Research Article

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On spherical shells containing all the zeros of a quaternionic polynomial

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Abstract: The goal of this paper is to find spherical shells that contain all of the zeros of a unilateral polynomial with quaternionic coefficients. These bounds are derived using Geršgorin-type results for the norms of the eigenvalues of a quaternionic matrix and matrix similarity. In addition to yielding some interesting implications, our results also bring certain classical results of Diaz-Barrero, Egozcue, Bidkham, and others into the quaternionic setting.

Keywords: Quaternionic polynomial, Noncommutative divsion ring, Similar matrices, t-Fibonacci numbers, Pell numbers.

1 Introduction and preliminaries

In geometric function theory, locating the zeros of a polynomial in the plane using various techniques and approaches is classical and of great importance. This kind of research has previously been carried out by several mathematicians, who have produced bounds for the moduli of the zeros of polynomials for a number of classical problems. Gauss and Cauchy were among the first to work on this subject. Since then, the area has progressed further due to the many articles published with the aim of finding new bounds for the zeros of a polynomial; see, for instance, ([1], [3]). First, we give the following classical result, which is of practical importance and gives an upper bound on the moduli of the zeros of a complex coefficient polynomial. It is due to Cauchy (see [4, p. 122]).

Theorem 1. *Let $A(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n with complex coefficients. Then all the zeros of $A(z)$ lie in $|z| \leq r$, where r is the unique positive root of the equation*

$$|a_n|z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|z - |a_0| = 0.$$

We suggest the reader to consult the comprehensive books of Marden [4] and Milovanović et al. [5] for a thorough analysis of explicit upper bounds for the zeros of a polynomial, which peaked in the early twentieth century. The elegant conclusion cited above has led to several comparable investigations, for example, see [6], that have since been published in the literature and provide insight into the zero bounds of a complex coefficient polynomial in the plane. In [7], Diaz-Barrero improved the above result by establishing an annulus containing all the zeros of a polynomial, when the inner and outer radii are expressed in terms of binomial coefficients and Fibonacci numbers.

Theorem 2. *Let $A(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ $(a_\nu \neq 0,\ 1 \leq \nu \leq n)$ be a non-constant complex polynomial. Then all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \frac{3}{2} \min_{1 \leq \nu \leq n} \left\{ \frac{2^n F_\nu \binom{n}{\nu}}{F_{4n}} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

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and

$$r_2 = \frac{2}{3} \max_{1 \le \nu \le n} \left\{ \frac{F_{4n}}{2^n F_\nu \binom{n}{\nu}} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

Here F_n is the nth Fibonacci number, namely, $F_0 = 0$, $F_1 = 1$ and for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$. Furthermore, $\binom{n}{\nu} = \frac{n!}{\nu!(n-\nu)!}$ are the binomial coefficients.

Further, Diaz-Barrero and Egozcue in [8] used Pell numbers $P_n = 2P_{n-1} + P_{n-2}$ for $n \ge 2$, where $P_0 = 0$ and $P_1 = 1$, and determined an annulus in the complex plane containing all the zeros of a polynomial with complex coefficients in the form of following result.

Theorem 3. *Let $A(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ ($a_\nu \ne 0$, $1 \le \nu \le n$) be a polynomial with nonzero complex coefficients. Then all its zeros lie in the ring shaped region $C = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$, where*

$$r_1 = \frac{5}{2} \min_{1 \le \nu \le n} \left\{ \frac{2^n P_\nu \binom{n}{\nu}}{P_{3n}} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2 = \frac{2}{5} \max_{1 \le \nu \le n} \left\{ \frac{P_{3n}}{2^n P_\nu \binom{n}{\nu}} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

Remark 1. For different classes of special numbers and polynomials, including orthogonal polynomials, Humbert's polynomials, Horadam polynomials and their generalizations see Đorđević and Milovanović [9], as well as a recent paper by Ben Romdhane and Abdelkader [10].

