A kind of generalized quadrature formulae of maximal degree of precision for numerical integration of analytic functions is considered. Precisely, a general weighted quadrature of Birkhoff–Young type with $4n + 3$ nodes and degree of precision $6n + 5$ is studied. Its nodes are characterized by an orthogonality relation and a general numerical method for their computation is given. Special cases and numerical results are also included.

1. Introduction

Recently Acharya et al. [1] have considered numerical approximation of integrals

$$I(w; f) = \int f(z)\,dz,$$

over a directed line segment $L$ from the point $z_0 - h$ to the point $z_0 + h$ in the complex plane $\mathbb{C}$, where $f$ is an analytic function in the disk

$$\Omega = \{z \in \mathbb{C} : |z - z_0| \leq r, \ r > |h|\},$$

by means of a 7-point quadrature formula of the form

$$Q_7(w; f) = Af(z_0) + B[f(z_0 + th) + f(z_0 - th)] + C[f(z_0 + h) + f(z_0 - h)] + Df(z_0 + ih) + f(z_0 - ih)],$$

where $t$ is a positive parameter different from 1. Such a formula is exact for all odd degree monomials $f(z) = (z - z_0)^{2k+1}$. In order that the formula is also exact for even monomials $f(z) = (z - z_0)^{2k}$, $k = 0, 1, 2, 3$, the authors in [1] determine coefficients in $Q_7(w; f)$ as functions on $t$,

$$(A, B, C, D) = \frac{h}{105} \left(\frac{8(21t^2 - 5)}{t^2} - \frac{20}{t^2(1 - t^2)} - \frac{2(9 - 14t^2)}{1 - t^2} - \frac{3 - 7t^2}{1 + t^2}\right),$$

and obtain a general formula of degree of precision at least seven for any finite positive value of the real parameter $t \neq 1$. Letting $t \to \infty$, this rule reduces to the well-known five-point Birkhoff–Young formula of fifth degree precision [3], for which
A = 8h/5, (B = 0), C = 4h/15, D = −h/15, and its remainder term \( R^B_S(w; f) \) can be estimated as (see [18] or Davis and Rabinowitz [5, p. 136])

\[
|R^B_S(w; f)| \leq \frac{|h|^7}{1890} \max_{z \in S} |f^{(6)}(z)|,
\]

where \( S \) denotes the square with vertices \( z_0 + ikh \), \( k = 0, 1, 2, 3 \). The case of \( Q_M(w; f) \) with \( h = 1 \) was considered in [17]. This kind of Birkhoff–Young quadrature formulae has been investigated by several authors [7,16,15,12–14]. Such quadrature formulas can also be used to integrate real harmonic functions (see [3]).

We mention here that Lyness and Delves [8] and Lyness and Moler [9], and later Lyness [10], developed formulae for numerical integration and numerical differentiation of complex functions.

By an analysis of the remainder term for the general 7-point quadrature formula \( Q_M(w; f) \) with respect to the parameter \( t \), it can be obtained a quadrature rule of the maximal precision nine for \( t = \sqrt{7}/15 \) (see [11]). Some other rules of degree of precision seven can also be derived.

However, with a little modification of \( Q_M(w; f) \) we can obtain a modified 7-point quadrature rule \( Q^M_M(w; f) \) of degree precision eleven. Furthermore, using such an approach we derive a general \((4n + 3)\)-point quadrature formula of the maximal degree of precision for a weighted integral.

The paper is organized as follows. In Section 2 we give the modified quadrature formula \( Q^M_M(w; f) \). Section 3 is devoted to a general weighted quadrature of Birkhoff–Young type with \( 4n + 3 \) nodes and degree of precision \( 6n + 5 \). The nodes of such quadratures are characterized by an orthogonality relation. The corresponding weight coefficients of quadratures are given in Section 4. A general numerical method for determining nodes of such quadratures of maximal degree of precision is discussed in Section 5, including numerical results.

2. The modified quadrature formula \( Q^M_M(w; f) \)

For numerical calculating of the integral (1.1) of an analytic function in the disk (1.2), in this section we consider a modification of the quadrature formula \( Q_M(w; f) \) in the following form

\[
Q^M_M(w; f) = Af(z_0) + Bf(z_0 + th) + f(z_0 - th) + C[f(z_0 + th) + f(z_0 - th)] + D[f(z_0 + ith) + f(z_0 - ith)],
\]

where \( t \) and \( \ell \) are mutually different positive parameters. In this case, from the corresponding system of equations

\[
\begin{align*}
\frac{1}{2} A + B + C + D &= h, \\
Bt^2 + C\ell^2 - D\ell^2 &= \frac{h}{3}, \\
Bt^4 + C\ell^4 + D\ell^4 &= \frac{h}{5}, \\
Bt^6 + C\ell^6 - D\ell^6 &= \frac{h}{7},
\end{align*}
\]

we get

\[
\begin{align*}
A &= \frac{2h}{105} \frac{105t^2\ell^4 - 21t^2 - 35\ell^2 + 15}{t^2\ell^4}, & B &= \frac{h}{210} \frac{7\ell^4 - 3}{t^2(t^4 - \ell^4)}, \\
C &= \frac{h}{210} \frac{35t^2\ell^2 + 21t^2 - 21\ell^2 - 15}{t^2(t^4 - \ell^4)}, & D &= \frac{h}{210} \frac{-35t^2\ell^2 + 21t^2 + 21\ell^2 - 15}{t^2(t^4 - \ell^4)},
\end{align*}
\]

where \( 0 < t, \ell < 1, t \neq \ell \).

It is pertinent to note that the modified quadrature formula \( Q^M_M(w; f) \) boils down to the seventh degree rule due to Acharya et al. [1] for \( \ell = -1 \), and the modified Birkhoff–Young rule due to Tošić [16] for \( \ell = (3/7)^{1/4} \) and \( t \to \infty \).

