Orthogonality in the complex plane

Gradimir V. Milovanović

This is a review of orthogonal polynomials and rational functions in the complex plane. In addition to a brief introduction to orthogonality on the real line and the unit circle, we consider orthogonal polynomials on the radial rays in the complex plane, as well as non-Hermitian orthogonal polynomials and Laurent orthogonal polynomials on the upper semicircle in the complex plane.

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Introduction

Orthogonal polynomials play a fundamental role in mathematics (approximation theory, Fourier series, numerical analysis, special functions, probability theory and statistics, ...), but also in many other computational and applied sciences, physics, chemistry, engineering, etc. For example, they are the basic tool in quadrature formulas of highest degree of precision.

Preliminaries

The standard type of orthogonal polynomials on the real line $\mathbb{R}$ are defined by the inner product

$$(p,q) = \int_a^b p(x)q(x)w(x)\mathrm{d}x \quad (p,q \in \mathcal{P}), \tag{1}$$

where $(a,b) \subset \mathbb{R}$, w is a given weight function, and $\mathcal{P}$ is the space of all algebraic polynomials. By the way, with $\mathcal{P}_n$ we denote the subspace of all algebraic polynomials of degree at most n, while $\{\pi_k\}$ denotes the system of the *monic orthogonal polynomials* with respect to the inner product (1). Because of $(xp,q) = (p,xq)$, the monic orthogonal polynomials on $\mathbb{R}$ satisfy the three–term recurrence relation of the form

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0,1,\ldots, \tag{2}$$

with $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$. All zeros of π_n are real, mutually distinct, and lie in (a,b).

There are several generalizations of these standard polynomials on $\mathbb{R}$: s-orthogonal and σ-orthogonal polynomials, several types of Sobolev type orthogonal polynomials, multiple orthogonal polynomials, triginometric orthogonal systems, generalized exponential orthogonal systems, orthogonal rational functions, Müntz orthogonal polynomials, etc.

Orthogonal polynomials in the complex plane

Orthogonal on the unit circle, when

$$(p,q)=\int_{-\pi}^{\pi} p(\mathrm{e}^{\mathrm{i}\theta})\overline{q(\mathrm{e}^{\mathrm{i}\theta})}w(\theta)\,\mathrm{d}\theta \quad (p,q\in\mathcal{P}),$$

were introduced by Szegő (1920, 1921). This inner product has not the property $(zp,q)=(p,zq)$, so that the three-term recurrence relation like (2) does not exist! But, $(zp,zq)=(p,q)$, which is important in proving that all zeros of the corresponding orthogonal polynomials $\phi_k(z)$ are inside the unit circle $|z|=1$. Orthogonal polynomials on a rectifiable curve or arc lying in the complex plane can be also studied (Szegő, Geronimus, etc.).

Main results

Orthogonal polynomials on the radial rays

Polynomials orthogonal on the radial rays in the complex plane was introduced in [7] (see also [6]). Let $M\in\mathbb{N}$, $a_s>0$, $s=1,\ldots,M$, $0\le\theta_1\le\cdots\le\theta_M<2\pi$, $\varepsilon_s=\mathrm{e}^{\mathrm{i}\theta_s}$, and $z_s=a_s\varepsilon_s\in\mathbb{C}$, $s=1,\ldots,M$. Some of a_s (or all) may coincide and also can be ∞. The case $M=6$ is displayed in Figure 1. The hermitian inner product

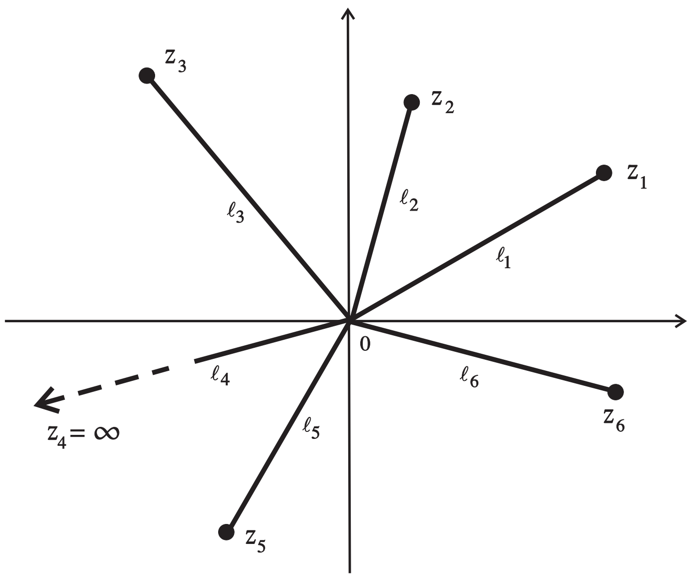

Figure 1: Radial rays $M=6$

was introduced by

$$(p,q)=\sum_{s=1}^{M}\mathrm{e}^{-\mathrm{i}\theta_s}\int_{\ell_s} p(z)\overline{q(z)}\,|w_s(z)|\mathrm{d}z \quad (p,q\in\mathcal{P}),$$