The bounds presented by the last two theorems above are a small selection from an extensive literature that had its peak in the early part of the twenty-first century. The section on polynomial zeros in [5] can be helpful for interested readers. However, we believe that sophisticated bounds, which may require the use of computational devices, are not particularly interesting because efficient zero-computation algorithms exist. While we have extremely useful and effective bounds for the zeros of a polynomial with complex coefficients, our main goal is to demonstrate how these elegant results can be extended to derive zero inclusion regions of polynomials with quaternionic variables and quaternionic coefficients. Such conclusions have applications not just in algebra, analysis, geometry, and other subjects, but also in modern mathematical topics such as computer graphics, control theory, signal processing, physics, and fluid dynamics.

We denote by $\mathbb{H}$, the noncommutative division ring of quaternions. It consists of elements of the form $q = x_0 + x_1 i + x_2 j + x_3 k$; $x_0, x_1, x_2, x_3 \in \mathbb{R}$, where the imaginary units i, j, k satisfy

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Every element $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ is composed by the real part $\mathrm{Re}(q) = x_0$ and the imaginary part $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$ and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. The inverse of each non zero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$. For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbb{H} : |q| < r\}$.

The functions we consider in this paper are polynomials of the form

$$T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu, \quad a_\nu \in \mathbb{H}, \quad \nu = 0, 1, 2, \ldots, n, \tag{1}$$

with quaternionic coefficients on the right and indeterminate on the left. Inspired by a work of Cullen [11], Gentili and Struppa introduced the definition of regularity for these functions of quaternionic variables in [12]. We refer the reader to [13–20] and the reference therein, for definitions and properties of quaternions and many aspects of the theory of quaternionic regular functions. In recent times, there has been a lot of activity in the study of regular functions, specifically polynomials of a quaternionic variable, with a focus on their zero bounds. As remarked above, this study finds many applications not just in geometric function theory but also in operator theory, quantum physics, and functional calculus. Polynomials with quaternionic coefficients located on only one side of the variable were also investigated by Janovská and Opfer in [21, 22]. It is noted that the zeros of a polynomial of type (1) are either spherical or isolated (e.g., see [14, 21]). The bounds for the zeros of a polynomial with restricted coefficients have been the subject of multiple recent publications in the literature (see, for instance, [23–32]).

In this study, bounds for the moduli of all zeros of a unilateral polynomial of type (1), without any restriction on its coefficients, are derived with the purpose of extending some classical results.

2 Main results

Our primary findings are presented in this section. Their proofs are presented in the next section. We first construct a spherical shell containing all zeros of a unilateral polynomial of type (1) using t-Fibonacci numbers $F_{t,n} = tF_{t,n-1} + F_{t,n-2}$, $n \geq 2$, with initial conditions $F_{t,0} = 0$, $F_{t,1} = 1$ and t is any positive real number.

Theorem 4. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ $(a_\nu \neq 0,\ 1 \leq \nu \leq n)$ be a non-constant quaternionic polynomial of degree n. Then all the zeros of $T(q)$ lie in the spherical shell $D = \{q \in \mathbb{H} : r_1 \leq |q| \leq r_2\}$, where*

$$r_1 = \min_{1\leq\nu\leq n} \left\{ \frac{(t^2+1)^\nu t^{n-\nu} F_{t,\nu} \binom{n}{\nu}}{F_{t,3n}} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2 = \max_{1\leq\nu\leq n} \left\{ \frac{F_{t,3n}}{(t^2+1)^\nu t^{n-\nu} F_{t,\nu} \binom{n}{\nu}} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

Observe that the t-Fibonacci number $F_{t,n}$ reduces to the Pell number P_n for $t = 2$. We obtain the following quaternionic analogue of Theorem 3 by taking $t = 2$ in Theorem 4.