Now, the error-term \( R^B_S(w; f) \) is \( I(w; f) - Q^M_M(w; f) \) for \( f(z) = (z - z_0)^{2k} \), \( k = 4, 5, 6 \), reduces to

\[
\begin{align*}
R^B_7((z - z_0)^8) &= \frac{2h^9}{315} 21t^4(5t^2 - 3) - 45t^2 + 35], \\
R^B_7((z - z_0)^{10}) &= \frac{2h^{11}}{231} [11t^4(7t^4 - 3) - 33t^4 + 21], \\
R^B_7((z - z_0)^{12}) &= \frac{2h^{13}}{1365} [91t^8(5t^2 - 3) + 65t^4(7t^4 - 3) - 195t^6 + 105].
\end{align*}
\]

Finally, from

\[
R^B_7((z - z_0)^8) = 0, \quad R^B_7((z - z_0)^{10}) = 0.
\]
we obtain two solutions for parameters:
\[ t^4 = \frac{5}{693} (57 + 4\sqrt{102}), \quad t^2 = \frac{1}{77} (45 - 2\sqrt{102}), \] 
(2.1)
and
\[ t^4 = \frac{5}{693} (57 - 4\sqrt{102}), \quad t^2 = \frac{1}{77} (45 + 2\sqrt{102}). \] 
(2.2)
Thus, there exist two modified 7-point quadratures \( Q^M_{7\nu}(w;f), \nu = 0, 1, \) of the maximal degree of precision eleven, with the following parameters:
\[
Q^M_{70}(w;f) : \quad \ell = \sqrt[4]{\frac{5}{693} (57 + 4\sqrt{102})} \approx 0.9155808999196944, \\
t = \sqrt[4]{\frac{1}{77} (45 - 2\sqrt{102})} \approx 0.5675304228160498, \\
A = \frac{256(198 - \sqrt{102})}{77175} = 0.6232915676809758h, \\
B = \frac{2939400 + 116087\sqrt{102}}{8680644} = 0.4736769794706059h, \\
C = 0.2151573287932331h, \quad D = -0.0004800921043269324h, \\
\]
and
\[
Q^M_{71}(w;f) : \quad \ell = \sqrt[4]{\frac{5}{693} (57 - 4\sqrt{102})} \approx 0.5883004297385740, \\
t = \sqrt[4]{\frac{1}{77} (45 + 2\sqrt{102})} \approx 0.9201849748878780, \\
A = \frac{256(198 + \sqrt{102})}{77175} = 0.6902944381499280h, \\
B = \frac{2939400 - 116087\sqrt{102}}{8680644} = 0.2035538803596094h, \\
C = 0.4582083249363621h, \quad D = -0.006909424370935494h. \\
\]
Expanding an analytic function in Taylor series
\[ f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad z \in \Omega, \]
and having in mind that \( Q^M_{7\nu}(w;f) \) has the maximal degree of precision eleven, the remainder is
\[ R^M_{\nu}(w;f) = \frac{f^{(12)}(z_0)}{12!} \cdot R^M_{\nu}(z - z_0)^{12} + \cdots \]
For the obtained parameters (2.1) and (2.2), its dominant error term reduces to
\[ R^M_{70}(w;f) \approx \frac{256(516 + 13\sqrt{102})}{3361743} \cdot \frac{h^{13}}{13!} f^{(12)}(z_0) \approx 7.92 \times 10^{-12} h^{13} f^{(12)}(z_0), \]
and
\[ R^M_{71}(w;f) \approx \frac{256(516 - 13\sqrt{102})}{3361743} \cdot \frac{h^{13}}{13!} f^{(12)}(z_0) \approx 4.70 \times 10^{-12} h^{13} f^{(12)}(z_0), \]
respectively. As we can see the second formula is slightly more accurate than the first one.

3. The generalized weighted quadrature formula \( Q^M_{8n+3}(w;f) \)
Let \( w : (-1, 1) \to \mathbb{R}^+ \) be an even positive weight function, for which all moments \( \mu_k = \int_{-1}^{1} z^k w(z) dz, \quad k = 0, 1, \ldots, \) exist. Without loss of generality, in this section we consider a weighted integration over \( L = [-1, 1] \) for analytic functions in the unit disk \( \Omega = \{ z : |z| \leq 1 \} \) by
\[ l(w;f) = \int_{-1}^{1} f(z) w(z) dz = Q_{8n+3}(w;f) + \Delta_{8n+3}(w;f). \]
(3.1)
where \( Q_{4n+3}(w; f) \) is the \((4n+3)\)-point quadrature formula of interpolatory type with nodes at the zeros of a monic polynomial of degree \( 4n + 3 \) \((n \in \mathbb{N})\),

\[
\Omega_{4n+3}(z) = z(z^2 - r_0)p_n(z^4) = z(z^2 - r_0) \prod_{k=1}^{n}(z^4 - r_k),
\]

(3.2)

where \( 0 < r_1 < \ldots < r_n < 1 \), \( r_0 \in (0, 1) \), and \( r_0 \neq r_k \) for each \( k = 1, \ldots, n \). Here, \( R_{4n+3}(w; f) \) is the corresponding remainder term.

According to (3.2) the quadrature formula in (3.1) has the form

\[
Q_{4n+3}(w; f) = Af(0) + Bf(x_0) + f(-x_0) + \sum_{k=1}^{n} \{ C_k[f(x_k) + f(-x_k)] + D_k[f(x_k) + f(-x_k)] \},
\]

(3.3)

where \( x_0 = \sqrt{r_0} \) and \( x_k = \sqrt[r_k]{r_0} \), \( k = 1, \ldots, n \).

**Theorem 3.1.** For any \( n \in \mathbb{N} \) there exist interpolatory quadratures \( Q_{4n+3}(w; f) \) with a maximal degree of precision \( d = 6n + 5 \). The nodes of such quadratures \( Q_{4n+3}(w; f) \) are characterized by the following orthogonality relation

\[
\int_{-1}^{1} \pi_k(z^2)(z^2 - r_0)p_n(z^4)w(z)dz = 0, \quad k = 0, 1, \ldots, n,
\]

(3.4)

where \( \{ \pi_k \}_{k \in \mathbb{N}_0} \) is a sequence of orthogonal polynomials with respect to the weight function on \((-1, 1)\).

**Proof.** Let \( \mathcal{P}_d \) denotes the set of algebraic polynomials of degree at most \( d \). For a given \( n \in \mathbb{N} \), suppose that \( f \in \mathcal{P}_d \), where \( d \geq 4n + 3 \). Then, it can be expressed in the form

\[
f(z) = u(z)\Omega_{4n+3}(z) + v(z) = u(z)z(z^2 - r_0)p_n(z^4) + v(z),
\]

where \( u \in \mathcal{P}_{d-4n-3} \) and \( v \in \mathcal{P}_{4n+2} \). Applying (3.1), we get

\[
I(w; f) = \int_{-1}^{1} u(z)z(z^2 - r_0)p_n(z^4)w(z)dz + I(v).
\]

Since this quadrature is of interpolatory type we have that \( I(w; v) = Q_{4n+3}(w; v) \) and also \( v(z) = f(z) \) at the zeros of the polynomial \( \Omega_{4n+3} \). Therefore, \( Q_{4n+3}(w; v) = Q_{4n+3}(w; f) \), so that for each \( f \in \mathcal{P}_d \) we have

\[
I(w; f) = \int_{-1}^{1} u(z)z(z^2 - r_0)p_n(z^4)w(z)dz + Q_{4n+3}(w; f).
\]

