Quadrature processes and numerical computation of the two dimensional exponential integrals

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Abstract

Integral representations of two-dimensional exponential integral (TDEI) functions and their numerical computation based on quadrature processes are treated. In addition to a general brief description of important quadrature processes, including some historical details, three methods for numerical calculation of TDEI functions are presented in particular. Precisely, the construction and application of the truncated Gauss-Christoffel quadrature formulas, the composite trapezoidal rule, and the method of integration between zeros of the integrand to the calculation of the values of TDEI functions in various situations are given. A series of numerical examples are given, including error analysis.

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1 Introduction

This paper is devoted to quadrature processes and their application in the numerical integration of two-dimensional exponential integrals (TDEI), which have been considered by several authors (*cf.* [1]-[3], [41]).

In the general case, numerical integration represents the approximation of a functional $I(f) := \int_{\mathbb{R}} f(t)\,\mathrm{d}\mu(t)$, defined on a class of functions X (with respect to the measure $\mathrm{d}\mu(t)$), by another functional (quadrature rule) $Q_n(f) := \sum_{\nu=1}^{n} A_\nu f(\tau_\nu)$, which is defined using only the values of the function f at selected n points τ_ν, $\nu = 1, \dots, n$. These points are called nodes, and the corresponding coefficients A_ν are weights. The difference $R_n(f) := I(f) - Q_n(f)$ represents the remainder term of the quadrature formula, which is usually constructed so that $R_n(f)$ is equal to zero on some subset of X.

The set of all algebraic polynomials of degree at most n will be denoted by $\mathcal{P}_n$ ($\subset \mathcal{P}$), where $\mathcal{P}$ be the set of all algebraic polynomials.

This paper is organized as follows. In Section 2 we give a brief overview of the most important quadrature processes, including some historical details. The main attention is paid to the integral representations of TDEI functions (Section 3), as well as to the numerical calculation of their values using numerical integration (Section 4).

2 Preliminaries on quadrature processes

In this section we give a short account on numerical integration using quadrature rules. There are two main approaches, originating from Isaac Newton (1647–1727) and Carl Friedrich Gauss (1777–1855).

2.1 Newton approach

Newton's idea from 1676 about the integration of an interpolation polynomial for the function $f : [a, b] \mapsto \mathbb{R}$, on a set of equidistant points τ_k on $[a, b]$, as well as later refinements by Roger Cotes, led to the well-known Newton-Cotes formulas for the numerical integration of functions on the interval $[a, b]$, which, in addition to their theoretical significance, also have practical applications in the construction of so-called composite quadrature formulas (trapezoidal, Simpson's, ...), as well as in adaptive integration.

The well-known and the simplest composite trapezoidal rule for calculating the integral of the function f over the interval $[a, b]$, using $n + 1$ values $f_k = f(\tau_k)$ at equidistant nodes $\tau_k = a + kh$, $k = 0, 1, \ldots, n$, with the step size $h = (b - a)/n$, is defined by

$$I(f) = \int_a^b f(t)\,\mathrm{d}t \approx T_n(f; h) := h\left(\frac{1}{2}f_0 + f_1 + \cdots + f_{n-1} + \frac{1}{2}f_n\right). \tag{1}$$

For sufficiently continuously-differentiable functions, the well-known Euler-Maclaurin summation formula

$$T_n(f; h) - I(f) = \sum_{\nu=1}^{m} \frac{h^{2\nu} B_{2\nu}}{(2\nu)!}\big(f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a)\big) + E_m(f)$$

holds, where B_k ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, ...) are Bernoulli numbers and $E_m(f)$ is the corresponding remainder term, which can be represented as

$$E_m(f) = (b - a)\frac{B_{2m+2}h^{2m+2}}{(2m + 2)!} f^{(2m+2)}(\xi), \quad a < \xi < b.$$

As we can see the trapezoidal rule $T_n(f; h)$, given by (1), with the error term

$$T_n(f; h) - I(f) = E_0(f) = \frac{1}{12}(b - a)f''(\xi)h^2 \quad (a < \xi < b),$$

converges very slowly with respect to step refinement as $O(h^2)$. Something better convergence $O(h^4)$ has the classical composite Simpson rule $S_n(f; h)$. In general, both of these formulas do not allow achieving high accuracy.

However, if we restrict our analysis to the class of analytic functions with all derivatives f vanishing at $x = a$ and $x = b$, then the discretization error is given only by the remainder $E_m(f)$ as $m \to +\infty$. Then the convergence with respect to step refinement is faster than any finite order and the trapezoidal rule becomes the method of choice. This kind of convergence is known as *exponential convergence*. Such a method, known in the literature as *IMT rule* was proposed in 1969 by Iri, Moriguti and Takasawa, but an English translation of the original Japanese paper was published in 1987 (*cf.* [16]). Error estimation for analytic functions and quadrature rules obtained by transformations of the integration variable were considered by Takahasi and Mori [34, 35]. In a survey paper Mori [31] particularly emphasized the so-called *double exponential integration rule* (DE-rule), which is characterized by a double exponential decay near the endpoints of the transformed integration interval. Particularly interesting are the cases when $(a, b) = (-\infty, +\infty)$ and $(0, +\infty)$.

In an interesting paper Waldvogel [38] proposed as a standard the trapezoidal rule on the entire real line $\mathbb{R}$ for numerical integration of analytic functions and suggested the choice of an elementary transformation $t = \phi(x)$ from the interval (a, b) to $(-\infty, +\infty)$. In addition, slow decay at infinity can be accelerated by the sinh-transformation, as well as some other transformations (*cf.* [16, 25, 30, 34, 35, 38]).

In the case $(a, b) = (-\infty, +\infty)$, formula (1) can be slightly modified as the *shifted trapezoidal sum* with step h and offset s (*cf.* [38]),

$$I(f) = \int_{-\infty}^{+\infty} f(t)\,\mathrm{d}t \approx T(f; h, s) := h \sum_{k=-\infty}^{+\infty} f(s + kh). \tag{2}$$

We note that $T(f; h, s) = T(f; h, s + h)$ and

$$T\Big(f; \frac{h}{2}, s\Big) = \frac{1}{2}\Big[T(f; h, s) + T\Big(f; \frac{h}{2}, s + \frac{h}{2}\Big)\Big].$$

The latter relation is useful for the efficient transition from step h to step $h/2$.

Remark 1. With the same Newton idea, one can also consider the weighted integration $\int_a^b f(t)w(t)\,\mathrm{d}t$, where $w : [a, b] \mapsto \mathbb{R}^+$ is a given weight function. Thus, the weighted Newton-Cotes formulas are given by

$$\int_a^b f(t)w(t)\,\mathrm{d}t = \sum_k A_k f(\tau_k) + R_n(f), \tag{3}$$

where the equidistant points (nodes) are taken by the step $h = (b - a)/n$ usually as

$$\tau_k = a + kh,\ k = \overline{0, n}; \quad \tau_k = a + kh,\ k = \overline{1, n-1}; \quad \tau_k = a + \Big(k - \frac{1}{2}\Big),\ k = \overline{1, n}.$$

Such quadrature formulas are of the interpolation type, for which the remainder term $R_n(f) = 0$ whenever $f \in \mathcal{P}_d$, where d is the algebraic degree of exactness depending of the number of nodes. The weight coefficients A_k (Cotes numbers) can be expressed using the corresponding interpolation formula (*cf.* [23, §5.1]). Closed expressions for Cotes numbers A_k were derived in terms of moments and Stirling numbers of the first kind in [22].

Remark 2. A type of interpolatory quadrature formulas, whose nodes are geometricaly distributed in the form $\tau_k = aq^k$, $k = 0, 1, \ldots, n$, were introduced in [21]. The explicit expressions for the coefficients A_k are also obtained using the q-binomial theorem.