Corollary 1. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ $(a_\nu \neq 0,\ 1 \leq \nu \leq n)$ be a non-constant quaternionic polynomial of degree n. Then all the zeros of $T(q)$ lie in the spherical shell $D = \{q \in \mathbb{H} : r_1 \leq |q| \leq r_2\}$, where*

$$r_1 = \frac{5}{2} \min_{1\leq\nu\leq n} \left\{ \frac{2^n P_\nu \binom{n}{\nu}}{P_{3n}} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2 = \frac{2}{5} \max_{1\leq\nu\leq n} \left\{ \frac{P_{3n}}{2^n P_\nu \binom{n}{\nu}} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

Next, we obtain the following result: a spherical shell containing all the zeros of a unilateral polynomial with quaternionic coefficients. A special case of this result gives the quaternionic analogue of Theorem 2.

Theorem 5. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ $(a_\nu \neq 0,\ 1 \leq \nu \leq n)$ be a non-constant quaternionic polynomial of degree n. Then all the zeros of $T(q)$ lie in the spherical shell $D = \big\{q \in \mathbb{H} : r_1 \leq |q| \leq r_2\big\}$, where*

$$r_1 = \min_{1\leq\nu\leq n} \left\{ \frac{(t^2+1)^{n-\nu} (t^3+2t)^\nu F_{t,\nu} \binom{n}{\nu}}{F_{t,4n}} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2 = \max_{1\leq\nu\leq n} \left\{ \frac{F_{t,4n}}{(t^2+1)^{n-\nu} (t^3+2t)^\nu F_{t,\nu} \binom{n}{\nu}} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

We now discuss some consequences of Theorem 5.

For $t = 1$ in Theorem 5, we get the following quaternionic analogue of Theorem 2.

Corollary 2. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ $(a_\nu \neq 0,\ 1 \leq \nu \leq n)$ be a non-constant quaternionic polynomial of degree n. Then all the zeros of $T(q)$ lie in the spherical shell $D = \{q \in \mathbb{H} : r_1 \leq |q| \leq r_2\}$, where*

$$r_1 = \frac{3}{2} \min_{1\leq\nu\leq n} \left\{ \frac{2^n F_\nu \binom{n}{\nu}}{F_{4n}} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2 = \frac{2}{3} \max_{1\leq\nu\leq n} \left\{ \frac{F_{4n}}{2^n F_\nu \binom{n}{\nu}} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

Remark 2. Recall that for any $t > 0$, the t-Fibonacci sequence $\{F_{t,n}\}_{n\in\mathbb{N}}$ is defined by

$$F_{t,n+1} = tF_{t,n} + F_{t,n-1} \quad \text{for} \quad n \geq 1, \tag{2}$$

with initial conditions $F_{t,0} = 0$ and $F_{t,1} = 1$. Furthermore, for $t = 2$, we get $F_{t,n} = P_n$, where $\{P_n\}$ is the Pell sequence of polynomials defined by

$$P_{n+1} = 2P_n + P_{n-1} \quad \text{for} \quad n \geq 1,$$

with initial conditions $P_0 = 0$ and $P_1 = 1$.

One obtains easily from Binet's formula (see [33]), the following relation:

$$F_{t,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{3}$$

where r_1 and r_2 are the roots of the characteristic equation

$$r^2 = tr + 1,$$

associated with the recurrence relation (2). For $t = 2$, we get $r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$, which on using in (3), gives

$$\begin{aligned} P_{4n} &= (r_1^{2n} + r_2^{2n})(r_1 + r_2) \\ &= (r_1^{2n} + r_2^{2n})P_{2n} \\ &= \left[(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}\right] P_{2n}. \end{aligned}$$

Hence, for $t = 2$ in Theorem 5, we get the following quaternionic analogue of a result due to Bidkham and Shashahani [34].

Corollary 3. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ $(a_\nu \neq 0,\ 1 \leq \nu \leq n)$ be a non-constant quaternionic polynomial of degree n. Then all the zeros of $T(q)$ lie in the spherical shell $D = \{q \in \mathbb{H} : r_1 \leq |q| \leq r_2\}$, where*

$$r_1 = \frac{12}{5} \min_{1\leq\nu\leq n} \left\{ \frac{5^n P_\nu \binom{n}{\nu}}{\left[(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}\right] P_{2n}} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2 = \frac{5}{12} \max_{1\leq\nu\leq n} \left\{ \frac{\left[(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}\right] P_{2n}}{5^n P_\nu \binom{n}{\nu}} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

3 Auxiliary results

We need the following lemmas to prove our main results. The following Geršgorin-type result for the distribution of eigenvalues of a quaternionic matrix is due to Zhang [35, Theorem 2].