It is clear that the quadrature formula \( Q_{4n+3}(w; f) \) becomes \( Q_{4n+3}^{d}(w; f) \), i.e., it has a maximal degree of precision, if and only if

\[
\int_{-1}^{1} u(z)z(z^2 - r_0)p_n(z^4)w(z)dz = 0
\]

(3.5)

for a maximal degree of polynomials \( u \in \mathcal{P}_{d-4n-3} \). Evidently, (3.5) is true for every even polynomial. Taking \( u \) as an odd polynomial \( u(z) = zh(z) \), where \( h \in \mathcal{P}_n \), the previous “orthogonality conditions” can be represented as

\[
\int_{-1}^{1} h(z^2)z^2(z^2 - r_0)p_n(z^4)w(z)dz = 0, \quad h \in \mathcal{P}_n.
\]

(3.6)

Since the maximal degree of the polynomial \( u \in \mathcal{P}_{d-4n-3} \) is

\[
d_{\text{max}} = 4n - 3 = 1 + 2n + 1,
\]

we conclude that the maximal degree of precision of such a quadrature \( Q_{4n+3}(w; f) \) is \( d_{\text{max}} = 6n + 5 \), i.e., \( Q_{4n+3}(w; f) = Q_{4n+3}^{d_{\text{max}}}(w; f) \).

Introducing the inner product in a usual way as

\[
(f, g) = \int_{-1}^{1} f(z)g(z)w(z)dz,
\]

the last orthogonality conditions (3.6) can be expressed in terms of orthogonal polynomials \( \{ \pi_k \}_{k \in \mathbb{N}_0} \) with respect to this inner product in the form \( (z\pi_k, \Omega_{4n+3}) = 0 \), \( 0 \leq k \leq n \), i.e., (3.4). \( \square \)

According to (3.2), the polynomial \((z^4 - rz^2)p_n(z^4)\) can be expressed in the form

\[
(z^2 - r_0)p_n(z^4) = \sum_{j=0}^{n} (-1)^j \sigma_j (z^{4(n-j-1)} - r_0z^{4(2n-2j-1)}),
\]

(3.7)
where $\sigma_j$ are the so-called elementary symmetric functions, defined by
\[
\sigma_j = \sum_{(k_1, \ldots, k_j)} r_{k_1} \cdots r_{k_j}, \quad j = 1, \ldots, n,
\]
and the summation is performed over all combinations $(k_1, \ldots, k_j)$ of the basic set $\{1, \ldots, n\}$. Thus,
\[
\sigma_1 = r_1 + r_2 + \cdots + r_n, \quad \sigma_2 = r_1r_2 + \cdots + r_{n-1}r_n, \quad \sigma_n = r_1r_2 \cdots r_n,
\]
and for the convenience we put $\sigma_0 = 1$. Also, we put $r$ instead of $r_0$.

Using the orthogonality conditions (3.4) and the expansion (3.7) we get the following system of nonlinear equations
\[
\sum_{j=0}^{n} (-1)^j \sigma_j \{ s_{k,2n-2j+2} - r s_{k,2n-2j+1} \} = 0, \quad k = 0, 1, \ldots, n, \tag{3.8}
\]
with respect to unknowns $r$, $\sigma_1, \ldots, \sigma_n$, or equivalently to $r_k$, $k = 0, 1, \ldots, n$, where $r = r_0$ and $s_{k,j} = (\pi_{2k}, z^{2j})$, $k, j > 0$.

Introducing the notations $\sigma = [\sigma_1 \sigma_2 \cdots \sigma_n]^T$, $A = A(r) = [a_{k,j}]_{k=0,j=1}^{n,n}$, $b = b(r) = [b_0, b_1, \ldots, b_n]^T$,
The system of $n + 1$ nonlinear equations (3.8) can be written in the matrix form
\[
A\sigma = b. \tag{3.10}
\]
We have seen that the problem for $n = 1$ (and $w(z) = 1$) has two solutions. Numerical experiments show that for an arbitrary $n$, the number of solutions is $n + 1$. This hypothesis can be checked numerically for some reasonable values of $n$ (e.g. $n \leq 10$) in the following way.

If we take a fixed value of $r \in (0, 1)$, then the overdetermined system of $n + 1$ linear equations
\[
A(r)\sigma = b(r), \tag{3.11}
\]
with $n$ unknowns $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, can be solved as a least squares problem (in the 2-norm)
\[
\min_\sigma \|A(r)\sigma - b(r)\|_2 = \|A(r)\tilde{\sigma} - b(r)\|_2, \quad \text{where the vector } A(r)\tilde{\sigma} - b(r) \text{ is the corresponding least squares residual and the solution } \tilde{\sigma} \text{ can be expressed in terms of Moore–Penrose inverse.}
\]

Only when for some $r = \tilde{r}_0$, the vector $A(r)\tilde{\sigma} - b(r)$ becomes zero, we can identify the existence of a solution $(\tilde{r}_0, \tilde{\sigma}) = (\tilde{r}_0, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ of our original (nonlinear) system of Eqs. (3.10).

Consider now the case $w(z) = 1$. Following [2, §10.10] we get
\[
(P_{2k}, z^{2j}) = k! \binom{j}{k} \frac{\Gamma(j+1/2)}{\Gamma(k+1/2)} = k! \binom{j}{k} \prod_{r=0}^{k} \frac{2j+2r+1}{2j+2r+1}, \tag{3.12}
\]
where $P_{2k}(z)$ is the Legendre polynomial of degree $2k$. Taking (3.12) instead of $s_{k,j}$, the corresponding norm $\|A(r)\tilde{\sigma} - b(r)\|_2$ as a function of $r$ is presented in Fig. 1 for $1 \leq n \leq 4$ and $n = 10$.

As we can see, for the weight function $w(z) = 1$ and these values of $n$, the 2-norm vanishes only at $n + 1$ points in $(0,1)$, which means that for a given $n$, there exist $n + 1$ different quadratures $Q_{4n+1,1,v}^M$; $v = 0, 1, \ldots, n$, each of degree of precision $d_{\text{max}} = 6n + 5$.

In a general case, if we have a solution of our nonlinear problem, say $\tilde{r}_0$, then in order to construct the corresponding quadrature formula $Q_{4n+1,1,v}^M$ for such a $r = \tilde{r}_0$, we should solve a system of $n$ linear equations from (3.11) in order to get the values $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$, and then the zeros $(\tilde{z}_1, \ldots, \tilde{z}_n)$ by solving the equation
\[
(z - \tilde{z}_1) \cdots (z - \tilde{z}_n) = z^n - \tilde{\sigma}_1 z^{n-1} + \tilde{\sigma}_2 z^{n-2} - \cdots + (-1)^n \tilde{\sigma}_n = 0.
\]
Then, the nodes in the corresponding quadrature formula (3.3) are
\[
x_0 = \sqrt{\tilde{r}_0} \quad \text{and} \quad x_k = \sqrt[4]{\tilde{r}_k}, \quad k = 1, \ldots, n.
\]
A determination of the weight coefficients $A$, $B$, $C_k$, and $D_k$, $k = 1, \ldots, n$, in the corresponding quadrature formula (3.3) is a linear problem and it is considered in the next section.