2.2 Gaussian approach

The most significant discovery in numerical analysis in the 19th century was Gauss's quadrature formulas of 1814. Gauss [9] dramatically improved Newton's method, increasing the algebraic degree of exactness of the n-point quadrature formula (3) (for $w(t) = 1$ on $(a, b) = (0, 1)$) from $n - 1$ to $2n - 1$, using only his result on continued fractions associated with hypergeometric series. It is interesting to note that Gauss determined the numerical values of nodes and weights for each $n \leq 7$, with almost 16 significant decimal digits, which are otherwise the zeros of the shifted Legendre polynomial $P_n(2t - 1)$. Twelve years after Gauss, Jacobi [17] gave an elegant alternative derivation of Gauss's formulas. Further contributions and the development of this discovery into a theory during the second half of the nineteenth century were made by Mehler, Heine, Radau and many others, among whom Christoffel stands out in particular, who in 1877 gave a significant generalization of Gauss's formulas for arbitrary weight functions or measures, thus providing a fundamental connection with orthogonal polynomials and continued fractions. Such formulas with a maximum (algebraic) degree of exactness are now known as the *Gauss-Christoffel quadrature formulas.* Markov, Stiltjes, Uspensky, etc. are credited with analyzing the error $R_n(f)$ of these formulas in various classes of functions, as well as the convergence of a sequence of quadrature formulas. A nice survey of Gauss-Christoffel quadrature formulae was written by Gautschi [10].

The main tool for constructing and analyzing Gauss-Christoffel quadrature rules are orthogonal polynomials $\pi_k(t)$ related to the inner product defined by

$$(p, q) = \int_{\mathbb{R}} p(t)q(t)\,\mathrm{d}\mu(t) \quad (p, q \in \mathcal{P}).$$

It is well-known that the monic polynomials $\pi_k(t)$, $k = 0, 1, 2, \ldots$, satisfy the three-term recurrence relation of the form

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0, 1, 2, \ldots, \tag{4}$$

where $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$ (by definition).

Remark 3. The coefficient β_0 in (4) may be arbitrary, but is conveniently defined by

$$\beta_0 = \mu_0 = \int_{\mathbb{R}} \mathrm{d}\mu(t).$$

The following result was proved by Jacobi [17]:

Theorem 2.1. *Given a positive integer m $(\leq n)$, the quadrature formula*

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\mu(t) = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f), \tag{5}$$

has degree of exactness $d = n - 1 + m$ if and only if the following conditions are satisfied:

1° *Formula* (5) *is interpolatory;*

2° *The node polynomial $\omega_n(t) = (t - \tau_1)\cdots(t - \tau_n)$ satisfies*

$$(\forall p \in \mathcal{P}_{m-1}) \qquad (p, \omega_n) = \int_{\mathbb{R}} p(t)\omega_n(t)\,\mathrm{d}\mu(t) = 0.$$

According to this result, the n-point quadrature formula (5) with respect to the positive measure $\mathrm{d}\mu(t)$ has the maximal algebraic degree of exactness $2n - 1$. In other words, setting $m = n$ is optimal ($\omega_n = \pi_n$).

Higher values of m $(> n)$ are impossible. Indeed, according to 2°, the case $m = n + 1$ would require the orthogonality condition

$$(\forall p \in \mathcal{P}_n) \qquad (p, \omega_n) = 0.$$

Choosing $p = \omega_n$ (which has degree n) immediately gives $(\omega_n, \omega_n) = 0$. Since μ is a positive measure, the inner product is strictly positive for any non-zero polynomial of degree at most n, leading to a contradiction.

The cases with lower m are well known:

- when $m = n - 1$, the resulting formula is a *Gauss–Radau* quadrature rule (one of the endpoints a or b is included among the nodes);
- when $m = n - 2$, it becomes a *Gauss–Lobatto* quadrature rule ($\tau_1 = a$ and $\tau_n = b$).

These rules have lower degrees of exactness than the classical Gaussian formula (typically $2n-2$ for Radau and $2n-3$ for Lobatto with n nodes), which is the price paid for enforcing the boundary conditions.

The first significant progress in the construction of Gauss-Christoffel formulas (i.e., the nodes τ_k and the weight coefficients A_k) for an arbitrary positive measure $\mathrm{d}\mu$ on $\mathbb{R}$ with finite or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k \,\mathrm{d}\mu$, $k = 0, 1, \ldots$, exist and are finite and $\mu_0 > 0$, was made in 1969 by Golub and Welsch [14]. They reduced the construction to the eigenvalue problem for a symmetric tri-diagonal, the so-called Jacobi matrix,

$$J_n(\,\mathrm{d}\mu) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}, \qquad (6)$$

where the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are the coefficients in a three-term recurrence relation (4). The nodes $\tau_1, \ldots, \tau_n$ are the eigenvalues of the matrix (6), and the first components $v_{k,1}$ of the corresponding normalized eigenvectors $\mathbf{v}_k = \begin{bmatrix} v_{k,1} & \cdots & v_{k,n}\end{bmatrix}^T$ ($\mathbf{v}_k^T\mathbf{v}_k = 1$) give the weight coefficients (Christoffel numbers) $A_k = \mu_0 v_{k,1}^2$, $k = 1, \ldots, n$.

These sequences $\{\alpha_k\}$ and $\{\beta_k\}$ depend only on the measure, i.e., of the weight function $w(t) = \mathrm{d}\mu/\,\mathrm{d}t$ if the measure is absolutely continuous, but unfortunately they are known, in an explicit form, only for some narrow classes of weight functions, such as, for example, classical weights (Jacobi weight on $(-1, 1)$, generalized Laguere weight on $(0, \infty)$ and Hermite weight on $\mathbb{R}$).

Another significant progress occurred at the beginning of the eighties of the last century, when Walter Gautschi, in a series of papers, recognizing the recursive sequences $\{\alpha_k\}$ and $\{\beta_k\}$ as fundamental quantities, and developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$ (*cf.* [11, 12]).

Thanks to the methods developed within the constructive theory of orthogonality, the calculation of the coefficients $\{\alpha_k\}$ and $\{\beta_k\}$, in a general case, is realized by numerical methods (see [11, 12, 27]). During their numerical construction, the problem of high instability arises, in relation to small perturbations of the input quantities. However, the progress that has been made in the last thirty years in symbolic calculation and in the so-called arithmetic of variable precision, today allows the generation of sequences of recursive coefficients sometimes by direct application of the original Chebyshev method of moments, with the use of arithmetic of sufficiently high precision, which enables numerical instability to be overcome! Such software for orthogonal polynomials and quadrature formulas are available today:

• MATLAB package `SOPQ` (Gautschi: `https://www.cs.purdue.edu/archives/2002/wxg/`);

• MATHEMATICA package `OrthogonalPolynomials` (*cf.* [6, 29]: available on the website of the Mathematics Institute SASA `http://www.mi.sanu.ac.rs/~gvm/`).

Remark 4. A new representation of Hermite's osculator interpolation was presented in [20], with the aim of constructing a weighted Hermite quadrature formula. Explicit forms for several special cases of quadratures are obtained, including the weighted Hermite quadrature rule with arithmetic and geometric knots, as well as the standard Gauss-Christoffel quadrature rule and the Gaussian quadrature rule using only derivatives of functions (*cf.* [19, 28]).

In many situations, Gauss-Christoffel quadrature formulas give very accurate approximations of integrals. However, in some cases these formulas have slow convergence or give very similar results compared to simpler methods. In a review paper, Trefethen [36] compared the accuracy of the Gauss-Christoffel and Clenshaw-Curtis methods and showed that there are numerous cases where these two methods give results with almost equal errors, but the construction of the Clenshaw-Curtis method is much simpler. His recent work [37] on the accuracy of quadrature formulas is also interesting.

3 Two dimensional exponential integrals

The exponential integral $E_1(x)$ is defined by

$$E_1(z) = \int_1^{+\infty} \frac{\mathrm{e}^{-zt}}{t}\,\mathrm{d}t,$$

and its generalization $E_n(x)$ by

$$E_n(z) = \int_1^{+\infty} \frac{\mathrm{e}^{-zt}}{t^n}\,\mathrm{d}t, \quad n > 0. \tag{7}$$

The exponential integral plays an important role in many subjects of physics, quantum chemistry, theory of fluid flow, etc. Integrals having as weight function this integral on the positive real line $\mathbb{R}^+$ (or on a finite part $[0, c]$, $c > 0$) are of interest in radiative transfer. W. Gautschi [13] has considered polynomials orthogonal with respect to (7).

There is also a generalisation of (7), defined by (*cf.* [3])

$$\left.\begin{aligned} \varepsilon_1(\tau,\beta) &= \int_1^{+\infty} (t^2+\beta^2)^{-1/2} \exp\bigl[-\tau(t^2+\beta^2)^{1/2}\bigr]\,\mathrm{d}t,\\ \varepsilon_2(\tau,\beta) &= \int_1^{+\infty} t^{-2} \exp\bigl[-\tau(t^2+\beta^2)^{1/2}\bigr]\,\mathrm{d}t,\\ \varepsilon_3(\tau,\beta) &= \tau\int_1^{+\infty} \varepsilon_2\Bigl(\tau t, \frac{\beta}{t}\Bigr)\,\mathrm{d}t, \end{aligned}\right\} \tag{8}$$

which appears in the study of the radiative transfer in a multi-dimensional medium. Note that for $\beta = 0$, $\varepsilon_1(\tau, 0) = E_1(\tau)$ and $\varepsilon_1(\tau, 0) = E_2(\tau)$.