Lemma 1. *Let $A = [a_{ij}]_{n\times n}$ be a quaternionic matrix and let $\lambda \in \mathbb{H}$ be a left or right eigenvalue of A. Then*

$$|\lambda| \le \max_i \sum_{j=1}^{n} |a_{ij}|.$$

The following lemma is due to Brenner [36, Theorem 13].

Lemma 2. *Similar matrices of quaternions have the same characteristic roots.*

Lemma 3. *Let $T(q) = q^n a_n + q^p a_p + q^{p-1}a_{p-1} + \cdots + qa_1 + a_0$, $0 \le p \le n-1$, be a polynomial of degree n with quaternionic coefficients. Then for every $t > 0$, all the zeros of $T(q)$ lie in the ball*

$$|q| \le \max\left\{t, \sum_{\nu=0}^{p}\left|\frac{a_\nu}{a_n}\right|\frac{1}{t^{n-\nu-1}}\right\}.$$

Proof. Let C_T be the companion matrix of the polynomial $T(q)$, then

$$C_T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1\\ -\dfrac{a_0}{a_n} & -\dfrac{a_1}{a_n} & -\dfrac{a_2}{a_n} & \cdots & -\dfrac{a_p}{a_n} & \cdots & 0 \end{bmatrix}.$$

Taking a diagonal matrix $A = \operatorname{diag}\left(\dfrac{1}{t^{n-1}}, \dfrac{1}{t^{n-2}}, \ldots, \dfrac{1}{t}, 1\right)$, where $t > 0$, we form the matrix

$$A^{-1}C_TA = \begin{bmatrix} 0 & t & 0 & \cdots & 0 & \cdots & 0\\ 0 & 0 & t & \cdots & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \cdots & t\\ -\dfrac{a_0}{a_n}\dfrac{1}{t^{n-1}} & -\dfrac{a_1}{a_n}\dfrac{1}{t^{n-2}} & -\dfrac{a_2}{a_n}\dfrac{1}{t^{n-3}} & \cdots & -\dfrac{a_p}{a_n}\dfrac{1}{t^{n-p-1}} & \cdots & 0 \end{bmatrix}.$$

Now, applying Lemma 1 to the rows, it follows that the left eigenvalues of $A^{-1}C_TA$ lie in the ball

$$|q| \le \max\left\{t, \sum_{\nu=0}^{p}\left|\frac{a_\nu}{a_n}\right|\frac{1}{t^{n-v-1}}\right\}. \tag{4}$$

The matrix $A^{-1}C_TA$ is similar to the matrix C_T, it follows by using Lemma 2 that all the zeros of $T(q)$ which are the left eigenvalues of C_T lie in the ball defined by (4). This completes the proof of Lemma 3. □

Lemma 4. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ $(a_\nu \ne 0,\ 1 \le \nu \le n)$ be a non-constant quaternionic polynomial of degree n. If $b_\nu \in \mathbb{H}$, $\nu = 1, 2, \ldots, n$, such that*

$$\sum_{\nu=1}^{n} |b_\nu| = 1,$$

then all the zeros of $T(q)$ *lie in the spherical shell* $E = \{q \in \mathbb{H} : r_1 \le |q| \le r_2\}$, *where*

$$r_1 = \min_{1\le\nu\le n} \left| b_\nu \frac{a_0}{a_\nu} \right|^{1/\nu}$$

and

$$r_2 = \max_{1\le\nu\le n} \left| \frac{1}{b_\nu} \frac{a_{n-\nu}}{a_n} \right|^{1/\nu}.$$