A general numerical method for solving our nonlinear problem (3.8) and finding all solutions for $r_0$, for a given $n$ and for an arbitrary weight function $w(z)$, is given in Section 5. Also, a general numerical method for calculating the necessary inner products $s_{k,j} = (\pi_{2k}, z^{2j})$, $0 \leq k \leq j$, is given. Furthermore, analytic expressions for the generalized Gegenbauer weight are derived.
In order to determine the weight coefficients \( W_r \) in a quadrature formula of interpolatory type \( Q_{4n+3}^{M}(w,f) \), we use the Lagrange polynomial constructed at the set of simple nodes \( Z = \{z_r\} \). In our case,

\[
Z = \{0, \pm x_0, \pm x_k, \pm i x_k, k = 1, \ldots, n\},
\]

where \( x_0 = \sqrt{r_0} \) and \( x_k = \sqrt{r_k}, k = 1, \ldots, n \), and the node polynomial is

\[
\Omega_{4n+3}(z) = z(z^2 - r_0) \prod_{k=1}^{n} (z^2 - r_k).
\]

The corresponding Lagrange polynomial is

\[
L_{4n+3}(f;z) = \sum_{z_r \in Z} \frac{\Omega_{4n+3}(z)}{(z - z_r)\Omega_{4n+3}(z_r)} f(z_r),
\]

so that

\[
W_r = \frac{1}{\Omega_{4n+3}(z_r)} \int_{-1}^{1} \frac{\Omega_{4n+3}(z)w(z)}{z - z_r} dz.
\]

(4.1)
Theorem 4.1. Let \( r_0 \) and \( r_k, \ k = 1, \ldots, n \), be determined according to Theorem 3.1. Then the weight coefficients in the quadrature formula \( Q_{4n+3}^M(w; f) \) with the maximal degree of precision \( d = 6n + 5 \) are given by

\[
A = -\frac{1}{r_0 p_n(0)} \int_{-1}^{1} (z^2 - r_0) p_n(z^4) w(z) \, dz,
\]
\[
B = \frac{1}{2r_0 p_n(r_0^2)} \int_{-1}^{1} z^2 p_n(z^4) w(z) \, dz,
\]
\[
C_k = \frac{1}{4r_k(\sqrt{r_k} - r_0)p_n(r_k)} \int_{-1}^{1} z^2(z^2 - r_0)p_n(z^4) \frac{w(z)}{z^2 - \sqrt{r_k}} \, dz, \quad k = 1, \ldots, n,
\]
\[
D_k = \frac{-1}{4r_k(\sqrt{r_k} + r_0)p_n(r_k)} \int_{-1}^{1} z^2(z^2 - r_0)p_n(z^4) \frac{w(z)}{z^2 + \sqrt{r_k}} \, dz, \quad k = 1, \ldots, n,
\]

where \( p_n(z) = \prod_{i=1}^{n}(z - r_i) \).

Proof. Let \( x_0 = \sqrt{r_0} \) and \( x_k = \sqrt[r_k]{r_k}, k = 1, \ldots, n \). According to

\[
\Omega_{4n+3}^M(z) = (3z^2 - r_0) \prod_{r=1}^{n}(z^2 - r_r) + 4z^4(z^2 - r_0) \prod_{j=1}^{n} \prod_{r \neq j} (z^2 - r_r),
\]

we have

\[
\Omega_{4n+3}^M(0) = -r_0 \prod_{r=1}^{n} (-r_r) = -r_0 p_n(0),
\]
\[
\Omega_{4n+3}^M(\pm x_0) = 2r_0 \prod_{r=1}^{n} (x_0^2 - r_r) = 2r_0 p_n(r_0^2),
\]
\[
\Omega_{4n+3}^M(\pm x_k) = 4r_k(\sqrt{r_k} + r_0) \prod_{r \neq k} (r_k - r_r) = 4r_k(\sqrt{r_k} - r_0)p_n(r_k),
\]
\[
\Omega_{4n+3}^M(\pm ix_k) = 4r_k(-\sqrt{r_k} - r_0) \prod_{r \neq k} (r_k - r_r) = -4r_k(\sqrt{r_k} + r_0)p_n(r_k),
\]

where \( k = 1, \ldots, n \). Now, applying (4.1) and using notations for coefficients as in (3.3), we get desired results. \( \square \)

For analytic functions we can give an explicit formula for the interpolation error \( E_{4n+3}(z) = f(z) - I_{4n+3}(f; z) \) in the form (see [11, pp. 55–56])

\[
E_{4n+3}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Omega_{4n+3}^M(z) f(\zeta)\,d\zeta}{\Omega_{4n+3}^M(z) - z} \quad (z \in \text{int}\Gamma),
\]

where \( \Gamma \) is a simple closed contour in \( C \), such that all interpolation nodes belong to \( \text{int}\Gamma \). Then, the remainder term in (3.1) can be expressed in the integral form

\[
R_{4n+3}(w; f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\Omega_{4n+3}^M(\zeta) \left( \int_{-1}^{1} \frac{\Omega_{4n+3}^M(z) w(z)}{\zeta - z} \, dz \right) d\zeta}. \quad (4.2)
\]

Some estimate of (4.2) will be given elsewhere.

5. Numerical methods for constructing quadratures

In the sequel we need the inner products \( s_{j,k} = (\pi_{2j}, z^{2j}) \). \( 0 \leq k \leq j \). For \( k = 0 \) these products reduce to the moments of the weight function \( w \), i.e.,

\[
s_{0,j} = (1, z^{2j}) = \int_{-1}^{1} z^{2j} w(z) \, dz = \mu_{2j}, \quad j = 0, 1, \ldots.
\]

First, we give an analytic expression of \( s_{j,k} \) for a wide class of weight functions, and after that we introduce a general numerical method for easy calculation of \( s_{j,k} \) for every even weight function.