where $w_s(z)$ are suitable complex (weight) functions on these rays. Using the characteristic function of a set A, defined by $\chi(A;z)=1$ if $z\in A$, and 0 for $z\notin A$, and putting $L=\ell_1\cup\ell_2\cup\cdots\cup\ell_M$, the previous inner product can be expressed in the usual form

$$(p,q)=\int_L p(z)\overline{q(z)}\mathrm{d}\mu(z) \quad (p,q\in\mathcal{P}),$$

where the measure $\mu(z)$ is given by $\mathrm{d}\mu(z) = \sum_{s=1}^{M} \varepsilon_s^{-1}|w_s(z)|\,\chi(\ell_s; z)\mathrm{d}z$. In the symmetric case with even numbers of rays ($M = 2m$) we obtained analytic results for the recurrence coefficients for all classical weight functions (Jacobi, generalized Laguerre, Hermite).

Assuming a logarithmic potential, an electrostatic interpretation of zeros of these polynomials in the case of the generalized Gegenbauer weights was given in [8] and [9].

Non-hermitian orthogonal polynomials on the semicircle in the complex plane

In a joint paper with W. Gautschi [2] we introduced (and later generalized together with H. Landau [3]) and studied orthogonal polynomials on the semicircle with respect to *the quasi-inner product* given by

$$\langle p, q\rangle = \int_0^\pi p(\mathrm{e}^{\mathrm{i}\theta})q(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \quad (p, q \in \mathcal{P}),$$

where the second factor is not conjugated, and w is a weight function on the open interval $(-1, 1)$, with possible singularities at ± 1, and which can be extended to a holomorphic function $w(z)$ in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$. The product is *not hermitian*, but has the property $(zp, q) = (p, zq)$, so that the corresponding monic polynomials satisfy the three-terms recurrence relation like (2).

Detailed investigations of these polynomials (recurrence relation, zeros, differential equation, etc.) and some applications in numerical integration and numerical differentiation were given in several works ([1, 4, 5, 10]).

This concept of orthogonality can be treated in general with respect to a complex moment functional

$$\mathcal{L}[z^k] = \mu_k = \langle 1, z^k\rangle = \int_0^\pi \mathrm{e}^{\mathrm{i}k\theta} w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta, \quad k = 0, 1, 2, \dots . \tag{3}$$

The corresponding (monic) orthogonal polynomials π_k exist uniquely under the mild restriction $\operatorname{Re}\mu_0 = \operatorname{Re}\int_0^\pi w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \neq 0$ (see [3]).

The Laurent system of orthogonal polynomials in $\mathbb{C}$

Recently we considered the same concept of orthogonality, which includes positive and negative exponents in (3), i.e., $-n + 1 \le k \le n$ (see [10, 11]). We developed an orthogonal system $\{R_\nu(z)\}$ on the linear space $\Lambda_{-n+1,n}$, generated by the monomial basis $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$. These Laurent polynomials satisfy two three-term recurrence relations,

$$\left.\begin{aligned} R_{2k+1}(z) &= (z - a_{2k})R_{2k}(z) + b_{2k}R_{2k-1}(z), \\ R_{2k+2}(z) &= \left(1 - \frac{a_{2k+1}}{z}\right) R_{2k+1}(z) + b_{2k+1}R_{2k}(z), \end{aligned}\right\} \tag{4}$$

where $R_0(z) = 1$ and $R_{-1}(z) = 0$, and $\{a_k\}$ and $\{b_k\}$ are sequences of complex numbers depending only on the weight function $w(z)$. If we take $R_m(z) = Q_m(z)/z^{[m/2]}$, then these numerators of the Laurent orthogonal polynomials satisfy the three-term recurrence relation

$$Q_{k+1}(z) = (z - a_k)Q_k(z) + b_k z\, Q_{k-1}(z), \quad k = 0, 1, \dots,$$

with $Q_0(z) = 1$ and $Q_{-1}(z) = 0$. These coefficients a_k and b_k are given as in (4). The following characterization can be done:

Theorem 1. *The unique monic polynomials $Q_n(z)$ are characterized by the following orthogonality relations*

$$\int_0^{\pi} \mathrm{e}^{-\mathrm{i}k\theta} Q_n\big(\mathrm{e}^{\mathrm{i}\theta}\big) w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\, \mathrm{d}\theta = 0, \quad k = 0, 1, \ldots, n-1. \tag{5}$$

For the even weight function $w(-z) = w(z)$, by the uniqueness of Q_n, we conclude that $\overline{Q}_n(-z) = (-1)^n Q_n(z)$, from which we conclude that if $\tau \in \mathbb{C}$ is a zero of $Q_n(z)$, then $-\overline{\tau} \in \mathbb{C}$ is also a zero of this polynomial $Q_n(z)$. In this important case, the coefficients a_k and b_k are pure imaginary.