Proof. By Lemma 3, it follows by taking $p = n-1$ and replacing ν by $n-\nu$, that all the zeros of $T(q)$ lie in the ball

$$|q| \le \max\left\{ t, \sum_{\nu=1}^{n} \left| \frac{a_{n-\nu}}{a_n} \frac{1}{t^{v-1}} \right| \right\}, \tag{5}$$

for every $t > 0$. We take

$$t = \max_{1\le\nu\le n} \left| \frac{1}{b_\nu} \frac{a_{n-\nu}}{a_n} \right|^{\frac{1}{\nu}},$$

then

$$\left| \frac{1}{b_\nu} \frac{a_{n-\nu}}{a_n} \right| \le t^\nu, \quad \nu = 1, 2, \ldots, n.$$

Hence

$$\begin{aligned} \sum_{\nu=1}^{n} \left| \frac{a_{n-\nu}}{a_n} \right| \frac{1}{t^{\nu-1}} &\le t \sum_{\nu=1}^{n} |b_\nu| \\ &= t. \end{aligned}$$

Using this in (5), it follows that all the zeros of $T(q)$ lie in the ball $|q| \le r_2$. This proves the second part of Lemma 4. To prove the first part of Lemma 4, we use the second part. If $a_0 = 0$, then there is nothing to prove. Assume that $a_0 \ne 0$, and consider the reciprocal polynomial

$$\begin{aligned} F(q) &= q^n * T\left(\frac{1}{q}\right) \\ &= q^n a_0 + q^{n-1} a_1 + \cdots + q a_{n-1} + a_n, \end{aligned}$$

of degree n. By the second part of Lemma 4, we get all the zeros of $F(q)$ lie in the ball

$$\begin{aligned} |q| &\le \max_{1\le\nu\le n} \left| \frac{1}{b_\nu} \frac{a_\nu}{a_0} \right|^{1/\nu} \\ &= \max_{1\le\nu\le n} \left| \frac{1}{b_\nu \frac{a_0}{a_\nu}} \right|^{1/\nu} \\ &= \max_{1\le\nu\le n} \left\{ \frac{1}{\left| b_\nu \frac{a_0}{a_\nu} \right|^{1/\nu}} \right\} \\ &= \frac{1}{\min\limits_{1\le\nu\le n} \left| b_\nu \frac{a_0}{a_\nu} \right|^{1/\nu}} \\ &= \frac{1}{r_1}. \end{aligned}$$

As $T(q) = q^n * F(1/q)$, it follows that all the zeros of $T(q)$ lie in

$$|q| \geq r_1 = \min_{1\leq\nu\leq n} \left| b_\nu \frac{a_0}{a_\nu a_\nu} \right|^{1/\nu}. \tag{6}$$

The desired result follows by combining (5) and (6). □

The following lemma is due to Diaz-Barrero and Egozcue [37, Theorem 1].

Lemma 5. *Let τ and σ be the roots of the quadratic equation $x^2 - tx - s = 0$, being t, s strictly positive real numbers. Define the two sequence $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ by $B_n = \sum_{\nu=0}^{n-1} \tau^\nu \sigma^{n-1-\nu}$ and $A_n = c\tau^n + d\sigma^n$, where c and d are real constants. If $\eta \geq 2$, then*

$$\sum_{\nu=0}^{n} \binom{n}{\nu} (sB_{\eta-1})^{n-\nu} B_\eta^\nu A_\nu = A_{\eta n}.$$

4 Proofs of the main results

Proof of Theorem 4. After setting $s = 1$ in Lemma 5, we have τ, σ be the roots of $x^2 - tx - 1 = 0$. Also taking $c = 1/\sqrt{t^2+4}$, $d = -1/\sqrt{t^2+4}$, so that the t-Fibonacci number $F_{t,n}$ is given by

$$A_n = c\tau^n + d\sigma^n = F_{t,n}.$$

Further for $\eta = 3$, we have $B_2 = t$, $B_3 = t^2 + 1$, and note that $F_{t,0} = 0$, we get the identity

$$\sum_{\nu=1}^{n} (t^2+1)^\nu t^{n-\nu} F_{t,\nu} \binom{n}{\nu} = F_{t,3n}.$$

Let

$$b_\nu = \frac{(t^2+1)^\nu t^{n-\nu} F_{t,\nu} \binom{n}{\nu}}{F_{t,3n}}, \quad \nu = 1, 2, \ldots, n.$$

