PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 3, July 1985

AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS

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ABSTRACT. Let W_n be the set of all algebraic polynomials of exact degree n whose coefficients are all nonnegative. For the norm in $L^2[0,\infty)$ with generalized Laguerre weight function $w(x) = x^\alpha e^{-x}$ $(\alpha > -1)$, the extremal problem $C_n(\alpha) = \sup_{P \in W_n}(\|P'\|/\|P\|)^2$ is solved, which completes a result of A. K. Varma [7].

1. In this paper we give the complete solution of a problem which has been investigated recently by A. K. Varma (see [**7**, **8**]). This problem is related to some previous integral inequalities of Varma [**9**, **10**] and also to the classical inequalities of A. Markov [**4**], P. Erdös [**1**], G. G. Lorentz [**2**, **3**], G. Szegö [**5**], and P. Turan [**6**].

Let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which are nonnegative, i.e.,

$$W_n = \left\{ P_n | P_n(x) = \sum_{k=0}^{n} a_k x^k,\, a_k \geqq 0\ (k = 0, 1, \ldots, n) \right\}.$$

We denote by W_n^0 the subset of W_n for which $a_0 = 0$ (i.e., $P_n(0) = 0$).

Let $w(x) = x^\alpha e^{-x}$ $(\alpha > -1)$ be a weight function on $[0,\infty)$, and let $\|f\|^2 = (f, f)$, where

$$(f, g) = \int_0^\infty w(x) f(x) g(x)\, dx \qquad (f, g \in L^2[0,\infty)).$$

In [7] Varma has investigated the problem of determining the best constant in the inequality

$$\|P_n'\|^2 \leqq C_n(\alpha) \|P_n\|^2, \tag{1.1}$$

where $P_n \in W_n$. In fact, he has proved

THEOREM A. *Let $P_n(x)$ be an algebraic polynomial of exact degree n with nonegative coefficients. Then for $\alpha \geqq (\sqrt{5} - 1)/2$,*

$$\int_0^\infty \left(P_n'(x)\right)^2 x^\alpha e^{-x}\, dx \leqq \frac{n^2}{(2n+\alpha)(2n+\alpha-1)} \int_0^\infty P_n^2(x) x^\alpha e^{-x}\, dx,$$

equality holding for $P_n(x) = x^n$. For $0 \leqq \alpha \leqq 1/2$ we have

$$\int_0^\infty \left(P_n'(x)\right)^2 x^\alpha e^{-x}\, dx \leqq \frac{1}{(2+\alpha)(1+\alpha)} \int_0^\infty P_n^2(x) x^\alpha e^{-x}\, dx. \tag{1.2}$$

Received by the editors March 29, 1984.
1980 *Mathematics Subject Classification*. Primary 26C05, 41A44.

Moreover, (1.2) *is also best possible in the sense that for* $P_n(x) = x^n + \lambda x$ *the expression on the left can be made arbitrarily close to the expression on the right by choosing* λ *positive and sufficiently large.*

The case $\alpha = 1$ was considered in **[8]**. The cases $\alpha \in (-1, 0)$ and $\alpha \in (1/2, (\sqrt{5} - 1)/2)$ are still open.

2. The object of this paper is to determine

$$C_n(\alpha) = \sup_{P_n \in W_n} \frac{\|P_n'\|^2}{\|P_n\|^2} \tag{2.1}$$

for all $\alpha \in (-1, \infty)$ and, thus, to give a complete solution of the extremal problem (1.1). Note that the supremum in (2.1) is attained for some $P_n \in W_n^0$. Indeed,

$$\sup_{P_n \in W_n} \frac{\|P_n'\|}{\|P_n\|} = \sup_{\substack{P_n \in W_n^0 \\ a_0 \geqq 0}} \frac{\|P_n'\|}{\|P_n + a_0\|} = \sup_{P_n \in W_n^0} \frac{\|P_n'\|}{\|P_n\|}.$$

We begin by proving three lemmas:

LEMMA 1. *If* $P_n \in W_n$ *then for every* $x \geqq 0$ *the inequality*

$$x\big(P_n'(x)^2 - P_n(x)P_n''(x)\big) \leqq P_n'(x)P_n(x) \tag{2.2}$$

holds.

PROOF. Let $P_n \in W_n$, i.e., $P_n(x) = \sum_{k=0}^n a_k x^k$ with $a_k \geqq 0$ $(k = 0, 1, \dots, n)$. Using the Cauchy-Schwarz inequality

$$\left|\sum_{k=0}^n x_k y_k\right|^2 \leqq \left(\sum_{k=0}^n |x_k|^2\right)\left(\sum_{k=0}^n |y_k|^2\right)$$

for $x_k = a_k^{1/2} x^{k/2}$ and $y_k = k a_k^{1/2} x^{k/2}$ $(x \geqq 0)$, we obtain

$$\left(\sum_{k=0}^n k a_k x^k\right) \leqq \left(\sum_{k=0}^n a_k x^k\right)\left(\sum_{k=0}^n k^2 a_k x^k\right),$$

which is equivalent to (2.2). □

LEMMA 2. *If* $P_n \in W_n^0$, *then for the integrals*

$$J_n(\alpha) = \int_0^\infty x^\alpha e^{-x} P_n'(x)^2\, dx,$$

$$I_{n,i}(\alpha) = \int_0^\infty x^\alpha e^{-x} P_n(x) P_n^{(i)}(x)\, dx \qquad (i = 0, 1, 2)$$

the following recurrence relations hold:

$$I_{n,2}(\alpha) = I_{n,1}(\alpha) - \alpha I_{n,1}(\alpha - 1) - J_n(\alpha) \qquad (\alpha > -1),$$

$$2I_{n,1}(\alpha) = I_{n,0}(\alpha) - \alpha I_{n,0}(\alpha - 1) \qquad (\alpha > -2).$$

The proof of this lemma is a simple application of integration by parts and will be omitted. We note that the integrals $I_{n,1}(\alpha)$ and $I_{n,0}(\alpha - 1)$ exist for $\alpha > -2$ because $P_n(0) = 0$.