5.1. Analytic expression of \( s_{j,k} \) for the generalized Gegenbauer weight

We consider the so-called generalized Gegenbauer weight function defined by \( w(z) = |z|^\gamma (1 - z^2)^\alpha \). \( \gamma, \alpha > -1 \), on \( (-1, 1) \). The monic polynomials \( W_{n}^{\alpha, \beta}(z) \). \( \nu = 0, 1, \ldots \), orthogonal with respect to this weight function, where \( \beta = (\gamma + 1)/2 \), were introduced by Laščenov [6] (see, also, Chihara [4, pp. 155–156] and Mastroianni and Milovanović [11, pp. 147–148]). These
polynomials can be expressed in terms of the Jacobi polynomials $P^{(\alpha,\beta)}_v(z)$, $v = 0, 1, \ldots$, which are orthogonal on $(-1, 1)$ with respect to the weight function $w^{(\alpha,\beta)}(z) = (1-z)^{\alpha}(1+z)^{\beta}$, $\alpha, \beta > -1$. Namely,

$$
W^{(2\beta)}_{2k}(z) = \frac{k!}{(k + \alpha + \beta + 1)_k} P^{(\alpha,\beta)}_k(2z^2 - 1),
$$

$$
W^{(2\beta+1)}_{2k+1}(z) = \frac{k!}{(k + \alpha + \beta + 2)_k} zP^{(\alpha,\beta+1)}_k(2z^2 - 1).
$$

(5.2)

Notice that $W^{(2\beta)}_{2k+1}(z) = zW^{(2\beta+1)}_{2k}(z)$. These polynomials satisfy the following three-term recurrence relation

$$
W^{(2\beta)}_{2k+1}(z) = zW^{(2\beta)}_v(z) - \beta_v W^{(2\beta)}_{v+1}(z), \quad v = 0, 1, \ldots,
$$

$$
W^{(2\beta)}_{v+1}(z) = 0, \quad W^{(2\beta)}_0(z) = 1,
$$

where

$$
\beta_{2k} = \frac{k(k + \alpha)}{(2k + \alpha + \beta)(2k + \alpha + \beta + 1)}, \quad \beta_{2k+1} = \frac{(k + \beta)(k + \alpha + \beta)}{(2k + \alpha + \beta - 1)(2k + \alpha + \beta)},
$$

for $k = 1, 2, \ldots$, except when $\alpha + \beta = -1$; then $\beta_1 = (\beta + 1)/(\alpha + \beta + 2)$.

Now, we want to find an explicit expression for the products

$$
s_{k,j} = (W^{(2\beta)}_{2k+1}, z^j) = \int_0^1 W^{(2\beta)}_{2k+1}(z)|z^j|^2 z^2 dz, \quad 0 \leq k \leq j.
$$

(5.3)

**Lemma 5.1.** Let $\alpha, \beta > -1$. Then the products defined in (5.3) are

$$
s_{k,j} = \frac{k!}{(k + \alpha + \beta + 1)_k} \binom{j}{k} \frac{\Gamma(k + \alpha + 1)\Gamma(j + \beta + 1)}{\Gamma(k + j + \alpha + \beta + 2)}
$$

(5.4)

for $0 \leq k \leq j$. Otherwise, $s_{k,j} = 0$.

In order to prove this lemma we need an auxiliary result.

**Lemma 5.2.** For $\alpha, \beta > -1$ and $j \in \mathbb{N}_0$, the expansion in Jacobi polynomials $P^{(\alpha,\beta)}_v(t), v = 0, 1, \ldots, j$,

$$
(1 + t)^j = 2^j \Gamma(j + \beta + 1) \sum_{v=0}^{j} \binom{j}{v} \frac{(2v + \alpha + \beta + 1)\Gamma(v + \alpha + \beta + 1)}{\Gamma(v + \beta + 1)\Gamma(v + j + \alpha + \beta + 2)} P^{(\alpha,\beta)}_v(t)
$$

holds.

The proof of this expansion can be given by induction in $j$, having in mind that

$$
(1 + t)P^{(\alpha,\beta)}_v(t) = \frac{2(v + 1)(v + \alpha + \beta + 1)}{(2v + \alpha + \beta + 1)(2v + \alpha + \beta + 2)} P^{(\alpha,\beta)}_v(t) + \left(1 + \frac{\beta^2 - \alpha^2}{(2v + \alpha + \beta)(2v + \alpha + \beta + 1)}\right) P^{(\alpha,\beta)}_v(t)
$$

$$
+ \frac{2(v + \alpha)(v + \beta)}{(2v + \alpha + \beta)(2v + \alpha + \beta + 1)} P^{(\alpha,\beta)}_{v-1}(t).
$$

We mention that the corresponding expansion in Bateman and Erdélyi [2, pp. 212] has a mistake.

**Proof of Lemma 5.2.** According to (5.3) and (5.2) we have

$$
s_{k,j} = \frac{2k!}{(k + \alpha + \beta + 1)_k} \int_0^1 P^{(2\beta)}_k(2z^2 - 1)z^{2j - 2\beta + 1}(1 - z^2)^2 dz.
$$

Changing the variables $t = 2z^2 - 1$ it reduces to

$$
s_{k,j} = \frac{k!}{2^{2\beta + 1}(k + \alpha + \beta + 1)_k} \int_{-1}^1 P^{(2\beta)}_k(t)(1 - t)^j(1 + t)^{\beta + 1} dt.
$$

Now, using the expansion from Lemma 5.2 and the orthogonality of Jacobi polynomials,

$$
(P^{(2\beta)}_k, P^{(2\beta)}_v) = ||P^{(2\beta)}_k||^2 \delta_{kv} = \frac{2^{2\beta + 1}\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}{k!(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)} \delta_{kv},
$$

$$
(5.5)
$$

and the recurrence relation for $W^{(2\beta)}_k(z)$.
where \(\delta_{k,\gamma}\) is Kronecker’s delta, we obtain
\[
s_{k,j} = \frac{k!2^j \Gamma (j + \beta + 1) k! \left( \begin{array}{c} j \\ k \end{array} \right)}{2^k + j + 1 \alpha} \frac{2^k + j + 1 \beta + 1}{\Gamma (k + j + 1) \Gamma (k + j + 1) \Gamma (k + j + 1)} \|p_j^\alpha,\beta\|^2,
\]
i.e., (5.4). For \(k > j\), because of orthogonality, \(s_{k,j} = 0\) □

We mention some special cases:

1° \(w(z) = 1\), i.e., \(\alpha = 0\), \(\gamma = 0\) \((\beta = -1/2)\): In this Legendre case we have
\[
s_{k,j} = \frac{(k)!^2}{(k + 1/2)!} \left( \begin{array}{c} k \\ j \end{array} \right) \Gamma (j + 1/2) \Gamma (j + 3/2),
\]

Compared with (3.12), the additional factor \(k!/(k + 1/2)_k\) \((= 1/a_{2k})\) comes from the leading coefficient in the Legendre polynomial \(P_{2k}(z) = a_{2k}z^{2k}\) terms of lower degree.