Altaç [2] (see also [1]) considered the n^{th} order generalized exponential integral functions $\varepsilon_n(\tau,\beta)$ $(n \in \mathbb{N})$ in the following form

$$\varepsilon_n(\tau,\beta) = \frac{1}{(n-1)!}\int_\tau^{+\infty} (t-\tau)^{n-1} \frac{\exp\bigl[-\sqrt{t^2+(\tau\beta)^2}\,\bigr]}{\sqrt{t^2+(\tau\beta)^2}}\,\mathrm{d}t,$$

or, after a change of variables $t := t\tau$, as

$$\varepsilon_n(\tau,\beta) = \frac{\tau^{n-1}}{(n-1)!}\int_1^{+\infty} (t-1)^{n-1} \frac{\exp\bigl[-\tau\sqrt{t^2+\beta^2}\,\bigr]}{\sqrt{t^2+\beta^2}}\,\mathrm{d}t. \tag{9}$$

He derived several series expansions, recurrence relations, as well as the other properties of this kind of integrals.

Since (*cf.* [33, p. 189] for $\nu = 0$)

$$\int_0^{+\infty} x\,\frac{\exp\bigl[-p\sqrt{x^2+z^2}\,\bigr]}{\sqrt{x^2+z^2}} J_0(cx)\,\mathrm{d}x = \frac{\exp\bigl[-z\sqrt{p^2+c^2}\,\bigr]}{\sqrt{p^2+c^2}},$$

where $J_\nu(z)$ is the Bessel function of the first kind and order ν, defined by

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)!}\Bigl(\frac{z}{2}\Bigr)^{2k+\nu}, \tag{10}$$

Eq. (9) can be expressed in the form

$$\varepsilon_n(\tau,\beta) = \frac{\tau^{n-1}}{(n-1)!}\int_1^{+\infty}(t-1)^{n-1}\left(\int_0^{+\infty}\varrho\,\frac{\exp\bigl[-t\sqrt{\varrho^2+\tau^2}\,\bigr]}{\sqrt{\varrho^2+\tau^2}}J_0(\beta\varrho)\,\mathrm{d}\varrho\right)\mathrm{d}t.$$

After changing the order of integration and using

$$\int_1^{+\infty}(t-1)^{n-1}\mathrm{e}^{-at}\,\mathrm{d}t = \frac{(n-1)!\mathrm{e}^{-a}}{a^n}, \quad a=\sqrt{\varrho^2+\tau^2},$$

we finally get

$$\varepsilon_n(\tau,\beta) = \tau^{n-1}\int_0^{+\infty}J_0(\beta\varrho)\frac{\varrho\,\mathrm{e}^{-\sqrt{\varrho^2+\tau^2}}}{(\varrho^2+\tau^2)^{(n+1)/2}}\,\mathrm{d}\varrho. \tag{11}$$

On the other side in 2015 Yardımcı et al. [41] considered the two-dimensional exponential integral (TDEI) functions given in the form

$$\varepsilon_n(\tau,\beta) = \frac{\tau^{n-1}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-r}}{r^{n+1}}\,\mathrm{e}^{-\mathrm{i}\beta x}\,\mathrm{d}x\,\mathrm{d}y, \tag{12}$$

where $r^2 = x^2+y^2+\tau^2$ and $n \in \mathbb{N}$. They proved that the function $(\tau,\beta)\mapsto\varepsilon_n(\tau,\beta)$ is uniformly convergent on

$$D(\epsilon) = \bigl\{(\tau,\beta)\in\Omega \,|\, \tau\in[\epsilon,+\infty),\ \beta\in\mathbb{R}\bigr\},$$

and nonuniformly convergent on $\Omega := [0,+\infty)\times\mathbb{R}$. Furthermore, using these facts, the authors concluded that for $(\tau,\beta)\in D(\varepsilon)$ the function $(\tau,\beta)\mapsto\varepsilon_n(\tau,\beta)$ satisfies the following asymptotic formulas

$$\begin{aligned}
&\varepsilon_n(\tau,\beta) = o(1), && \tau\to+\infty;\\
&\varepsilon_n(\tau,\beta) = o(1), && \beta\to\pm\infty;\\
&\varepsilon_n(\tau,\beta) = E_n(\tau)+o(1), && \beta\to+\infty.
\end{aligned}$$

Beside the problem of convergence and asymptotic behaviour, they investigated also numerical computations of the TDEI functions.

Note that the previous functions (8)–(12) are two-dimensional analogs of the exponential integral (7).

Now, we prove that the formulas (12) and (9) are equivalent.

Proposition 3.1. *The formulas for $\varepsilon_n(\tau,\beta)$, given by* (12) *and* (9) *are equivalent.*

Proof. If we introduce the polar coordinates (ϱ,θ) in the integral (12): $x=\varrho\cos\theta$, $y=\varrho\sin\theta$, then (12) reduces to

$$\begin{aligned}
\varepsilon_n(\tau,\beta) &= \frac{\tau^{n-1}}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\frac{\mathrm{e}^{-\sqrt{\varrho^2+\tau^2}}}{(\varrho^2+\tau^2)^{(n+1)/2}}\,\mathrm{e}^{-\mathrm{i}\beta\varrho\cos\theta}\varrho\,\mathrm{d}\theta\,\mathrm{d}\varrho\\
&= \tau^{n-1}\int_0^{+\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\beta\varrho\cos\theta}\,\mathrm{d}\theta\right)\frac{\mathrm{e}^{-\sqrt{\varrho^2+\tau^2}}\varrho}{(\varrho^2+\tau^2)^{(n+1)/2}}\,\mathrm{d}\varrho,
\end{aligned}$$

i.e.,

$$\varepsilon_n(\tau,\beta) = \tau^{n-1}\int_0^{+\infty}J_0(\beta\varrho)\frac{\mathrm{e}^{-\sqrt{\varrho^2+\tau^2}}\varrho}{(\varrho^2+\tau^2)^{(n+1)/2}}\,\mathrm{d}\varrho, \tag{13}$$

because of (*cf.* [32, p. 223])

$$J_n(z) = \frac{1}{2\pi \mathrm{i}^n} \int_0^{2\pi} \mathrm{e}^{\mathrm{i} z \cos\theta} \cos(n\theta)\, \mathrm{d}\theta$$

and $J_0(-z) = J_0(z)$, where $J_\nu(z)$ is the Bessel function of the first kind and order ν, defined by (10). It is obvious that (13) coincides with (11), which is obtained from (9), thus completing the proof. □

In the sequel we give a few equivalent integral representations of $\varepsilon_n(\tau,\beta)$.

Proposition 3.2. *For $\tau > 0$, $\beta > 0$, and $n \in \mathbb{N}$, we have*

$$\varepsilon_n(\tau,\beta) = \mathrm{e}^{-\tau} \int_0^{+\infty} \frac{J_0(\tau\beta\sqrt{z(z+2)})}{(z+1)^n}\, \mathrm{e}^{-\tau(z+1)}\, \mathrm{d}z, \tag{14}$$

$$\varepsilon_n(\tau,\beta) = \int_0^{+\infty} J_0\big(\tau\beta\sqrt{\mathrm{e}^{2x}-1}\big)\mathrm{e}^{-(n-1)x-\tau\mathrm{e}^x}\, \mathrm{d}x \tag{15}$$

and

$$\varepsilon_n(\tau,\beta) = \int_0^{+\infty} \frac{J_0(\tau\beta\sinh t)}{\cosh^n t}\, \mathrm{e}^{-\tau\cosh t} \sinh t\; \mathrm{d}t. \tag{16}$$

Proof. For a given τ we introduce a new variable z (≥ 0) by means $\varrho = \tau\sqrt{z(z+2)}$, so that

$$\sqrt{\varrho^2+\tau^2} = \tau(z+1) \qquad \text{and} \qquad \mathrm{d}\varrho = \frac{\tau(z+1)}{\sqrt{z(z+2)}}\, \mathrm{d}z.$$

Then, the integral (11) becomes

$$\varepsilon_n(\tau,\beta) = \tau^{n-1} \int_0^{+\infty} J_0(\beta\tau\sqrt{z(z+2)})\, \frac{\mathrm{e}^{-\tau(z+1)}\tau\sqrt{z(z+2)}}{\tau^{n+1}(z+1)^{n+1}} \cdot \frac{\tau(z+1)}{\sqrt{z(z+2)}}\, \mathrm{d}z,$$

i.e., (14).

