively. Using the moment determinants, we can express the (monic) polynomials π_k and π_k^* as

$$\pi_k(z) = \frac{1}{\Delta_k}\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & & \mu_{k+1} \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-1} \\ 1 & z & & z^k \end{vmatrix}$$

and

$$\pi_k^*(z) = \frac{1}{\Delta_k^*} \begin{vmatrix} \mu_0^* & \mu_1^* & \cdots & \mu_k^* \\ \mu_1^* & \mu_2^* & & \mu_{k+1}^* \\ \vdots & & & \\ \mu_{k-1}^* & \mu_k^* & & \mu_{2k-1}^* \\ 1 & z & & z^k \end{vmatrix},$$

where

$$\Delta_k = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{k-1} \\ \mu_1 & \mu_2 & & \mu_k \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-2} \end{vmatrix}, \qquad \Delta_k^* = \begin{vmatrix} \mu_0^* & \mu_1^* & \cdots & \mu_{k-1}^* \\ \mu_1^* & \mu_2^* & & \mu_k^* \\ \vdots & & & \\ \mu_{k-1}^* & \mu_k^* & & \mu_{2k-2}^* \end{vmatrix}.$$

We can prove that $\pi_k^*(\overline{z}) = \overline{\pi_k(z)}$, as well as the relation

$$\pi_k^*(z) = p_k(z) - i\theta_{k-1}^* p_{k-1}(z), \qquad k = 0, 1, 2, \dots,$$

where

$$\theta_{k-1}^* = \frac{(\pi_k^*, \pi_k^*)^*}{[p_{k-1}, p_{k-1}]}, \quad k = 1, 2, \dots, \qquad \theta_{-1}^* = \mu_0^*.$$

Here, $\theta_{k-1}^* = -\overline{\theta}_{k-1}$, where θ_{k-1} is the corresponding coefficient in the polynomial π_k.

Also, the following theorem holds:

Theorem 4.1. *The dual (monic) orthogonal polynomials $\{\pi_k^*\}$ satisfy the three-term recurrence relation*

$$\begin{aligned} \pi_{k+1}^*(z) &= (z - i\alpha_k^*)\pi_k^*(z) - \beta_k^*\pi_{k-1}^*(z), \qquad k = 0, 1, 2, \dots, \\ \pi_{-1}^*(z) &= 0, \quad \pi_0^*(z) = 1, \end{aligned}$$

with $\alpha_k^* = -\overline{\alpha}_k$ and $\beta_k^* = \overline{\beta}_k$, where α_k and β_k are the coefficients in the corresponding recurrence relation for the polynomials $\{\pi_k\}$.

Using dual polynomials we can give a very short proof that $\theta_{k-1} > 0$ ($k \geq 0$) for a symmetric weight $w(z) = w(-z)$. Namely, since $(\pi_k, \pi_k) = \theta_{k-1}[p_{k-1}, p_{k-1}]$ it is enough to prove that $(\pi_k, \pi_k) > 0$. In this *symmetric case*, θ_{k-1} is real and we have $\theta_{k-1}^* = -\theta_{k-1}$ and

$$(\pi_k, \pi_k) = (\pi_k, \pi_k^*) = \int_{\Gamma_R} G(z)w(z)(iz - R)^{-1}\, dz = -\int_{-1}^{1} G(x)\frac{w(x)}{ix - R}\, dx,$$

where $G(z) = p_k(z)^2 + \theta_{k-1}^2 p_{k-1}(z)^2$. Then

$$(\pi_k, \pi_k) = R \int_{-1}^{1} G(x) \frac{w(x)}{R^2 + x^2}\, dx + i \int_{-1}^{1} xG(x) \frac{w(x)}{R^2 + x^2}\, dx.$$

Since $x \mapsto G(x)$ is an even positive function, the second integral on the right-hand side vanishes and $(\pi_k, \pi_k) > 0$.

One complicated proof of the previous result was given in [7].