2° \(w(z) = 1/\sqrt{1 - z^2}\), i.e., \(\alpha = -1/2\), \(\gamma = 0\) \((\beta = -1/2)\): In the Chebyshev case of the first kind, the inner product (5.4) reduces to
\[
s_{k,j} = \frac{\pi}{2^{2j + 2k - 1}} \left( \begin{array}{c} 2j \\ j - k \end{array} \right).
\]

3° \(w(z) = \sqrt{1 - z^2}\), i.e., \(\alpha = 1/2\), \(\gamma = 0\) \((\beta = -1/2)\): In the Chebyshev case of the second kind, it becomes
\[
s_{k,j} = \frac{\pi}{2^{2j + 2k - 1}} \left( \begin{array}{c} 2k + 1 \\ 2j + 1 \end{array} \right) \left( \begin{array}{c} j \\ j - k \end{array} \right).
\]

4° \(w(z) = (1 - z^2)^2\), i.e., \(\alpha > -1\), \(\gamma = 0\) \((\beta = -1/2)\): In this Gegenbauer case, we have
\[
s_{k,j} = \frac{k!}{(k + \alpha + 1)_k} \left( \begin{array}{c} j \\ k \end{array} \right) \Gamma (k + \alpha + 1) \Gamma (j + 1/2) \Gamma (k + j + \alpha + 3/2).
\]

5° \(w(z) = |z|\), i.e., \(\alpha = 0\), \(\gamma = 1\) \((\beta = 0)\): In this case
\[
s_{k,j} = \frac{1}{k + j + 1} \left( \begin{array}{c} 2j \\ j \end{array} \right) \left( \begin{array}{c} 2k \\ k \end{array} \right) \left( \begin{array}{c} j \\ j - k \end{array} \right).
\]

5.2. General numerical method for calculating \(s_{k,j}\)

It is well-known that monic polynomials \(\{\pi_v\}_{v \in N_0}\) orthogonal with respect to an even weight function satisfy the three-term recurrence relation of the form
\[
\pi_{v+1}(z) = 2z\pi_v(z) - \beta_v \pi_{v-1}(z), \quad v = 0, 1, \ldots,
\]
with \(\pi_0(z) = 1\) and \(\pi_{-1}(z) = 0\). It is convenient to put \(\beta_0 = \mu_0\).

**Lemma 5.3.** Let \(\beta_v, \nu \geq 0\), be recursion coefficients in (5.5) for polynomials orthogonal with respect to the even weight function \(w\) on \((-1, 1)\), with the moments \(\mu_v = \int_{-1}^{1} z^\nu w(z) dz\), \(\nu \geq 0\). For the inner products \(s_{k,j} = (\pi_{2k}, z^{2j})\) the following recurrence relation
\[
s_{k+1,j} = s_{k,j} + (\beta_{2k} + \beta_{2k+1}) s_{k,j} + \beta_{2k} \beta_{2k+1} s_{k-1,j}
\]
holds, with \(s_{0,j} = \mu_{2j}, \quad j = 0, 1, \ldots\), and \(s_{k,j} = 0\) for \(k > j\).

**Remark 5.4.** Coefficients from the relation (5.6) appear in the recurrence relation for polynomials \(\{\pi_{2k}(\sqrt{t})\}_{k \in N_0}\) orthogonal with respect to the weight function \(w(\sqrt{t})/\sqrt{t}\) on \((0, 1)\) (see [11, pp. 101–103]).

**Proof of Lemma 5.3.** Because of orthogonality, it is clear that \(s_{k,j} = (\pi_{2k}, z^{2j}) = 0\) for \(k > j\). When \(k = 0\), for the boundary values \(s_{0,j}\) we have (5.1). For diagonal elements we have
\[
s_{j,j} = (\pi_{2j}, z^{2j}) = \left\| \pi_{2j} \right\|^2 = \prod_{\nu=0}^{2j} \beta_{\nu}.
\]
In order to get (5.6), we start with the recurrence relation (5.5). Thus,

\[ s_{k+1,j} = \int_{-1}^{1} \pi_{2k+1}(z) w(z) dz = \int_{-1}^{1} \left[ z \pi_{2k+1}(z) - \beta_{2k+1} \pi_{2k}(z) \right] z^j w(z) dz \]

\[ = \int_{-1}^{1} \pi_{2k+1}(z) z^{j+1} w(z) dz - \beta_{2k+1} \int_{-1}^{1} \pi_{2k}(z) z^j w(z) dz = \int_{-1}^{1} \left[ z \pi_{2k}(z) - \beta_{2k} \pi_{2k-1}(z) \right] z^{j+1} w(z) dz - \beta_{2k+1} s_{k,j} \]

Expanding \( z \pi_{2k-1}(z) \) as a linear combination of \( \pi_{2k}(z) \) and \( \pi_{2k-2}(z) \), we get

\[ s_{k,j+1} = s_{k+1,j} + \beta_{2k} \int_{-1}^{1} \left[ \pi_{2k}(z) + \beta_{2k-1} \pi_{2k-2}(z) \right] z^j w(z) dz + \beta_{2k+1} s_{k,j} = s_{k+1,j} + \beta_{2k} s_{k,j} + \beta_{2k} \beta_{2k-1} s_{k-1,j} + \beta_{2k+1} s_{k,j} \]

i.e., (5.6).

**Fig. 2** displays the triangular array of the inner products \( s_{k,j} = (\pi_{2k}, z^j) \), \( 0 \leq k \leq j \), and the computing stencil showing that the circled entry is computed in terms of the three other entries. The entries in the boxes are the known boundary values

\[ s_{0,j} = \mu_{2j}, \quad s_{j,j} = \beta_0 \beta_1 \cdots \beta_{2j}, \quad s_{j+1,j} = 0, \quad j = 0, 1, \ldots, 2n+2. \]

Zero entries \( s_{j,1,j} \) are displayed as white circles (in the boxes).