In order to get (15) we use a change of variables $z = \mathrm{e}^x - 1$ in (14), so that $z(z+2) = (z+1)^2 - 1 = \mathrm{e}^{2x} - 1$. In this way we obtain

$$\begin{aligned}\varepsilon_n(\tau,\beta) &= \mathrm{e}^{-\tau} \int_0^{+\infty} \frac{J_0(\tau\beta\sqrt{\mathrm{e}^{2x}-1})}{\mathrm{e}^{nx}}\, \mathrm{e}^{-\tau(\mathrm{e}^x-1)}\mathrm{e}^x\, \mathrm{d}x \\ &= \int_0^{+\infty} J_0(\tau\beta\sqrt{\mathrm{e}^{2x}-1})\mathrm{e}^{-(n-1)x-\tau\mathrm{e}^x}\, \mathrm{d}x.\end{aligned}$$

Finally, using the $1-1$ transformation $t \mapsto x = \log(\cosh t)$, which maps the interval $[0,+\infty)$ into itself, the integral (15) reduces to (16). □

Remark 5. According to (15) or (16) we can conclude that $\varepsilon_n(\tau,\beta) \leq \varepsilon_1(\tau,\beta)$. Since $|J_0(t)| \leq 1$ for $t \geq 0$, we have

$$\big|J_0\big(\tau\beta\sqrt{\mathrm{e}^{2x}-1}\big)\mathrm{e}^{-(n-1)x-\tau\mathrm{e}^x}\big| \leq \mathrm{e}^{-\tau\mathrm{e}^x},$$

so that

$$\varepsilon_1(\tau,\beta) < \varepsilon_1(\tau,0) = \int_0^{+\infty} \mathrm{e}^{-\tau\mathrm{e}^x}\, \mathrm{d}x = \int_\tau^{+\infty} \frac{\mathrm{e}^{-t}}{t}\, \mathrm{d}t = \Gamma(0,\tau),$$

where $\Gamma(a,z)$ is the incomplete gamma function, defined by

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1}\mathrm{e}^{-t}\, \mathrm{d}t.$$

Remark 6. We note that the function $z \mapsto J_0\big(\omega\sqrt{z(z+2)}\big)$ ($\omega = \tau\beta$), which appears in the integral (14), can be highly oscillatory function (see Fig. 1), so this formula is not useful for calculating the function $\varepsilon_n(\tau,\beta)$. On the other side, the expressions (15) and (16) are acceptable for numerically calculating the value of $\varepsilon_n(\tau,\beta)$.

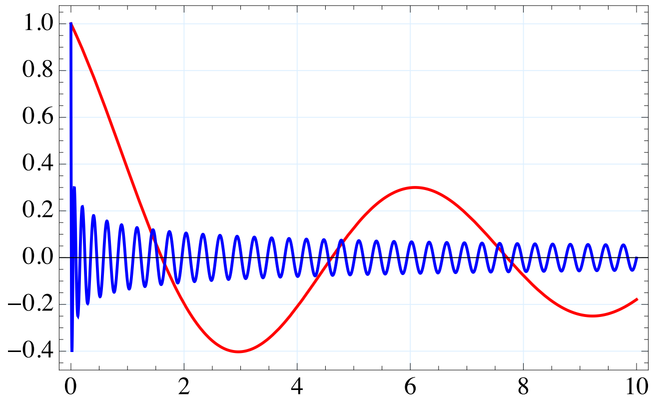

Figure 1: Graphics of the function $z \mapsto J_0\big(\omega\sqrt{z(z+2)}\big)$ for $\omega = 1$ (red line) and $\omega = 20$ (blue line)

4 Numerical calculation of the TDEI functions

In this section we give some alternative methods based on quadrature processes (*cf.* [23]) for numerical computation of the values of the two dimensional exponential integral (TDEI) functions. We analyze quadrature processes based on the application of the Gauss-Laguerre formula, the composite trapezoidal formula, and the method of so-called *integration between zeros of the integrand* for oscillatory functions (*cf.* [7, p. 230]). Special methods for the integration of fast oscillatory functions, such as those in the papers [4, 5, 15, 18, 26, 39, 40] and the book [8], will be the subject of future research.

4.1 Application of Gauss-Christoffel quadrature formulas

Computing weighted integrals of the Bessel function $J_0(\omega x)$ over $(0,+\infty)$ is a complex topic with results that depend heavily on the chosen weight function. Only in some cases such integrals have the closed forms, e.g.,

$$\int_0^{+\infty} \mathrm{e}^{-ax} J_0(\omega x)\,\mathrm{d}x = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \int_0^{+\infty} x\mathrm{e}^{-ax} J_0(\omega x)\,\mathrm{d}x = \frac{a}{(a^2+\omega^2)^{3/2}}, \quad \dots ,$$

while in most cases need specialized numerical methods are required.

First, we consider $\varepsilon_n(\tau,\beta)$ in the integral form (11). Let $\Phi(\varrho) \equiv \Phi(\varrho;\tau,\beta,n)$ denote its integrand, which can be written as a product $F(\varrho)w(\varrho)$, where w is the Laguerre weight function on $(0,+\infty)$, given by $w(\varrho) = \mathrm{e}^{-\varrho}$, and

$$F(\varrho) \equiv F(\varrho;\tau,\beta,n) = \Phi(\varrho;\tau,\beta,n)\mathrm{e}^{\varrho} = \tau^{n-1} J_0(\beta\varrho) \frac{\varrho\,\mathrm{e}^{-\tau^2/\left(\varrho+\sqrt{\varrho^2+\tau^2}\right)}}{(\varrho^2+\tau^2)^{(n+1)/2}}. \tag{17}$$

This allows us to apply the Gauss-Laguerre quadrature formula, so we have

$$\varepsilon_n(\tau,\beta) = \int_0^{+\infty} F(\varrho)w(\varrho)\,\mathrm{d}\varrho = \sum_{k=1}^{N} A_k F(\varrho_k) + R_N(F), \tag{18}$$

where $R_N(F)$ is the corresponding remainder term.

The functions $\Phi(\varrho)$ and $F(\varrho)$ for $\tau = 1$, $\beta = 10$ and $n = 2$ are displayed in Figures 2 and 3, respectively.

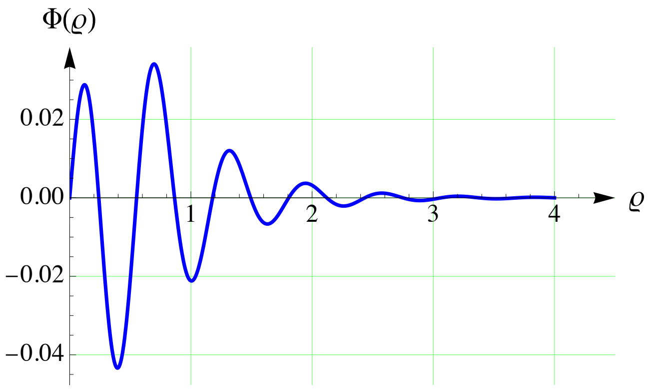

Figure 2: Graphic of the function $\varrho \mapsto \Phi(\varrho) \equiv \Phi(\varrho; 1, 10, 2)$ on $[0, 4]$

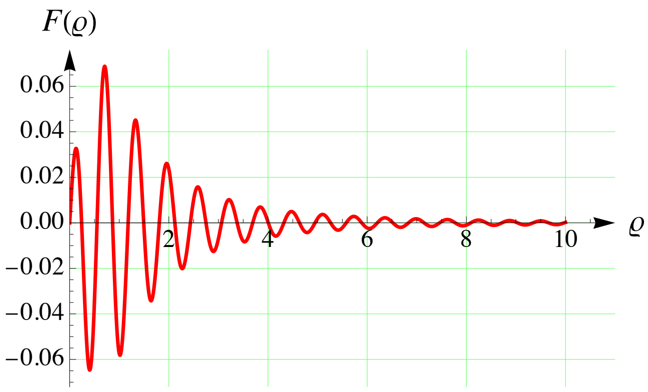

Figure 3: Graphic of the function $\varrho \mapsto F(\varrho) \equiv F(\varrho; 1, 10, 2)$ on $[0, 10]$

For calculating quadrature parameters (the nodes ϱ_k and the weights A_k) we use our MATHEMATICA Package `OrthogonalPolynomials` (*cf.* [6, 29]). Taking into account the oscillatory character and slow decay of the function $\varrho \mapsto F(\varrho)$ as $\varrho \to +\infty$, we will construct Gauss-Laguerre quadrature formulas with a large number of nodes $N(=$ `nn`$)$, from 100 to 1000, with a step of 100 (and a working precision of 25 decimal digits). The corresponding commands are:

```
<< orthogonalPolynomials`
pq[n_] := aGaussianNodesWeights[n, {aLaguerre}, WorkingPrecision -> 25,
          Precision -> 20];
nw = Table[pq[nb],{nn,100,1000,100}];
```

Here, `nw[[v]]` gives the nodes `nw[[v]][[1]]` and the weights `nw[[v]][[2]]` for `nn` $= 100\,$`v`, where `v` $= 1, 2, \ldots, 10$.