Several results on zero distribution for even weight functions, in particular for Legendre [10], Chebyshev [11] i Gegenbauer weights, are proved. It is interesting the Chebyshev weight of the first kind, for which

$$a_k = \mathrm{i} \quad (k \geq 0), \quad b_1 = \mathrm{i}, \quad b_k = \frac{\mathrm{i}}{2} \quad (k \geq 2).$$

The corresponding polynomials are (see [11, Proposition 4.1])

$$Q_n(z) = Q_n^T(z) = \frac{1}{2^n}\left[\left(z + \sqrt{z^2-1} - \mathrm{i}\right)^n + \left(z - \sqrt{z^2-1} - \mathrm{i}\right)^n\right]$$

for each $n \geq 0$. Here, $|z + \sqrt{z^2-1}| = r > 1$, whenever $z \in \mathbb{C} \setminus [-1, 1]$.

The zeros of $Q_n^T(z)$ are on the upper semicircle (see [11, Theorem 4.3]) $\zeta_k = \cos\varphi_k \sqrt{1 + \sin^2\varphi_k} + \mathrm{i}\sin^2\varphi_k = \mathrm{e}^{\mathrm{i}\theta_k}$, where

$$|\zeta_k| = 1, \quad \varphi_k = \frac{(2k-1)\pi}{2n}, \quad \sin\theta_k = \sin^2\varphi_k, \quad k = 1, 2, \ldots, n.$$

The zeros of the polynomial of degree $n = 20$ are displayed in Figure 2.

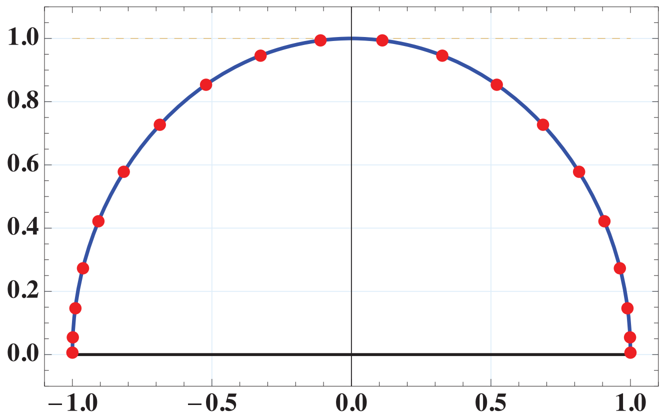

Figure 2: Zeros of $Q_{20}^T(z)$

An application of the Laurent orthogonal polynomials in constructing quadrature formulas of the highest degree of precision on the space $\Lambda_{n+1,n}$ were also considered. Such kind of quadratures can be applied for fast and efficient calculation of quasi-singular integrals, which appear in many problems in phisics and engineering.

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References

[1] F. Calio', M. Frontini and G.V. Milovanović, *Numerical differentiation of analityc functions using quadratures on the semicircle*, Comput. Math. Appl. **22**, 99–106, 1991.

[2] W. Gautschi and G.V. Milovanović, *Polynomials orthogonal on the semicircle*, J. Approx. Theory **46**, 230–250, 1986.

[3] W. Gautschi, H.J. Landau and G.V. Milovanović, *Polynomials orthogonal on the semicircle. II*, Constr. Approx. **3**, 389–404, 1987.

[4] G.V. Milovanović, *Some applications of the polynomials orthogonal on the semicircle*, In: Numerical Methods (Miskolc, 1986), pp. 625–634, Colloquia Mathematica Societatis Janos Bolyai, Vol. 50, North-Holland, Amsterdam – New York, 1987.

[5] G.V. Milovanović, *Complex orthogonality on the semicircle with respect to Gegenbauer weight: theory and applications*, In: Topics in Mathematical Analysis (Th. M. Rassias, ed.), pp. 695–722, Ser. Pure Math., 11, World Sci. Publ., Teaneck, NJ, 1989.

[6] G.V. Milovanović, *Generalized Hermite polynomials on the radial rays in the complex plane*, In: Theory of Functions and Applications, Collection of Works Dedicated to the Memory of M.M. Djrbashian (H.B. Nersessian, ed.), pp. 125–129, Yerevan, Louys Publishing House, 1995.

[7] G.V. Milovanović, *A class of orthogonal polynomials on the radial rays in the complex plane*, J. Math. Anal. Appl. **206** (1997), 121–139.

[8] G.V. Milovanović, *Orthogonal polynomials on the radial rays and an electrostatic interpretation of zeros*, Publ. Inst. Math. (Beograd) (N.S.) **64** (78) (1998), 53–68;

[9] G.V. Milovanović, *Orthogonal polynomials on the radial rays in the complex plane and applications*, Rend. Circ. Mat. Palermo, Serie II, Suppl. **6**8 (2002), 65–94.

[10] G.V. Milovanović, *Special cases of orthogonal polynomials on the semicircle and applications in numerical analysis*, Bull. Cl. Sci. Math. Nat. Sci. Math. **44**, 1–28, 2019.

[11] G.V. Milovanović, *Orthogonality on the semicircle: old and new results*, Electron. Trans. Numer. Anal. **59** 99–115, 2023.

Serbian Academy of Sciences and Arts, Kneza Mihaila 35, 11000 Belgrade, Serbia

University of Niš, Faculty of Sciences and Mathematics, 18000 Niš, Serbia

E-mail(s): gvm@mi.sanu.ac.rs; gradimir.milovanovic@mac.com