Therefore, we have

$$\sum_{\nu=1}^{n} |b_\nu| = \sum_{\nu=1}^{n} b_\nu = \sum_{\nu=1}^{n} \frac{(t^2+1)^\nu t^{n-\nu} F_{t,\nu} \binom{n}{\nu}}{F_{t,3n}} = 1,$$

and, hence from Lemma 4, all the zeros of $T(q)$ lie in the spherical shell $D = \{q \in \mathbb{H} : r_1 \leq |q| \leq r_2\}$, where

$$r_1 = \min_{1\leq\nu\leq n} \left| b_\nu \frac{a_0}{a_\nu} \right|^{1/\nu} = \min_{1\leq\nu\leq n} \left| \frac{(t^2+1)^\nu t^{n-\nu} F_{t,\nu} \binom{n}{\nu}}{F_{t,3n}} \frac{a_0}{a_\nu} \right|^{1/\nu}$$

and

$$r_2 = \max_{1\leq\nu\leq n} \left| \frac{1}{b_\nu} \frac{a_{n-\nu}}{a_n} \right|^{1/\nu} = \max_{1\leq\nu\leq n} \left| \frac{F_{t,3n}}{(t^2+1)^\nu t^{n-\nu} F_{t,\nu} \binom{n}{\nu}} \frac{a_{n-\nu}}{a_n} \right|^{1/\nu}.$$

This completes the proof of Theorem 4. □

Proof of Theorem 5. For $s = 1$ and $\eta = 4$, we proceed as in the proof of Theorem 4, and obtain $B_3 = t^2 + 1, B_4 = t^3 + 2t$. Additionally, take note that $F_{t,0} = 0$ yields the identity

$$\sum_{\nu=1}^{n} (t^2+1)^{n-\nu} (t^3+2t)^\nu F_{t,\nu} \binom{n}{\nu} = F_{t,4n}.$$

Let

$$b_\nu = \frac{(t^2+1)^{n-\nu}(t^3+2t)^\nu F_{t,\nu}\binom{n}{\nu}}{F_{t,4n}}, \quad \nu = 1,2,\ldots,n.$$

Therefore, we have

$$\sum_{\nu=1}^{n} |b_\nu| = \sum_{\nu=1}^{n} b_\nu = \sum_{\nu=1}^{n} \frac{(t^2+1)^{n-\nu}(t^3+2t)^\nu F_{t,\nu}\binom{n}{\nu}}{F_{t,4n}} = 1,$$

and, hence from Lemma 4, all the zeros of $T(q)$ lie in the spherical shell $D = \{q \in \mathbb{H} : r_1 \le |q| \le r_2\}$, where

$$r_1 = \min_{1\le\nu\le n} \left| b_\nu \frac{a_0}{a_\nu} \right|^{1/\nu} = \min_{1\le\nu\le n} \left| \frac{(t^2+1)^{n-\nu}(t^3+2t)^\nu F_{t,\nu}\binom{n}{\nu}}{F_{t,4n}} \frac{a_0}{a_\nu} \right|^{1/\nu}$$

and

$$r_2 = \max_{1\le\nu\le n} \left| \frac{1}{b_\nu} \frac{a_{n-\nu}}{a_n} \right|^{1/\nu} = \max_{1\le\nu\le n} \left| \frac{F_{t,4n}}{(t^2+1)^{n-\nu}(t^3+2t)^\nu \binom{n}{\nu}} \frac{a_{n-\nu}}{a_n} \right|^{1/\nu}.$$