Applying the quadrature rule (18) to the function $F(\varrho; \tau, \beta, n)$, we obtain the approximative values of $\varepsilon_n(\tau, \beta)$, in the notation $\tilde{\varepsilon}_n^{(N)}(\tau, \beta)$. The corresponding values in the case $\tau = 1$, $\beta = 10$ and $n = 2$ are given in the second column of Table 1, and exact figures are underlined. Numbers in parentheses indicate decimal exponents, for example $1.24(-2) = 1.24 \times 10^{-2}$.

The exact value of this TDEI function in the point $(\tau, \beta) = (1, 10)$ is

$$\varepsilon_2(1, 10) = 2.98693042768590728497420377043\ldots \times 10^{-5},$$

obtained in Wolfram Mathematica, ver. 14.3, using the high precision arithmetics with 100 decimal digits.

N	$\tilde{\varepsilon}_2^{(N)}(1,10)$	$\mathrm{Err}(N)$	j_N (15%)	$\bar{\varepsilon}_2^{(j_N)}(1,10)$
100	$-2.5526893352569225(-4)$	9.55	15	$-2.5333617783794170(-5)$
200	$3.0240940409582615(-5)$	$1.24(-2)$	30	$3.0243994432526185(-5)$
300	$2.9882400642087742(-5)$	$4.38(-4)$	45	$2.9882404837193715(-5)$
400	$2.9869202414602691(-5)$	$3.41(-6)$	60	$2.9869202417527040(-5)$
500	$2.9869302932774400(-5)$	$4.50(-8)$	75	$2.9869302932765853(-5)$
600	$2.9869304302724077(-5)$	$8.66(-10)$	90	$2.9869304302724031(-5)$
700	$2.9869304276972022(-5)$	$3.78(-12)$	105	$2.9869304276972022(-5)$
800	$2.9869304276852608(-5)$	$2.16(-13)$	120	$2.9869304276852608(-5)$
900	$2.9869304276859088(-5)$	$5.17(-16)$	135	$2.9869304276859088(-5)$
1000	$2.9869304276859074(-5)$	$5.11(-17)$	150	$2.9869304276859074(-5)$

Table 1: Gauss-Laguerre approximations in numerical computation of $\varepsilon_2(1,10)$

From this example, we can see the very slow convergence of the Gauss-Laguerre quadrature formula. Relative errors

$$\mathrm{Err}(N) = \left| \frac{\tilde{\varepsilon}_n^{(N)}(\tau,\beta) - \varepsilon_n(\tau,\beta)}{\varepsilon_n(\tau,\beta)} \right|$$

are presented in the third column in Table 1. However, we can use the idea of Mastroianni and Monegato [24] to reduce the numerical work in computing quadrature sums. Namely, they proposed a truncated version where the last part of its nodes is omitted from the classical Gauss-Laguerre quadrature formula, i.e.,

$$\varepsilon_n(\tau,\beta) = \int_0^{+\infty} F(\varrho) w(\varrho)\,\mathrm{d}\varrho = \sum_{k=1}^{j_N} A_k F(\varrho_k) + R_{j_N}(F), \tag{19}$$

where $j_N < N$. Truncated sums $\bar{\varepsilon}_n^{(N)}(\tau,\beta)$, by taking only the first 15% of terms, are presented in the same table (last column). The number of terms taken in quadrature sums is given in the penultimate column. As we can see, the values of the truncated sums, with only $\lfloor 15N/100 \rfloor$ terms, are almost the same as those (with N terms) in the second column of Table 1.

Thus, in calculating the approximations of TDEI functions, we use a maximum of 150 quadrature nodes, so the numerical work is reduced many times. However, the construction of the high-order Gaussian formulas remains.

Remark 7. Instead of the Gauss-Laguerre quadrature rule in (18) one can use the generalized Gauss-Laguerre rule with the weight function $\varrho \mapsto \varrho\,\mathrm{e}^{-\varrho}$ on $(0,+\infty)$, and extracting the factor "ϱ" from (17), but some improvement cannot be achieved.

In the sequel we provide graphics (see Figure 4) for the TDEI functions $\varepsilon_n(\tau,\beta)$, $n = 1,2,3$, obtained using the previously described approach.

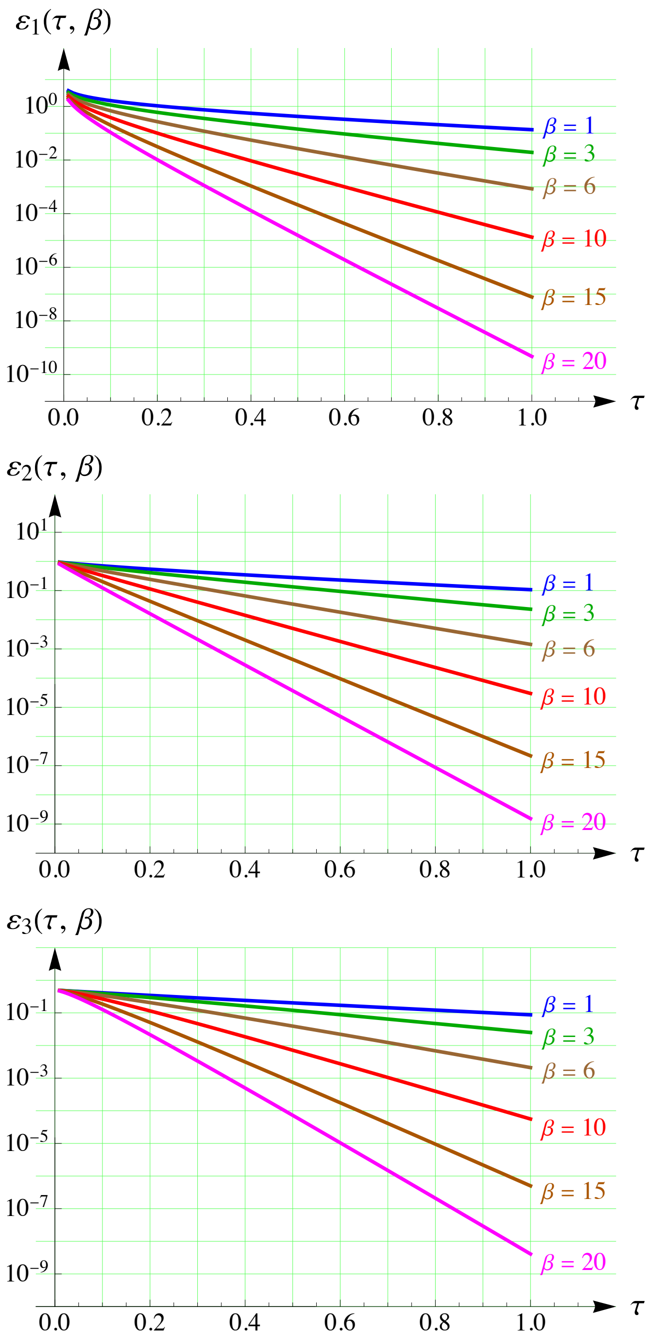

Figure 4: Graphics of the TDEI functions $\tau \mapsto \varepsilon_n(\tau, \beta)$, $n = 1, 2, 3$, in log-scale for some selected values of the parameter β, when τ runs over $(0, 1)$

The graphics for the TDEI functions $\tau \mapsto \varepsilon_n(\tau, \beta)$ are given in log-scale for $\tau \in (0, 1)$ and some selected values of the parameter $\beta \in \{1, 3, 6, 10, 15, 20\}$.