From Lemmas 1 and 2 there immediately follows

LEMMA 3. *If $P_n \in W_n^0$, then for $\alpha > -1$,*

$$J_n(\alpha) \leqq \tfrac{1}{4}\left\{I_{n,0}(\alpha) + (1-2\alpha)I_{n,0}(\alpha-1) + (\alpha-1)^2 I_{n,0}(\alpha-2)\right\}.$$

THEOREM. *The best constant $C_n(\alpha)$ defined in* (2.1) *is*

$$C_n(\alpha) = \begin{cases} 1/(2+\alpha)(1+\alpha) & (-1 < \alpha \leqq \alpha_n), \\ n^2/(2n+\alpha)(2n+\alpha-1) & (\alpha_n \leqq \alpha < +\infty), \end{cases} \tag{2.3}$$

where

$$\alpha_n = \tfrac{1}{2}(n+1)^{-1}\left((17n^2+2n+1)^{1/2} - 3n + 1\right). \tag{2.4}$$

PROOF. Let $P_n \in W_n^0$, i.e., $P_n(x) = \sum_{k=1}^n a_k x^k$ $(a_k \geqq 0)$. Then

$$P_n(x)^2 = \sum_{k=2}^{2n} b_k x^k \qquad (b_k \geqq 0)$$

and

$$\|P_n\|^2 = I_{n,0}(\alpha) = \sum_{k=2}^{2n} b_k \Gamma(k+\alpha+1),$$

where Γ is the gamma function. Using Lemma 3 we obtain

$$4J_n(\alpha) \leqq \sum_{k=2}^{2n} b_k\left\{\Gamma(k+\alpha+1) + (1-2\alpha)\Gamma(k+\alpha) + (\alpha-1)^2\Gamma(k+\alpha-1)\right\},$$

i.e.,

$$J_n(\alpha) \leqq \sum_{k=2}^{2n} H_k(\alpha) b_k \Gamma(k+\alpha+1), \tag{2.5}$$

where

$$H_k(\alpha) = \tfrac{1}{4}\cdot k^2/(k+\alpha)(k+\alpha-1).$$

From (2.5) it follows that

$$\|P_n'\|^2 \leqq \left(\max_{2\leqq k\leqq 2n} H_k(\alpha)\right)\|P_n\|^2,$$

so

$$C_n(\alpha) \leqq \max_{2\leqq k\leqq 2n} H_k(\alpha).$$

Determining the maximum of $f(x) = x^2/(x+\alpha)(x+\alpha-1)$ on the interval $[2, 2n]$, we find that

$$\max_{2\leqq k\leqq 2n} H_k(\alpha) = \begin{cases} H_2(\alpha) & \text{if } -1 < \alpha \leqq \alpha_n, \\ H_{2n}(\alpha) & \text{if } \alpha_n \leqq \alpha < +\infty, \end{cases}$$

where α_n is given by (2.4).

In order to show that $C_n(\alpha)$ defined in (2.3) is best possible, i.e. that $C_n(\alpha) = \max_{2 \leqq k \leqq 2n} H_k(\alpha)$, we consider $\tilde{P}_n(x) = x^n + \lambda x$ ($\lambda \geqq 0$) and set

$$Q_n(\lambda) = \|\tilde{P}_n'\|^2 / \|\tilde{P}_n\|^2.$$

By a simple computation we find that

$$Q_n(\lambda) = \frac{n^2\Gamma(2n+\alpha-1) + 2\lambda n\Gamma(n+\alpha) + \lambda^2\Gamma(\alpha+1)}{\Gamma(2n+\alpha+1) + 2\lambda\Gamma(n+\alpha+2) + \lambda^2\Gamma(\alpha+3)}.$$

Since

$$Q_n(0) = n^2/(2n+\alpha)(2n+\alpha-1) = H_{2n}(\alpha)$$

and

$$\lim_{\lambda \to \infty} Q_n(\lambda) = 1/(\alpha+1)(\alpha+2) = H_2(\alpha),$$

we conclude that $\tilde{P}_n(x) = x^n$ is an extremal polynomial for $\alpha \geqq \alpha_n$. Furthermore, if $-1 < \alpha \leqq \alpha_n$, there exists a sequence of polynomials, for example, $p_{n,k}(x) = x^n + kx$, $k = 1, 2, \ldots$, for which

$$\lim_{k \to \infty} \frac{\|p_{n,k}'\|^2}{\|p_{n,k}\|^2} = C_n(\alpha). \quad \square$$

REMARK. From (2.4) we have $\alpha_1 = (\sqrt{5} - 1)/2$, $\alpha_2 = (\sqrt{73} - 5)/6$, $\alpha_3 = (\sqrt{10} - 2)/2$, etc. The sequence (α_k) is decreasing, i.e., $\alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_\infty$, where $\alpha_\infty = \lim_{n \to \infty} \alpha_n = (\sqrt{17} - 3)/2 \cong 0.56155$.

ACKNOWLEDGMENT. The author is grateful to Professor W. Gautschi for his careful reading of the paper and useful suggestions for better and more complete formulations of the material.

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