This completes the proof of Theorem 5. □

Remark 3. All the main results of this study are largely dependent on the Lemma 4. It has applications that go beyond the findings of this investigation. It can yield various conclusions about spherical shells containing all the zeros of a polynomial with quaternionic coefficients if the sequence $\{b_\nu\}$ is chosen appropriately. The Lucas sequence $\{L_n\}$, for example, is defined by

$$L_n = L_{n-2} + L_{n-1} \quad \text{for} \quad n \ge 2,$$

with the initial conditions $L_0 = 2, L_1 = 1$. Using an identity (see [38, p. 54]) concerning these numbers, namely:

$$\sum_{\nu=1}^{n} L_\nu = L_{n+2} - 3,$$

and take

$$b_\nu = \frac{L_\nu}{L_{n+2}-3}, \quad \nu = 1,2,\ldots,n,$$

then by the above identity, we have

$$\sum_{\nu=1}^{n} |b_\nu| = \sum_{\nu=1}^{n} \frac{L_\nu}{L_{n+2}-3} = 1.$$

On using this in Lemma 4, we see that all the zeros of $T(q) = \sum_{\nu=1}^{n} q^\nu a_\nu$ lie in the spherical shell $F = \{q \in \mathbb{H} : r_1^* \le |q| \le r_2^*\}$, where

$$r_1^* = \min_{1\le\nu\le n} \left\{ \frac{L_\nu}{L_{n+2}-3} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2^* = \max_{1\le\nu\le n} \left\{ \frac{L_{n+2}-3}{L_\nu} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

Remark 4. We end this section by using the identity (see [38, p. 55]), namely:

$$\sum_{\nu=1}^{n} L_\nu^2 = L_n L_{n+1} - 2, \quad n \ge 1,$$

and take

$$b_\nu = \frac{L_\nu^2}{L_n L_{n+1} - 2}, \quad \nu = 1, 2, \ldots, n.$$

Then by the above identity, we have

$$\sum_{\nu=1}^{n} |b_\nu| = \sum_{\nu=1}^{n} \frac{L_\nu^2}{L_n L_{n+1} - 2} = 1.$$

On using this in Lemma 4, we see that all the zeros of $T(q) = \sum_{\nu=1}^{n} q^\nu a_\nu$ lie in the spherical shell $F = \{q \in \mathbb{H} : r_1^{**} \le |q| \le r_2^{**}\}$, where

$$r_1^{**} = \min_{1 \le \nu \le n} \left\{ \frac{L_\nu^2}{L_n L_{n+1} - 2} \left| \frac{a_0}{a_\nu} \right| \right\}^{1/\nu}$$

and

$$r_2^{**} = \max_{1 \le \nu \le n} \left\{ \frac{L_n L_{n+1} - 2}{L_\nu^2} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu}.$$

Remark 5. The famous Geršgorin theorem is one of the fundamental theorems in complex matrix theory. It ensures that all the eigenvalues of a matrix over $\mathbb{C}$ are contained in the Geršgorin discs. The main obstacle in the study of quaternionic matrices is the non-commutative multiplication of quaternions. Firstly, $Ax = \lambda x$ and $Ax = x\lambda$, in the quaternionic setting (where A is a square quaternionic matrix and λ is a quaternion), are two very different systems of equations; in fact, they are so unlike from one another that there is typically no relationship between them. The fact is that a quaternion λ is a left eigenvalue of a square matrix A if and only if $A - \lambda I$ is singular, since $Ax = \lambda x$ is equivalent to $(A - \lambda I)x = 0$, while this is not true in case of right eigenvalues. Secondly, it is nowadays well-known that the right eigenvalue problem (which is not defined by a linear operator since $A - I\lambda$ is not linear when A is a square quaternionic matrix and λ a quaternion) is in reality coming from the notion of S-spectrum (for reference see [2]) which boils down to the search of right eigenvalues when dealing with matrices.

5 Conclusions

The classical and fundamental approaches dealing with the derivation of zero inclusion regions of regular polynomials have their own intrinsic value in geometric function theory. They play an equally significant role in contemporary studies that address these kinds of issues. Here, by using the generalized t-Fibonacci numbers and their various properties, we constructed a framework to establish various zero inclusion regions in the form of spherical shells of a regular unilateral polynomial of a quaternionic variable.

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