5. Geronimus' version of orthogonality

In the paper [21], J. W. Jayne considered the Geronimus' concept of orthogonality for recursively generated polynomials. Ya. L. Geronimus proved that a sequence of polynomials $\{p_k\}$, which is orthogonal on a finite interval on real line, is also orthogonal in the sense that there is a weight function $z \mapsto \chi(z)$ having one or more singularities inside a simple curve C and such that

$$\langle p_k, p_m \rangle = \frac{1}{2\pi i} \oint_C p_k(z) p_m(z) \chi(z)\, dz = \begin{cases} 0, & k \neq m, \\ h_m, & k = m. \end{cases} \tag{5.1}$$

Following Geronimus [19] and Jayne [21], Milovanović and Rajković [32] determined such a complex weight function $z \mapsto \chi(z)$, for (monic) polynomials $\{\pi_k\}$ orthogonal on the semicircle Γ, and also for the corresponding polynomials $\{\pi_k^R\}$ orthogonal on the circular arc Γ_R $(R > 0)$.

Denoting by C any positively oriented simple closed contour surrounding some circle $|z| = r > 1$, we assume that

$$\chi(z) = \sum_{k=1}^{\infty} \omega_k z^{-k}, \quad \omega_1 = 1, \tag{5.2}$$

for $|z| > 1$, and express z^n as a linear combination of the monic polynomials π_m, $m = 0, 1, \ldots, n$, which are orthogonal on the semicircle Γ, with respect to the inner product (3.1). Thus,

$$z^n = \sum_{m=0}^{n} \gamma_{n,m} \pi_m(z), \tag{5.3}$$

where $(z^n, \pi_m) = \gamma_{n,m}(\pi_m, \pi_m)$, $m = 0, 1, \ldots, n$. Using the inner product (5.1) and the representation (5.2), we obtain

$$\langle z^n, 1 \rangle = \frac{1}{2\pi i} \oint_C z^n \chi(z)\, dz = \frac{1}{2\pi i} \oint_C \sum_{k=1}^{\infty} \omega_k z^{n-k}\, dz = \omega_{n+1}.$$

On the other hand, because of (5.3) and the orthogonality condition (5.1), we find

$$\langle z^n, 1\rangle = \langle \sum_{m=0}^{n} \gamma_{n,m}\pi_m(z), 1\rangle = \sum_{m=0}^{n} \gamma_{n,m}\langle \pi_m, 1\rangle,$$

i.e., $\langle z^n, 1\rangle = \gamma_{n,0}\langle \pi_0, \pi_0\rangle = \gamma_{n,0}h_0$. Thus, we have $w_{n+1} = \gamma_{n,0}h_0 = \gamma_{n,0}$, because $h_0 = \omega_1 = 1$.

Finally, using the moments $\mu_n = (z^n, 1)$, we obtain $\omega_{n+1} = \mu_n/\mu_0$, $n \geq 0$, and

$$\chi(z) = \frac{1}{\mu_0}\sum_{k=1}^{\infty} \mu_{k-1}z^{-k}, \qquad |z| > 1, \tag{5.4}$$

where we need the convergence of this series for $|z| > r > 1$.

Suppose that w be a weight function, nonnegative on $(-1, 1)$, holomorphic in $D_+ = \{z \in \mathbf{C} \,|\, |z| < 1, \operatorname{Im} z > 0\}$, integrable over ∂D_+, and such that (3.3) is satisfied. Then the moments μ_k can be expressed in the form

$$\mu_0 = \int_\Gamma w(z)(iz)^{-1}\,dz = \frac{1}{i}\left(i\pi w(0) - \text{v.p.}\int_{-1}^{1} \frac{w(x)}{x}\,dx\right)$$

and

$$\mu_k = \int_\Gamma z^k w(z)(iz)^{-1}\,dz = i\int_{-1}^{1} x^{k-1}w(x)\,dx, \quad k \geq 1.$$

These moments are included in the series (5.4).

Supposing that the weight function w has such moments μ_k, which provide the convergence of the series (5.4), for all z outside some circle $|z| = r > 1$ lying interior to C, Milovanović and Rajković [32] proved:

Theorem 5.1. *The monic polynomials* $\{\pi_k\}$, *which are orthogonal on the semicircle* Γ *with respect to the inner product* (3.1), *are also orthogonal in the sense of* (5.1), *where*

$$\chi(z) = \frac{1}{z}\left(1 + \frac{i}{\mu_0}\int_{-1}^{1} \frac{w(x)}{z - x}\,dx\right), \qquad |z| > r > 1,$$

and

$$\mu_0 = \pi w(0) + i\,\text{v.p.}\int_{-1}^{1} \frac{w(x)}{x}\,dx.$$

In Gegenbauer case they obtained the following result:

Corollary 5.2. *Let* $w(z) = (1-z^2)^{\lambda-1/2}$, $\lambda > -1/2$. *The monic polynomials* $\{\pi_k\}$, *which are orthogonal on the unit semicircle with respect to the inner product* (3.1), *are also orthogonal in the sense of* (5.1), *where*

$$\chi(z) = \frac{1}{z} + \frac{i}{\sqrt{\pi}\, z^2} \cdot \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)}\, F\Big(1, \frac{1}{2}, \lambda+1; \frac{1}{z^2}\Big),$$

where F *is the Gauss hypergeometric series and* Γ *is the gamma function.*

In Legendre case ($\lambda = 1/2$) we have

$$\chi(z) = \frac{1}{z} + \frac{i}{\pi z} \log \frac{z+1}{z-1},$$

where the interval from -1 to 1 on the real axis is considered as a branch cut.

The corresponding complex weight for polynomials $\{\pi_k^R\}$ ($R > 0$) orthogonal on the circular arc Γ_R was also derived in [32] in the form

$$\chi(z) = \frac{1}{\mu_0} \int_{-1}^{1} \frac{(R+ix)w(x)}{(R^2+x^2)(z-x)}\, dx, \qquad |z| > r > 1,$$

where

$$\mu_0 = \int_{-1}^{1} \frac{R+ix}{R^2+x^2}\, w(x)\, dx.$$

6. Orthogonality on the radial rays in the complex plane

In this section we start with a new type of nonstandard orthogonality on some radial rays in the complex plane. Suppose that we have M points in the complex plane, $z_s = a_s e^{i\varphi_s} \in \mathbb{C}$, $s = 0, 1, \ldots, M-1$, with different arguments φ_s. Some of a_s (or all) can be ∞. The case $M = 5$ is shown in Fig. 6.1. We can define an inner product on these radial rays ℓ_s in the complex plane which connect the origin $z = 0$ and the points z_s, $s = 0, 1, \ldots, M-1$. Namely,

$$(f,g) = \sum_{s=0}^{M-1} e^{-i\varphi_s} \int_{\ell_s} f(z)\overline{g(z)}\, |w(z)|\, dz,$$

where $z \mapsto w(z)$ is a suitable function (complex weight).

Since, this product can be expressed in the form

$$(f,g) = \sum_{s=0}^{M-1} \int_{0}^{a_s} f\big(xe^{i\varphi_s}\big)\overline{g\big(xe^{i\varphi_s}\big)}\, \big|w\big(xe^{i\varphi_s}\big)\big|\, dx,$$

we see that

$$(f,f)=\sum_{s=0}^{M-1}\int_0^{a_s}\left|f\left(xe^{i\varphi_s}\right)\right|^2\left|w\left(xe^{i\varphi_s}\right)\right|dx>0,$$

except when $f(z)=0$.