4.2 Application of the trapezoidal rule

As we mentioned in Section 3 (Remark 6) the expressions (15) and (16) are acceptable for numerically calculating the value of $\varepsilon_n(\tau, \beta)$. In this part we use the second one, denoting its integrand by

$$g(t) \equiv g(t; \tau, \beta, n) = \frac{J_0(\tau\beta \sinh t)}{\cosh^n t}\, \mathrm{e}^{-\tau \cosh t} \sinh t, \tag{20}$$

for which

$$|g(t)| \leq \frac{|J_0(\tau\beta \sinh t)| \sinh t}{\cosh^n t}\, \mathrm{e}^{-\tau \cosh t} \leq 2^{n-1} \exp\left(-\frac{\tau}{2}\, \mathrm{e}^t\right) \quad (t > 0).$$

Although this integrand g is oscillatory, its modulus $|g(t)|$ has a double exponential decay, so that we can apply the simplest rule, the so-called composite trapezoidal rule, for numerically computing the integral (16).

The trapezoidal sum (2) for integration over $(-\infty, +\infty)$ can be modified for this case of integration on $[0, +\infty)$ in the following way. For a given step h, we define the trapezoidal approximation

$$\int_0^{+\infty} g(t)\,\mathrm{d}t \approx T(g;h) := h \sum_{k=1}^{+\infty} g(kh), \tag{21}$$

where the initial term (for $k = 0$) vanishes because $g(0) = 0$. Also, if we define the shifted sum

$$\widetilde{T}(g;h) := h \sum_{k=0}^{+\infty} g\left(\frac{h}{2} + kh\right),$$

then the corresponding approximation for $h/2$ can be expressed as

$$T\left(g;\frac{h}{2}\right) = \frac{1}{2}\left[T(g;h) + \widetilde{T}(g;h)\right]. \tag{22}$$

Th relation (22) is useful for efficiently applying this method by successively reducing the step size to half the previous value. Due to the rapid decay of the modulus of integrand as $t \to +\infty$, the infinite sum is replaced by a finite one and the method becomes very simple. Therefore, the summation in (21) should be performed for $k \le M = \lfloor b/h \rfloor$, where b is such that $|g(t)| \ll \text{eps}$ for $t > b$, where eps is a suitably small value.

Here we consider two examples: (a) $(\tau, \beta, n) = (1, 10, 2)$ and (b) $(\tau, \beta, n) = (1/2, 1, 1)$.

(a) This case has been treated in the previous part of this section by the Gauss-Christoffel rule. Its integrand $g(t; 1, 10, 2)$ is presented in Figure 5 (red line). Figure 6 (left) shows the integrand on the interval $(3, 6)$. For example, $|g(t)|$ for $t = 4, 5, 6$ takes the values 8.73×10^{-16}, 2.30×10^{-36}, 1.86×10^{-92}, respectively, so that we can take $b = 5$.

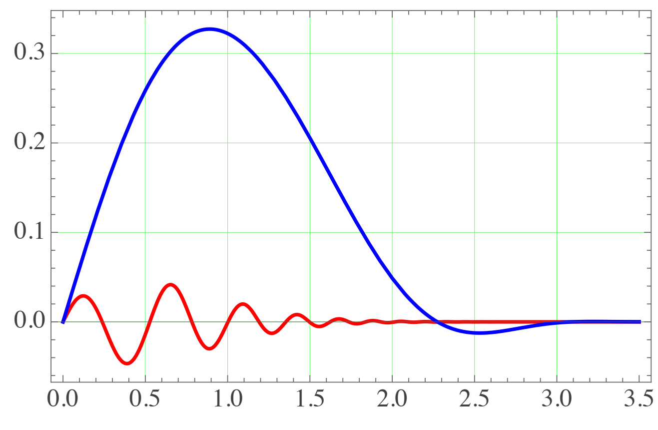

Figure 5: The integrand (20) for $(\tau, \beta, n) = (1, 10, 2)$ (red line) and $(\tau, \beta, n) = (1/2, 1, 1)$ (blue line)

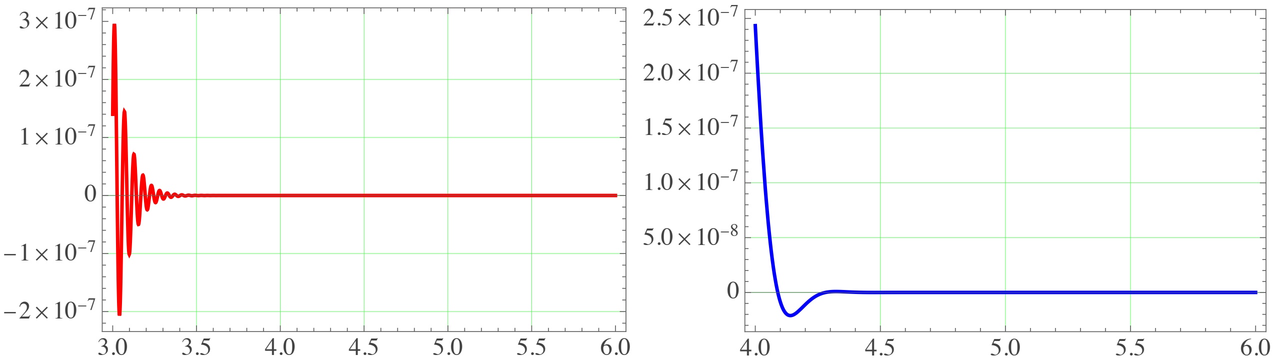

Figure 6: The integrand (20) for $(\tau, \beta, n) = (1, 10, 2)$ (left) and $(\tau, \beta, n) = (1/2, 1, 1)$ (right)

In our experiment, we use the steps $h_\nu = h/2^\nu$, $\nu = 0, 1, \ldots, 10$, where $h = 1/100$. The corresponding (finite) trapezoidal sums, denoted by $T_\nu(\tau, \beta, n)$, are given in Table 2, as well as their relative errors $\mathrm{Err}_\nu(\tau, \beta, n)$. Exact digits are underlined.

ν	2^ν	$T_\nu(1, 10, 2)$	$\mathrm{Err}_\nu(1, 10, 2)$	$T_\nu(1/2, 1, 1)$	$\mathrm{Err}_\nu(1/2, 1, 1)$
0	1	2.6802834709948373(−5)	1.03(−1)	0.42370204303593167	1.19(−5)
1	2	2.9102838314573682(−5)	2.57(−2)	0.42370583388315840	2.98(−6)
2	4	2.9677697247703535(−5)	6.41(−3)	0.42370678158922689	7.46(−7)
3	8	2.9821403110862998(−5)	1.60(−3)	0.42370701851538539	1.86(−7)
4	16	2.9857329022315141(−5)	4.01(−4)	0.42370707774690259	4.66(−8)
5	32	2.9866310465532772(−5)	1.00(−4)	0.42370709255478050	1.16(−8)
6	64	2.9868555824171853(−5)	2.51(−5)	0.42370709625674988	2.91(−9)
7	128	2.9869117163696290(−5)	6.26(−6)	0.42370709718224223	7.28(−10)
8	256	2.9869257498568941(−5)	1.57(−6)	0.42370709741361531	1.82(−10)
9	512	2.9869292582286575(−5)	3.92(−7)	0.42370709747145858	4.55(−11)
10	1024	2.9869301353215951(−5)	9.79(−8)	0.42370709748591940	1.14(−11)

Table 2: Approximations S_k in numerical computation of $\varepsilon_2(1, 10)$

(b) In the second example $(\tau, \beta, n) = (1/2, 1, 1)$ the function g is not rapidly oscillatory (see Figures 5 and 6 (blue line)).

In this case, $|g(t)|$ for $t = 5, 6, 7$ takes the values 1.85×10^{-18}, 1.12×10^{-45}, 4.14×10^{-121}, respectively, so that we can take $b = 6$. The corresponding values for $T_\nu(1/2, 1, 1)$ and $\mathrm{Err}_\nu(1/2, 1, 1)$ are presented again in Table 2. As we can see, in this case the convergence of the trapezoidal formula is significantly faster than in case (a).