As we can see, for generating the system of Eqs. (3.8), i.e., (3.10), we use only entries \( s_{k,j} \) for \( k \leq n \). Thus, we need the following matrix of the type \((n+1) \times (2n+3)\),

\[
S = \begin{bmatrix}
S_{0,1} & S_{0,1} & \cdots & S_{0,n} & \cdots & S_{0,2n+1} & S_{0,2n+2} \\
S_{1,1} & S_{1,1} & \cdots & S_{1,n} & \cdots & S_{1,2n+1} & S_{1,2n+2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
S_{n,1} & S_{n,1} & \cdots & S_{n,n} & \cdots & S_{n,2n+1} & S_{n,2n+2}
\end{bmatrix}
\]  

(5.7)

### 5.3. Method for calculating the nodes

Consider again the system of \( n+1 \) nonlinear equations (3.8). Taking only the last \( n \) equations of (3.8), we get

\[
C \sigma = d,
\]

(5.8)

where \( C = C(r) = [a_{k,j}]_{k=1,n, j=1}^{n} \) and \( d = d(r) = [b_1 \cdots b_n]^T \). According to (3.9), the elements \( a_{k,j} \) and \( b_k \) are expressed in terms of the elements of the matrix \( S \) given by (5.7). The determinant of the matrix \( C \) has the form

\[
\det C = (-1)^{n(n-1)/2} \begin{vmatrix}
S_{1,2n} - R S_{1,2n-1} & S_{1,2n-2} - R S_{1,2n-3} & \cdots & S_{1,2} - R S_{1,1} \\
S_{2,2n} - R S_{2,2n-1} & S_{2,2n-2} - R S_{2,2n-3} & \cdots & S_{2,2} - R S_{2,1} \\
\vdots & \vdots & \ddots & \vdots \\
S_{n,2n} - R S_{n,2n-1} & S_{n,2n-2} - R S_{n,2n-3} & \cdots & S_{n,2} - R S_{n,1}
\end{vmatrix},
\]

Fig. 2. The scheme for calculating the inner products \( s_{k,j} = (\pi_{2k}, z^j) \) for \( n = 3 \).
where its elements $s_{k,2n-2j+2} - rs_{0,2n-2j+1}$ are equal to zero for each $k, j \in \{1,2,\ldots,n\}$ such that $k + 2j > 2n + 2$, because of $s_{k,j} = 0$ for $k > j$. It is easy to see that such zero-elements are the last $n-2$ elements in the last $n$th column, the last $n-4$ elements in the $(n-1)$st column, etc. Otherwise, $\det C = \Delta_n(r)$ is a polynomial of degree $n$. Since the vector $d$ on the right side in (5.8) is given by

$$d = \begin{bmatrix} s_{1,2n+2} - rs_{1,2n+1} \\ s_{2,2n+2} - rs_{2,2n+1} \\ \vdots \\ s_{n,2n+2} - rs_{n,2n+1} \end{bmatrix},$$

the corresponding determinants in Cramer’s rule $\Delta_j^{(r)}$, $j = 1, \ldots, n$, are also polynomials of degree $n$. Thus, for a given $r$, such that $\Delta_n(r) \neq 0$, the unique solution of (5.8) is given by

$$\tilde{\alpha}_j = \frac{\Delta_j^{(r)}}{\Delta_n(r)}, \quad j = 1, \ldots, n.$$

Using Mathematica package it can be obtained in a symbolic form as rational functions in $r$. Substituting $\tilde{\alpha}_j(r)$, $j = 1, \ldots, n$, in

$$f_0 = \sum_{j=0}^{n} (-1)^j \tilde{\alpha}_j \{ s_{0,2n-2j+2} - rs_{0,2n-2j+1} \} = 0,$$

we obtain the following algebraic equation of degree $n + 1$,

$$\Phi^{(r)}_{n+1}(r) \equiv \sum_{j=0}^{n} (-1)^j \{ s_{0,2n-2j+2} - rs_{0,2n-2j+1} \} \Delta_j^{(r)} (r) = 0,$$

where $\Delta_j^{(0)} (r) \equiv \Delta_n(r)$.

Numerical experiments show that the (monic) polynomial $\Phi^{(r)}_{n+1}(r)$ has $n + 1$ different real zeros located in $(0,1)$.

In the case $n = 1$ we have seen in Section 2 that two different solutions exist for $r$ and they give two quadratures of the same precision eleven.

![Graphs of $\Phi^{(r)}_{n+1}(r)$, $n = 3, 4, 5$, for $w(z) = 1$.](image)

**Table 1**

Different solutions of $\Phi^{(r)}_{n+1}(r) = 0$ for $n = 2|5$ and $w(z) = 1$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.2044987378293505</td>
<td>0.1439216162367618</td>
<td>0.108189744669971</td>
<td>0.08510161904718037</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.6167356745407912</td>
<td>0.4619273121368076</td>
<td>0.3598672165580655</td>
<td>0.2897653961037322</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.9208470355936592</td>
<td>0.7593055545829755</td>
<td>0.6211046569905429</td>
<td>0.5148988061113188</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.9519683824480733</td>
<td>0.8360223612692</td>
<td>0.967823803414767</td>
<td>0.7211387868476094</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.961483073914880</td>
<td>0.881483073914880</td>
<td>0.976959264002607</td>
<td>0.976959264002607</td>
<td></td>
</tr>
</tbody>
</table>
The graphs of monic polynomials $\Phi_{n+1}(r)$ on $[0,1]$ for $n = 3, 4, 5$ are displayed in Fig. 3.

In Table 1 we present the corresponding numerical values of the zeros of the polynomials $\Phi_{n+1}(r)$ for $n = 2, 3, 4, 5$.

Similar results can be obtained for other weight functions. The cases with the Chebyshev weight function of the first kind $w(z) = \frac{1}{1 - z^2}$ and the weight function $w(z) = |z|$ are displayed in Figs. 4 and 5, respectively.

The previous method for finding all solutions for $\bar{r}_0$ works for some reasonable values of $n$, e.g., $n < 50$. For example, for the Chebyshev weight of the first kind and $n = 50$ we obtain 21 solutions for $\bar{r}_0$ (given to 50 decimal digits):

$$
\text{Out}[7] = \left\{ \begin{array}{c}
0.0150485864753572668744527330521, \\
0.55753266988225225298657439117182, \\
0.167824679058336195589651992843, \\
0.2925101898254294970529822, \\
0.36729586090725174227220507435, \\
0.504858032671624316488143131170, \\
0.5713094673106376091020367257, \\
0.635241286837459256261610097877, \\
0.6935827036177609655281197194, \\
0.758259042435763252928400288756, \\
0.80454161595365152530647156549, \\
0.851344214028606784515849170776, \\
0.8924279383595564814715995310, \\
0.92735121372080665019429733552, \\
0.95574216185269077871893880015, \\
0.9773005648403337796516284890742, \\
0.991799603894642762751920785079, \\
0.999087253927436018093044192512 \\
\end{array} \right\}
$$