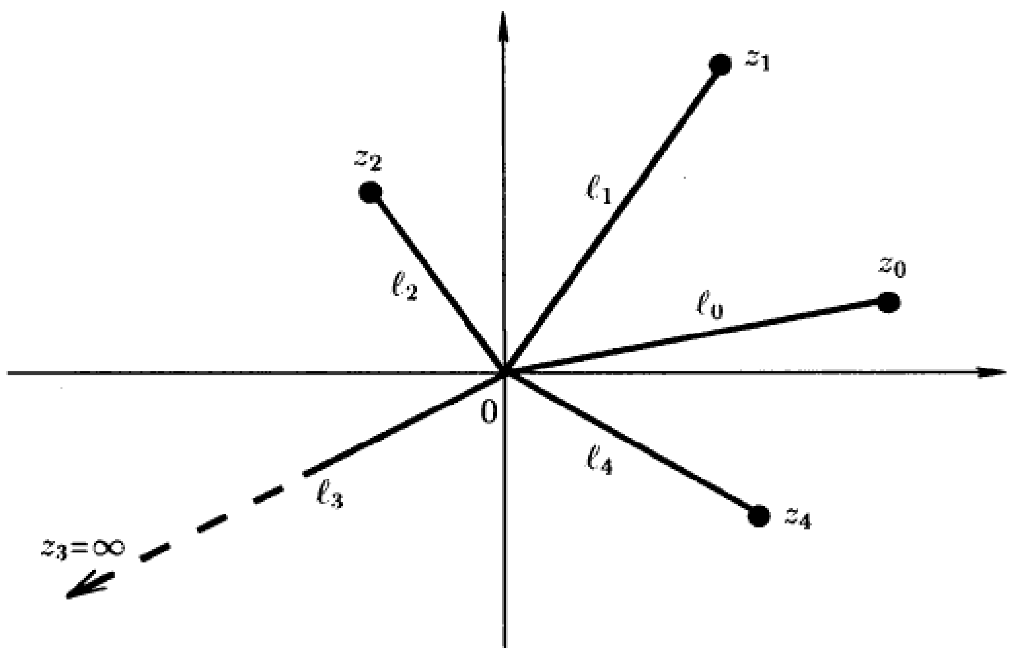

Fig. 6.1

We will consider here only the case when M is an even number and $\varphi_s=\pi s/m$, $s=0,1,\ldots,2m-1$. Thus, let $m\in\mathbf{N}$ and $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{2m-1}$ be $(2m)$th roots of unity, i.e., $\varepsilon_s=\exp(i\pi s/m)$, $s=0,1,\ldots,2m-1$. We will study orthogonal polynomials relative to the inner product

$$(f,g)=\sum_{s=0}^{2m-1}\varepsilon_s^{-1}\int_{\ell_s}f(z)\overline{g(z)}|w(z)|\,dz. \tag{6.1}$$

Suppose that $a_s=1$ for each s and let $z\mapsto w(z)$ be a holomorphic function such that

$$|w(x\varepsilon_s)|=w(x),\quad s=0,1,\ldots,2m-1,$$

and $x\mapsto w(x)$ be a weight function on $(0,1)$ (nonnegative on $(0,1)$ and $\int_0^1 w(x)\,dx>0$). Then, (6.1) can be written in the form

$$(f,g)=\int_0^1\left(\sum_{s=0}^{2m-1}f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\right)w(x)\,dx. \tag{6.2}$$

In the case $m = 1$, (6.2) becomes

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx,$$

so we have the standard case of polynomials orthogonal on $(-1,1)$ with respect to the weight function $x \mapsto w(x)$.

The inner product (6.2) has the following property:

Lemma 6.1. $(z^m f, g) = (f, z^m g)$.

Proof. Since $\varepsilon_s^m = \varepsilon_s^{-m} = (-1)^s$ we have

$$\begin{aligned}(z^m f, g) &= \int_0^1 \left(\sum_{s=0}^{2m-1} x^m \varepsilon_s^m f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\right) w(x)\,dx \\ &= \int_0^1 \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{x^m \varepsilon_s^m g(x\varepsilon_s)}\right) w(x)\,dx \\ &= (f, z^m g). \quad \square\end{aligned}$$

The moments are given by

$$\mu_{p,q} = (z^p, z^q) = \left(\sum_{s=0}^{2m-1} \varepsilon_s^{p-q}\right) \int_0^1 x^{p+q} w(x)\,dx, \quad p,q \geq 0. \tag{6.3}$$

If $p = 2mn + \nu$, $n = [p/(2m)]$, and $0 \leq \nu \leq 2m-1$, it is easy to verify that

$$\sum_{s=0}^{2m-1} \varepsilon_s^p = \sum_{s=0}^{2m-1} \varepsilon_s^\nu = \begin{cases} 2m & \text{if } \nu = 0, \\ 0 & \text{if } 1 \leq \nu \leq 2m-1. \end{cases}$$

Thus, $\mu_{p,q}$ in (6.3) is different from zero only if $p \equiv q \pmod{2m}$; otherwise $\mu_{p,q} = 0$. Using the moment determinants

$$\Delta_0 = 1, \quad \Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} \end{vmatrix}, \quad N \geq 1,$$

we can prove the following existence result for the (monic) orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ with respect to the inner product (6.2) (see Milovanović [30]):

Theorem 6.2. *If $\Delta_N > 0$ for all $N \geq 1$ the monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$, orthogonal with respect to the inner product* (6.2), *exist uniquely.*

It is well known that an orthogonal sequence of polynomials satisfies a three-term recurrence relation if the inner product has the property $(zf, g) = (f, zg)$. In our case the corresponding property is given by $(z^m f, g) = (f, z^m g)$ (see Lemma 6.1) and the following result holds:

Theorem 6.3. *Let the inner product $(\cdot,\cdot)$ be given by* (6.2) *and let the corresponding system of monic orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ exist. They satisfy the recurrence relation*

$$\tag{6.4} \begin{gathered} \pi_{N+m}(z) = z^m \pi_N(z) - b_N \pi_{N-m}(z), \quad N \geq m, \\ \pi_N(z) = z^N, \quad N = 0, 1, \ldots, 2m-1, \end{gathered}$$

where

$$\tag{6.5} b_N = \frac{(\pi_N, z^m \pi_{N-m})}{(\pi_{N-m}, \pi_{N-m})} = \frac{\|\pi_N\|^2}{\|\pi_{N-m}\|^2}.$$

In a simple case when $m = 2$ and $w(x) = 1$, i.e., when the inner product $(\cdot,\cdot)$ is given by

$$\tag{6.6} (f, g) = \int_0^1 \left[f(x)\overline{g(x)} + f(ix)\overline{g(ix)} + f(-x)\overline{g(-x)} + f(-ix)\overline{g(-ix)} \right] dx,$$

we can calculate directly the coefficient b_N in the recurrence relation (6.4). The moments are given by

$$\mu_{p,q} = (z^p, z^q) = \begin{cases} \dfrac{4}{p+q+1}, & p \equiv q \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if $p = 4i + \nu$ and $q = 4j + \nu$, $\nu \in \{0, 1, 2, 3\}$, we have

$$\mu_{4i+\nu,4j+\nu} = \frac{4}{4(i+j) + 2\nu + 1}, \qquad i, j \geq 0.$$

Our purpose is to evaluate the moment determinants

$$\Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} \end{vmatrix}, \quad N \geq 1.$$

In order to make it, for every $k \in \mathbf{N}$, we define the determinants

$$C_k = \begin{vmatrix} \mu_{00} & 0 & \mu_{40} & 0 & \cdots & \\ 0 & \mu_{22} & 0 & \mu_{62} & & \\ \mu_{04} & 0 & \mu_{44} & 0 & & \\ 0 & \mu_{26} & 0 & \mu_{66} & & \\ \vdots & & & & \ddots & \\ & & & & & \mu_{2k-2,2k-2} \end{vmatrix},$$

$$D_k = \begin{vmatrix} \mu_{11} & 0 & \mu_{51} & 0 & \cdots & \\ 0 & \mu_{33} & 0 & \mu_{73} & & \\ \mu_{15} & 0 & \mu_{55} & 0 & & \\ 0 & \mu_{37} & 0 & \mu_{77} & & \\ \vdots & & & & \ddots & \\ & & & & & \mu_{2k-1,2k-1} \end{vmatrix},$$

which can be expressed in terms of the determinants $E_0^{(\nu)} = 1$ and

$$E_n^{(\nu)} = \begin{vmatrix} \mu_{\nu,\nu} & \mu_{4+\nu,\nu} & \cdots & \mu_{4(n-1)+\nu,\nu} \\ \mu_{\nu,4+\nu} & \mu_{4+\nu,4+\nu} & \cdots & \mu_{4(n-1)+\nu,4+\nu} \\ \vdots & & & \\ \mu_{\nu,4(n-1)+\nu} & \mu_{4+\nu,4(n-1)+\nu} & \cdots & \mu_{4(n-1)+\nu,4(n-1)+\nu} \end{vmatrix},$$

where $\nu = 0, 1, 2, 3$.

Interpreting these determinants in terms of Hilbert-type determinants and using Cauchy's formula (see Muir [34, p. 345])

$$\det \left[\frac{1}{a_i + b_j} \right]_{i,j=1}^{n} = \frac{\prod_{i>j=1}^{n} (a_i - a_j)(b_i - b_j)}{\prod_{i,j=1}^{n} (a_i + b_j)}$$

with $a_i = 4i$ and $b_j = 4j + 2\nu - 7$, we obtain (see [30])

$$E_n^{(\nu)} = 4^{n^2} \frac{(0!1!\cdots(n-1)!)^2}{\prod_{i=0}^{n-1} \prod_{j=0}^{n-1} (4i + 4j + 2\nu + 1)}, \quad n \geq 1.$$

Also, we can prove that

$$C_k = E_{k/2}^{(0)} E_{k/2}^{(2)}, \quad k(\text{even}) \geq 2; \quad C_k = E_{(k+1)/2}^{(0)} E_{(k-1)/2}^{(2)}, \quad k(\text{odd}) \geq 1,$$

as well as

$$D_k = E^{(1)}_{k/2} E^{(3)}_{k/2}, \quad k(\text{even}) \geq 2; \quad D_k = E^{(1)}_{(k+1)/2} E^{(3)}_{(k-1)/2}, \quad k(\text{odd}) \geq 1.$$