4.3 Integration between zeros of the integrand

For computing the value of an integral whose integrand oscillates over $(0, +\infty)$, as is the case here, it may be useful to compute the positive and negative contributions separately, and then sum the resulting infinite series. For computing integrals between zeros, the Gauss-Lobatto quadrature rule at $N + 2$ points is suitable, since the integrand values at the endpoints of these subintervals are zero. Thus, this rule has the algebraic degree of exactness $2N + 1$, with calculations of the integrand at only N nodes.

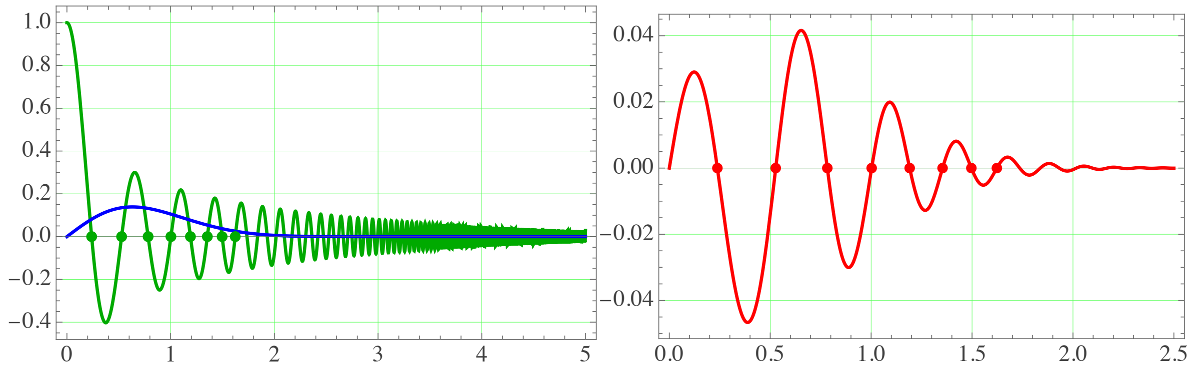

Figure 7: Graphics of the functions for $\tau = 1$, $\beta = 10$, $n = 2$. Left: $K(t;\tau\beta) = J_0(\tau\beta\sinh t)$ (green) and $g(t;\tau,n)$ (blue); Right: The product $F(t) = K(t;\tau\beta)g(t;\tau,n)$ (red)

Consider the TDEI function $\varepsilon_n(\tau,\beta)$ in the form (16). In general, it is an integral of the positive function $t \mapsto g(t;\tau,n)$ over $(0,+\infty)$, with the oscillatory kernel $K(t;\omega)$, where

$$g(t;\tau,n) = \frac{\mathrm{e}^{-\tau\cosh t}\sinh t}{\cosh^n t}, \quad K(t;\omega) = J_0(\omega\sinh t) \quad (\omega = \tau\beta). \tag{23}$$

The positive zeros of the Bessel function $J_n(x)$: $j_{n,k}$, $k = 1,2,\ldots$, are provided in Mathematica as a mathematical function `BesselJZero[n,k]`, suitable for both symbolic and numerical calculation (with arbitrary numerical precision).

In our case, for example, the first $M = 120$ positive zeros of the kernel $K(t;\omega) = J_0(\omega\sinh t)$, for $\omega = 10$ ($\tau = 1$, $\beta = 10$) can be obtained (with 25 decimal digits) very easy as

```
zeros = Table[ArcSinh[N[BesselJZero[0,k],25]/10], {k,1,120}];
```

The functions (23) (for $\tau = 1$, $\beta = 10$, $n = 2$) are presented in Figure 7. The first eight zeros are also shown.

Let z_k, $k = 1,\ldots,M$, be the first M zeros of the kernel $K(t;\tau\beta)$ and let

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = w_0 f(-1) + \sum_{\nu=1}^{N} w_k f(x_k) + w_{N+1} f(1) + R_{N+2}(f), \tag{24}$$

be the $(N+2)$-point Gauss-Lobatto quadrature rule (*cf.* [23, p. 330]) on the standard interval $[-1,1]$. In order to integrate a function F on $[z_{k-1}, z_k]$ (with $F(z_{k-1}) = F(z_k) = 0$), the formula (24) can be transformed to the corresponding quadrature rule

$$\int_{z_{k-1}}^{z_k} F(t)\,\mathrm{d}t = \frac{z_k - z_{k-1}}{2}\sum_{\nu=1}^{N} w_k F\left(\frac{z_k - z_{k-1}}{2}x_\nu + \frac{z_k + z_{k-1}}{2}\right) + R_N^{(k)}(F), \quad k = 2,\ldots,M.$$

If $F(0) = 0$, the previous rule can be also applied to the interval $[0, z_1]$, i.e., for $k = 1$, if we put $z_0 = 0$. In our case this condition for the function $F(t) = K(t;\tau\beta)g(t;\tau,n)$ is satisfied, because $g(0;\tau,n) = 0$.

The Gauss-Lobatto quadrature rule (24) can be constructed by using our Mathematica Package `OrthogonalPolynomials` (*cf.* [6, 29]). In this case, it is enough to take only $N = 10$ internal nodes. The corresponding commands are:

```
<< orthogonalPolynomials`
{al, be} = Transpose[aLegendre["ttr"] /@ Range[0, 20]];
{nL0, wL0} = aLobattoNodesWeights[10, al, be, -1, 1, WorkingPrecision -> 30,
             Precision -> 20]
(* Dropping the first and last nodes *)
nL = Delete[nL0, {{1}, {12}}]; wL = Delete[wL0, {{1}, {12}}];
```

For a given $\{\tau, \beta, n\}$, we calculate the values of the integrals over subintervals between zeros (the sequence `terms`):

```
F[t_,tau_,beta_,n_] :=
   BesselJ[0, tau beta Sinh[t]] Exp[-tau Cosh[t]] (Sinh[t]/Cosh[t]^n);
(* Set values for {tau,beta,n}  *)
seq = Table[0, {k,1,120}];
seq[[1]] = zeros[[1]]/2 wL.F[zeros[[1]]/2 (nL+1), 1,10,2];
Do[a = (zeros[[k]]-zeros[[k-1]])/2; b = (zeros[[k-1]]+zeros[[k]])/2;
   nT = a nL + b;
seq[[k]] = a wL.F[nT,1,10,2], {k,2,120}];
```

The obtained sequence is oscillatory and the corresponding numerical series may be slowly convergent, when it is necessary to apply some of the acceleration procedures (e.g., Euler-Abel transform, ε-transformation, Aitken's Δ^2 method, Levin's V-transform, etc.).

The partial sums S_k for $\varepsilon_2(1, 10)$, when $k = 20(10)120$, with relative errors, are presented in Table 3. Exact digits are underlined.

k	Partial sum S_k	Relative error
20	$\underline{2.9}4063387806154546(-5)$	$1.55(-2)$
30	$\underline{2.986}1740508523990(-5)$	$2.53(-4)$
40	$\underline{2.9869}142274627505(-5)$	$5.42(-6)$
50	$\underline{2.986930}0231936906(-5)$	$1.35(-7)$
60	$\underline{2.9869304}165409176(-5)$	$3.73(-9)$
70	$\underline{2.986930427}3570364(-5)$	$1.10(-10)$
80	$\underline{2.9869304276}757006(-5)$	$3.42(-12)$
90	$\underline{2.986930427685}5780(-5)$	$1.10(-13)$
100	$\underline{2.986930427685}8963(-5)$	$3.67(-15)$
110	$\underline{2.98693042768590}69(-5)$	$1.25(-16)$
120	$\underline{2.986930427685907}3(-5)$	$4.40(-18)$

Table 3: Approximations S_k in numerical computation of $\varepsilon_2(1, 10)$

A good feature of this method is that when breaking the series (by taking a partial sum), the error we make is always smaller than the first discarded term of the series. For example, in the observed case, if we discard the hundred and first term $1.87553789682425808 \times 10^{-19}$, the absolute error in the partial sum S_{100} will be smaller than 1.88×10^{-19}, which means that the relative error is smaller than 6.28×10^{-15}. From Table 3 we see that it is actually 3.67×10^{-15}.