In the case $w(z) = 1$ and $n = 50$, the following sequence of 51 solutions for $\bar{r}_0$ is given with 20 decimal digits:

$$
\text{Out}[11] = \left\{ \begin{array}{c}
0.040059914529631981612, \\
0.015198148896652530762, \\
0.047886461968069102119, \\
0.067502176479469289610, \\
0.088904351714626293601, \\
0.11701315198798583, \\
0.1356849220893164063, \\
0.1606082629787534527, \\
0.18637354088540880104, \\
0.21282382747632203945, \\
0.2393738459592023875, \\
0.267306391769399361, \\
0.2951294903080191880, \\
0.32321054714584841291, \\
0.35147693246320767511, \\
0.3793841928314226360, \\
0.4082102392094646313, \\
0.4365312624315445444, \\
0.46473035175990477588, \\
0.49273807747585617047, \\
0.520491228854959292527, \\
0.547928095622616851752, \\
0.57498920592171805202, \\
0.60161719533987947594, \\
0.62775672001012940416, \\
0.65335438628586067294, \\
0.67835870652605916475, \\
0.70272005934878642745, \\
0.72693065223892979455, \\
0.7493266968418699756, \\
0.77147801492965236242, \\
0.792800845872161408075, \\
0.8132757117531934504, \\
0.83284135040495289269, \\
0.85146888214350868901, \\
0.86912373257358626800, \\
0.88577330675487417323, \\
0.90138699659712564769, \\
0.91593621302745143452, \\
0.92939439805653787567, \\
0.94173707696323433174, \\
0.9529418537170192266, \\
0.96298845061167595446, \\
0.97185572249560409663, \\
0.97953667924407666148, \\
0.986008528088609667575, \\
0.99126256482379910043, \\
0.99528944820940483897, \\
0.9980820354293588811, \\
0.99963584456149960413 \\
\end{array} \right\}
$$

Fig. 4. Graphs of $\Phi_{n+1}(r)$. $n = 3, 4, 5$, for $w(z) = (1 - z^2)^{-1/2}$. 
From the theoretical point of view it would be nice to prove (or disprove) the following conjecture:

**Conjecture 5.5.** For a given weight function $w(z)$ and each $n \in \mathbb{N}$, all zeros of $U_{n+1}(r)$ are real and distinct and are located in $(0,1)$. Zeros of $U_{n+1}(r)$ and $U_{n+2}(r)$ interlace.

5.4. Numerical results

For quadratures $Q_{4n+3}^M(w;f)$ of degree of precision $d = 6n + 5$, in our MATHEMATICA procedure we obtain complete parameters in the form

$$\{ x_0, x_k, A, B, C_k, D_k \} = \{ x_0, \{ x_1, \ldots, x_n \}, A, B, \{ C_1, \ldots, C_n \}, \{ D_1, \ldots, D_n \} \}.$$

For example, let $w(z) = 1$ and $n = 2$. Then we have three quadratures $Q_{11}^M(1;f)$ of degree of precision $d = 17$ (see first column in Table 1). For these values of $r_0$, i.e.,

$$\{0.2044987378293505, 0.6167356745407912, 0.9208470355936592\}$$

we get the following parameters:

$$\{0.4522153666444237, 0.7754684395027309, 0.9570916645968834, 0.4880467095490914, 0.3936044844812900, 0.2527549012554169, 0.1099114468981711, -0.0003299665322107021, 0.00003577912278703123\},$$

$$\{0.7853252030469869, 0.4741479794169331, 0.95895312602622328, 0.54739797047003460, 0.2416521097533237, 0.3846699903497127, 0.1051207091442720, -0.005150757567968953, 8.995970487465769 \times 10^{-6}\},$$

$$\{0.9596077509032840, 0.4802111190778518, 0.7885463525798828, 0.5616568463150571, 0.1034616930531016, 0.3835087691311978, 0.2383909330938098, -0.006285348161458679, 0.0009552972582096275\},$$

respectively.

In the case of the Chebyshev weight $w(z) = 1/\sqrt{1-z^2}$ and $n = 2$, for the corresponding values of $r_0$,

$$\{0.2321837439443931, 0.6740206457250330, 0.960983110305740\},$$

the quadrature parameters are respectively:

$$\{0.4818544842007731, 0.8124087172755511, 0.9790447658917281, 0.5249337433901672, 0.4705720970208580, 0.4268191199148742, 0.4112815014700504, -0.000399170537455941, 0.00005590723150697970\}.
At the end of this paper we give a numerical example. The formula (3.3) could be interesting for real functions of the form $f(z) = g(z^2)$. According to (3.3), in that case each of quadratures $Q^M_{1,n-3}(w; f)$, $v = 0, 1, \ldots, n$ (for $n + 1$ different values of $\tau_0$), becomes a quadrature formula of the following form with $N = (n + 2)$ nodes,

$$I(w; f) = \int_{-1}^{1} f(x)w(x)dx \approx Af(0) + 2Bf(x_0) + \sum_{k=1}^{n} W_k f(x_k),$$

(5.9)

where $W_k = 2(C_k + D_k)$, $k = 1, \ldots, n$.

Let $f(x) = 1/(1 + x^8)$. We consider two integrals

$$\int_{-1}^{1} \frac{dx}{1 + x^8} = \frac{1}{4} \left[ \sin \frac{\pi}{8} \left( \pi + 2 \text{tanh}^{-1} \left( \frac{\sin \frac{\pi}{8}}{8} \right) \right) + \cos \frac{\pi}{8} \left( \pi + 2 \text{tanh}^{-1} \left( \frac{\cos \frac{\pi}{8}}{8} \right) \right) \right] \approx 1.849303411551076,$$

and

$$\int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} = \pi q F_3 \left( \frac{1}{8}, \frac{5}{8}, \frac{7}{8}; \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}; -1 \right) \approx 2.626270969212133.$$

For their calculation we apply all quadratures $Q^M_{11,v}(w; f)$, $v = 0, 1, 2$, whose parameters are given before. Here $n = 2$, $N = 4$, and $d_{\text{max}} = 17$. The corresponding relative errors

$$\left| \frac{Q^M_{11,v}(w; f) - I(w; f)}{I(w; f)} \right|, \quad v = 0, 1, 2,$$

are given in the last two columns in Table 2. Numbers in parentheses indicate decimal exponents.

We compare these results with corresponding ones obtained by $m$-point Gaussian quadratures (for even functions), with respect to Legendre and Chebyshev weights,

$$I(w; f) \approx Q^m_{11,v}(w; f) = \sum_{\tau} A_{\tau} f(\tau_v) = 2 \sum_{\tau} A_{\tau} f(\tau_v) + \left\{ A_{(m-1)/2} f(0), \quad \text{if } m \text{ is odd}, \right. \left. 0, \quad \text{if } m \text{ is even}. \right\}$$

where $A_{\tau}$ and $\tau_v$, $v = 1, \ldots, m$, are Christoffel numbers and nodes, respectively (see [11, pp. 324–325]), such that $1 > \tau_1 > \ldots > \tau_m > -1$. If we take $m = 7, 8, 9$, the number of nodes in the corresponding quadrature are $N = 4, 4, 5$, respectively.

As we can see, the quadratures $Q^M_{11,v}(w; f)$, $v = 0, 1, 2$, have a higher degree of precision than quadratures $Q^m_{11,v}(w; f)$, $m = 7, 8$ (with the same number of nodes), as well as that they give a better accuracy.

References