Using the same techniques we find that

$$\Delta_{2k} = C_k D_k \quad \text{and} \quad \Delta_{2k+1} = C_{k+1} D_k.$$

Combining these equalities we obtain:

Lemma 6.4. *We have*

$$\begin{aligned} \Delta_{4n} &= E^{(0)}_n E^{(1)}_n E^{(2)}_n E^{(3)}_n, \\ \Delta_{4n+1} &= E^{(0)}_{n+1} E^{(1)}_n E^{(2)}_n E^{(3)}_n, \\ \Delta_{4n+2} &= E^{(0)}_{n+1} E^{(1)}_{n+1} E^{(2)}_n E^{(3)}_n, \\ \Delta_{4n+3} &= E^{(0)}_{n+1} E^{(1)}_{n+1} E^{(2)}_{n+1} E^{(3)}_n. \end{aligned}$$

We note, first of all, that $\Delta_N > 0$ for all $N \geq 1$, and therefore, the orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ with respect to the inner product (6.6) exist uniquely, and

$$(\pi_N, \pi_N) = \|\pi_N\|^2 = \frac{\Delta_{N+1}}{\Delta_N} > 0.$$

Theorem 6.5. *The (monic) polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$, orthogonal with respect to the inner product (6.6), satisfy the recurrence relation*

$$\begin{aligned} &\pi_{N+2}(z) = z^2 \pi_N(z) - b_N \pi_{N-2}(z), \quad N \geq 2, \\ &\pi_N(z) = z^N, \quad N = 0, 1, 2, 3, \end{aligned} \tag{6.7}$$

where

$$b_{4n+\nu} = \begin{cases} \dfrac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 0, 1, \\[2ex] \dfrac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 2, 3. \end{cases} \tag{6.8}$$

Proof. Because of (6.5), the coefficients b_N can be expressed in the form

$$b_N = \frac{\|\pi_N\|^2}{\|\pi_{N-2}\|^2} = \frac{\Delta_{N+1}}{\Delta_N} \cdot \frac{\Delta_{N-2}}{\Delta_{N-1}}, \quad N \geq 2.$$

In order to find these quotients we need a quotient of the determinants $E_n^{(\nu)}$. According to the previous equalities we get

$$\frac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}} = \frac{4}{8n+2\nu+1}\left(\prod_{k=n}^{2n-1}\frac{4(k-n+1)}{4k+2\nu+1}\right)^2, \quad n \geq 1,$$

and $E_1^{(\nu)}/E_0^{(\nu)} = 4/(2\nu+1)$.

Then, for $\nu = 0, 1$ we find

$$b_{4n+\nu} = \frac{\Delta_{4n+\nu+1}/\Delta_{4n+\nu}}{\Delta_{4(n-1)+\nu+3}/\Delta_{4(n-1)+\nu+2}} = \frac{E_{n+1}^{(\nu)}/E_n^{(\nu)}}{E_n^{(\nu+2)}/E_{n-1}^{(\nu+2)}},$$

i.e.,

$$b_{4n+\nu} = \frac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)}.$$

Similarly, for $\nu = 2, 3$, we have

$$b_{4n+\nu} = \frac{\Delta_{4n+\nu+1}/\Delta_{4n+\nu}}{\Delta_{4n+\nu-1}/\Delta_{4n+\nu-2}} = \frac{E_{n+1}^{(\nu)}/E_n^{(\nu)}}{E_{n+1}^{(\nu-2)}/E_n^{(\nu-2)}},$$

i.e.,

$$b_{4n+\nu} = \frac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)}. \quad \square$$

From (6.8) we conclude that

$$b_N \to \frac{1}{4} \quad \text{as } N \to +\infty,$$

just like in Szegő's theory for orthogonal polynomials on the interval $(-1, 1)$.