5 Conclusion

Beside the general short description of important quadrature processes, including some historical details, we considered integral representations of two-dimensional exponential integral (TDEI) functions, as well as their numerical calculation based on quadrature processes. Three methods are proposed: (1) *Truncated Gauss-Christoffel quadrature formulas with respect to the Laguerre weight function on* $[0, +\infty)$; (2) *Composite trapezoidal rule*; (3) *Integration between zeros of the integrand.*

The simplest trapezoidal rule can be used when the integrand is not fast oscillating function and when high accuracy is not required. The truncated Gauss-Laguerre formulas are efficient to apply, but the construction of the basic high-order formulas can be demanding. However, this is successfully solved using the available MATHEMATICA Package `OrthogonalPolynomials`. Finally, as an alternative, integration between zero integrands is a very reliable and accurate method.

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References

[1] Z. Altaç, *Integrals involving Bickley and Bessel functions in radiative transfer, and generalized exponential integral functions*, J. Heat Transfer **118 (4)**, 789–792, 1996.

[2] Z. Altaç, *Exact series expansions, recurrence relations, properties and integrals of the generalized exponential integral functions*, J. Quant. Spectrosc. Radiat. Transfer **104 (2)**, 310–325, 2007.

[3] W. F. Breig and A. L. Crosbie, *Numerical computation of a generalized exponential integral function*, Math. Comp. **28 (128)**, 575–579, 1974.

[4] R. Chen, *Numerical approximations to integrals with a highly oscillatory Bessel kernel*, Appl. Numer. Math. **62 (5)**, 636–648, 2012.

[5] R. Chen, *On the evaluation of Bessel transformations with the oscillators via asymptotic series of Whittaker functions*, J. Comput. Appl. Math. **250**, 107–121, 2013.

[6] A. S. Cvetković and G. V. Milovanović, *The Mathematica package "OrthogonalPolynomials"*, Facta Univ. Ser. Math. Inform. **19**, 17–36, 2004.

[7] Ph. J. Davis and Ph. Rabinowitz, *Methods of numerical integration* (Second Edition), Comput. Sci. Appl. Math., Academic Press, Orlando, FL, 1984.

[8] A. Deaño, D. Huybrechs and A. Iserles, *Computing highly oscillatory integrals*, Soc. Indust. Appl. Math. (SIAM), Philadelphia, PA, 2018.

[9] C. F. Gauss, *Methodus nova integralium valores per approximationem inveniendi*, Comment. Soc. Regiae Sci. Gottingensis Recent. Werke III **3**, 163–196, 1814.

[10] W. Gautschi, *A survey of Gauss-Christoffel quadrature formulae*, In: *E. B. Christoffel – The influence of his work on mathematics and the physical sciences* (Ed. by P. L. Butzer and F. Fehér), Birkhäuser, Basel, 1981, 72–147.

[11] W. Gautschi, *On generating orthogonal polynomials*, SIAM J. Sci. Stat. Comput. **3 (3)**, 289–317, 1982.

[12] W. Gautschi, *Orthogonal polynomials: Applications and computation*, Acta Numer. **5**, 45–119, 1996.

[13] W. Gautschi, *Polynomials orthogonal with respect to exponential integrals*, Numer. Algorithms **70 (1)**, 215–226, 2015.

[14] G. H. Golub and J. H. Welsch, *Calculation of Gauss quadrature rules*, Math. Comp. **23 (106)**, 221–230, 1969.

[15] D. Huybrechs and S. Vandewalle, *On the evaluation of highly oscillatory integrals by analytic continuation*, SIAM J. Numer. Anal. **44 (3)**, 1026–1048, 2006.

[16] M. Iri, S. Moriguti and Y. Takasawa, *On certain quadrature formula*, J. Comput. Appl. Math. **17 (1)**, 3–20, 1987.

[17] C. G. J. Jacobi, *Über Gauß' neue Methode, die Werte der Integrale näherungsweise zu finden*, J. Reine Angew. Math. **30**, 301–308, 1826.

[18] D. Kılıç Kurtoğlu, A. I. Hasçelik and G. V. Milovanović, *A method for efficient computation of integrals with oscillatory and singular integrand*, Numer. Algorithms **85 (4)**, 1155–1173, 2020.

[19] M. Masjed-Jamei, *A new type of weighted quadrature rules and its relation with orthogonal polynomials*, Appl. Math. Comput. **188 (1)**, 154–165, 2007.

[20] M. Masjed-Jamei and G. V. Milovanović, *Weighted Hermite quadrature rules*, Electron. Trans. Numer. Anal. **45**, 476–498, 2016; ISSN 1068-9613.

[21] M. Masjed-Jamei, G. V. Milovanović and M. A. Jafari, *Explicit forms of weighted quadrature rules with geometric nodes*, Math. Comput. Modelling **53 (5–6)**, 1133–1139, 2011; https://doi.org/10.1016/j.mcm.2010.11.076.

[22] M. Masjed-Jamei, G. V. Milovanović and M. A. Jafari, *Closed expressions for coefficients in weighted Newton-Cotes quadratures*, Filomat **27 (4)**, 649–658, 2013; DOI: 10.2298/FIL1304649M.

[23] G. Mastroianni and G. V. Milovanović, *Interpolation processes: Basic theory and applications*, Springer Monographs in Mathematics, Springer-Verlag, Berlin–Heidelberg, 2008.

[24] G. Mastroianni and G. Monegato, *Truncated quadrature rules over* $(0,\infty)$ *and Nyström type methods*, SIAM J. Numer. Anal. **41 (5)**, 1870–1892, 2003.

[25] G. V. Milovanović, *Expansions of the Kurepa function*, Publ. Inst. Math. (Beograd) (N.S.) **57 (71)**, 81–90, 1995.

[26] G. V. Milovanović, *Numerical calculation of integrals involving oscillatory and singular kernels and some applications of quadratures*, Comput. Math. Appl. **36 (10)**, 19–39, 1998.

[27] G. V. Milovanović, *Orthogonal polynomials on the real line*, In: Walter Gautschi: Selected Works and Commentaries (Volume 2), (Ed. by C. Brezinski and A. Sameh), Birkhäuser, Basel, 2013.

[28] G. V. Milovanović and A. S. Cvetković, *Gaussian quadrature rules using function derivatives*, IMA J. Numer. Anal. **31 (2)**, 358–377, 2011.

[29] G. V. Milovanović and A. S. Cvetković, *Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type*, Math. Balkanica **26 (1–2)**, 169–184, 2012.

[30] G. Monegato and L. Scuderi, *Quadrature rules for unbounded intervals and their application to integral equations*, In: Approximation and computation: in honor of Gradimir V. Milovanović, (Ed. by W. Gautschi, G. Mastroianni and Th. M. Rassias), Springer Optim. Appl. **42**, Springer, New York, 185–208, 2011.

[31] M. Mori, *Quadrature formulas obtained by variable transformation and DE-rule*, J. Comput. Appl. Math. **12-13**, 119–130, 1985.

[32] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and Ch. W. Clark, *NIST handbook of mathematical functions*, Cambridge Univ. Press, New York, 2010.

[33] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and series, Vol. 2: special functions*, Gordon and Breach Sci. Publ., New York, 1986.

[34] H. Takahasi and M. Mori, *Error estimation in the numerical integration of analytic functions*, Rep. Comput. Centre Univ. Tokyo **3**, 41–108, 1970.

[35] H. Takahasi and M. Mori, *Quadrature formulas obtained by variable transformation*, Numer. Math. **21 (3)**, 206–219, 1973.

[36] L. N. Trefethen, *Is Gauss quadrature better than Clenshaw-Curtis?*, SIAM Rev. **50 (1)**, 67–87, 2008; https://doi.org/10.1137/060659831.

[37] L. N. Trefethen, *Exactness of quadrature formulas*, SIAM Rev. **64 (1)**, 132–150, 2022.

[38] J. Waldvogel, *Towards a general error theory of the trapezoidal rule*, In: *Approximation and computation: in honor of Gradimir V. Milovanović* (Ed. by W. Gautschi, G. Mastroianni and Th. M. Rassias), Springer Optim. Appl. (Volume 42), Springer, New York, 267–282, 2011.

[39] Z. Xu and G. V. Milovanović, *Efficient method for the computation of oscillatory Bessel transform and Bessel Hilbert transform*, J. Comput. Appl. Math. **308**, 117–137, 2016.

[40] Z. Xu, G. V. Milovanović and S. Xiang, *Efficient computation of highly oscillatory integrals with Hankel kernel*, Appl. Math. Comput. **261**, 312–322, 2015.

[41] Ş. Yardımcı, M. Olgun and Ç. Can, *Numerical computation and properties of the two dimensional exponential integrals*, Hacet. J. Math. Stat. **45 (4)**, 975–990, 2015.