Since

$$\|\pi_N\|^2 = \begin{cases} b_N b_{N-2} \cdots b_2 \|\pi_0\|^2, & N \text{ even}, \\ b_N b_{N-2} \cdots b_3 \|\pi_1\|^2, & N \text{ odd}, \end{cases}$$

and $\|\pi_0\|^2 = \Delta_1/\Delta_0 = \mu_{00} = 4$ $(\Delta_0 \equiv 1)$,

$$\|\pi_1\|^2 = \Delta_2/\Delta_1 = \mu_{00}\mu_{11}/\mu_{00} = \mu_{11} = 4/3,$$

we can define $b_0 = 4$, $b_1 = 4/3$, so that (6.7) holds for every $N \geq 0$, where

$$\pi_{-2}(z) = \pi_{-1}(z) = 0, \ \pi_0(z) = 1, \ \pi_1(z) = z.$$

Finally, we can determine the norms of the polynomials $\{\pi_N(z)\}$. Let $N = 4n + \nu$, $n = [N/4]$, $0 \le \nu \le 3$. Since

$$\|\pi_N\|^2 = \frac{\Delta_{N+1}}{\Delta_N} = \frac{\Delta_{4n+\nu+1}}{\Delta_{4n+\nu}} = \frac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}},$$

we have

$$\|\pi_N\|^2 = \frac{4}{2N+1}, \quad 0 \le N \le 3,$$

$$\|\pi_N\|^2 = \|\pi_{4n+\nu}\|^2 = \frac{4}{8n+2\nu+1}\left(\prod_{k=n}^{2n-1} \frac{4(k-n+1)}{4k+2\nu+1}\right)^2, \quad N \ge 4.$$

7. A representation of $\pi_N(z)$ and zeros

In this section we again consider the general case of the inner product (6.2) for which the corresponding system of the monic orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ exists and satisfies the recurrence relation (6.4). Based on this recurrence relation, we can conclude and easily prove that $\pi_N(z)$ are incomplete polynomials with the following representation (see Milovanović [30]):

Theorem 7.1. *The polynomials from Theorem 6.3 can be expressed in the form*

$$\pi_{2mn+\nu}(z) = z^\nu q_n^{(\nu)}(z^{2m}), \quad \nu = 0, 1, \ldots, 2m-1; \ n = 0, 1, \ldots, \tag{7.1}$$

where $q_n^{(\nu)}(t)$, $\nu = 0, 1, \ldots, 2m-1$, *are monic polynomials of exact degree* n, *which satisfy the three-term recurrence relation*

$$\begin{aligned} &q_{n+1}^{(\nu)}(t) = (t - a_n^{(\nu)})q_n^{(\nu)}(t) - b_n^{(\nu)} q_{n-1}^{(\nu)}(t), \quad n = 0, 1, \ldots, \\ &q_0^{(\nu)}(t) = 1, \quad q_{-1}^{(\nu)}(t) = 0. \end{aligned} \tag{7.2}$$

The recursion coefficients $a_n^{(\nu)}$ *and* $b_n^{(\nu)}$ *are given in terms of the b-coefficients as*

$$a_n^{(\nu)} = b_N + b_{N+m}, \quad b_n^{(\nu)} = b_{N-m} b_N, \quad N = 2mn + \nu.$$

The three-term recurrence relation (7.2) shows that the monic polynomial systems $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$, $\nu = 0, 2, \ldots, 2m-1$, are orthogonal. The following theorem gives this orthogonality:

Theorem 7.2. *Let $x \mapsto w(x)$ be a weight function in the inner product* (6.2) *which guarantees the existence of the polynomials $\pi_N(z)$, i.e., $q_n^{(\nu)}(t)$, $\nu = 0, 1, \ldots, 2m-1$, determined by* (7.1). *For any $\nu \in \{0, 1, \ldots, 2m-1\}$, the sequence of polynomials $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$ is orthogonal on* $(0, 1)$ *with respect to the weight function $t \mapsto w_\nu(t) = t^{(2\nu+1-2m)/2m} w(t^{1/2m})$.*

As we can see the question of the existence of the polynomials $\pi_N(z)$ is reduced to the existence of polynomials $q_n^{(\nu)}(t)$, orthogonal on $(0, 1)$ with respect to the weight function $w_\nu(t)$, for every $\nu = 0, 1, \ldots, 2m-1$.

The next result gives the zero distribution of the polynomials $\pi_N(z)$ (see [30]):

Theorem 7.3. *Let $N = 2mn + \nu$, $n = [N/2m]$, $\nu \in \{0, 1, \ldots, 2m-1\}$. All zeros of the polynomial $\pi_N(z)$ are simple and located symmetrically on the radial rays l_s, $s = 0, 1, \ldots, 2m-1$, with the possible exception of a multiple zero of order ν at the origin $z = 0$.*

At the end we mention that an analogue of the Jacobi polynomials and the corresponding problem with the generalized Laguerre polynomials were treated in [30].

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