ITERATIVE APPROXIMATION OF FIXED POINTS AND APPLICATIONS TO TWO-POINT SECOND-ORDER BOUNDARY VALUE PROBLEMS AND TO MACHINE LEARNING

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Abstract. In this paper, we revisit two recently published papers on the iterative approximation of fixed points by Kumam et al. [*Numer. Funct. Anal. Optim.* **40** (2019), 1644–1677] and Maniu [*Numer. Funct. Anal. Optim.* **41** (2020), 929-949] and reproduce convergence, stability, and data dependency results presented in these papers by removing some strong restrictions imposed on parametric control sequences. We confirm the validity and applicability of our results through various non-trivial numerical examples. We suggest a new method based on the iteration algorithm given by Thakur et al. [*J. Inequal. Appl.* 2014, **328** (2014), 15 pp.] to solve the two-point second-order boundary value problems. Furthermore, based on the above mentioned iteration algorithm and S-iteration algorithm, we propose two new gradient type projection algorithms and applied them to supervised learning. In both applications, we present some numerical examples to demonstrate the superiority of the newly introduced methods in terms of convergence, accuracy, and computational time against some earlier methods.

1. Introduction and preliminaries

Throughout this paper, K is a Banach space, C is a nonempty, closed, and convex subset of K, T is a self map of C, and $F(T) = \{p : Tp = p\}$ denotes the set of fixed points of T.

To recall a mapping $T : C \to C$ is called a contraction if for all $x, y \in C$, there exists a $\delta \in [0, 1)$ such that

$$\|Tx - Ty\| \leq \delta \|x - y\| . \tag{1.1}$$

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A mapping $T : C \to C$ belongs to the class $D(a, b, c, d, e)$ if for all $x, y \in C$,

$$\begin{aligned} \|Tx - Ty\| \leq a\,\|x - y\| + b\,\|x - Tx\| + c\,\|y - Ty\| \\ + d\,\|x - Ty\| + e\,\|y - Tx\|\,, \end{aligned} \tag{1.2}$$

where a, b, c, d, e are real numbers in $[0, 1]$ satisfying certain conditions (see [1]).

The iterative approximation of fixed points plays a crucial role in finding solutions to a large number of problems encountered in various research areas. The study of fixed point iteration algorithms, which has considerable literature in the field of fixed point theory begins with the Picard iterative algorithm [2]. Subsequently, due to the failure of the Picard iterative algorithm to converge to the fixed points of non-expansive mappings, the iteration algorithms were introduced by Mann [3], Krasnoselskij [4], Noor [5], and Ishikawa [6] and their various features were studied by many researchers within the framework of different structures.

In line with the extension of application fields of iteration algorithms, many iteration algorithms have been recently introduced with the claim of faster convergence rate and their qualitative features such as convergence, rate of convergence, stability, and data dependence have been intensively examined (see [7]–[14] and references therein).

In 2007, Agarwal et al. [15] introduced an S-iterative algorithm as follows

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n)\, Tx_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)\, x_n + \beta_n T x_n, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}\,, \{\beta_n\}$ are real sequences in $(0, 1)$ for all $n \in \mathbb{N}$.

S-iterative algorithm (1.3) is independent of the Mann, Krasnoselskij, and Ishikawa iteration algorithms, and performs better than the mentioned iteration algorithms when converging to the fixed points of various classes of mappings (see [15]). For this reason, S-iteration algorithm (1.3) has been studied in detail in terms of its various features in many works.

Motivated by the performance of S-iterative algorithm and on going research in this direction, Thakur et al. [16] have designed an iteration algorithm as follows

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)\, Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\, z_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)\, x_n + \gamma_n T x_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $(0,1)$ for all $n \in \mathbb{N}$.

It has been shown in [16] that iterative algorithm (1.4) converges faster than various iteration algorithms including the Picard, Mann, Ishikawa, and S-iteration algorithms for contractions in the sense of Berinde [17].

Very recently, Kumam et al. [1] and Maniu [18] have employed S-iterative algorithm (1.3) of a mapping belonging to the class $D(a,b,c,d,e)$ and the iterative algorithm (1.4) of a contraction mapping, respectively. In these papers, some convergence, stability, and data dependence results are obtained under certain strong conditions imposed on the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $(0,1)$.

The purpose of this paper is to revisit the above mentioned papers of Kumam et al. [1] and Maniu [18] and to reproduce their convergence, stability, and data dependence results without any conditions imposed on the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $(0,1)$. We present some non-trivial numerical examples to illustrate our theoretical results. Our theoretical results can be seen as refinement and substantial improvement of the corresponding results in [1, 18, 19]. Motivated by the performance and achievements of iterative algorithm (1.4), we propose a new method based on this iterative algorithm to approximate the exact solutions of two-point second order boundary value problems. We give several test examples that will demonstrate the superiority of the newly introduced method in terms of accuracy, the number of iterations to obtain the required accuracy and convergence rate in relation to another method defined by Bello et al. [19]. As a second application, we propose two new gradient type projection algorithms based on S-iterative algorithm (1.3) and iterative algorithm (1.4) and apply them to supervised learning which is a subfield of supervised machine learning. The numerical experiments presented herein show that much better results can be achieved when compared to the traditional gradient projection algorithm by applying new gradient type projection algorithms to the machine learning.

2. Convergence Theorems

In this section, we prove that the convergence of the iterative algorithm (1.4) for the contraction mappings and S-iterative algorithm (1.3) for the mappings belonging to the class $D(a,b,c,d,e)$ are independent of any choice of the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $(0,1)$.

Maniu [18] proved the following theorem:

Theorem 1 ([18]). *Let C be a nonempty, closed, and convex subset of a Banach space K, and $T: C \to C$ be a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by* (1.4)*, with $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in* $(0,1)$*, satisfying $\sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \infty$. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Remark 1. The following inequality

$$\|x_{n+1} - p\| \leq \delta^{(n+1)} \prod_{k=0}^{n} \left[1 - (1-\delta)^2 \alpha_k \beta_k \gamma_k\right] \|x_0 - p\|, \tag{2.1}$$

where $1 - (1-\delta)^2 \alpha_k \beta_k \gamma_k < 1$ as $\delta \in (0,1)$ and $\alpha_k, \beta_k, \gamma_k \in (0,1)$ for all k, was obtained in the proof of Theorem 4.5 in [18]. Since

$$\prod_{k=0}^{n} \left[1 - (1-\delta)^2 \alpha_k \beta_k \gamma_k\right] < 1$$

for all $n \in \mathbb{N}$, the above inequality becomes

$$\|x_{n+1} - p\| \leq \delta^{(n+1)} \|x_0 - p\|. \tag{2.2}$$

By passing to the limit in (2.2), we obtain $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$. Indeed, $\delta \in (0,1)$ and $\lim_{n\to\infty} \delta^{(n+1)} = 0$.

Hence, the condition $\sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \infty$ on the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ in Theorem 1 is superfluous.

The following example shows that the convergence of the iterative algorithm (1.4) is independent of any choice of the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$.

Example 1. Let K be the Banach space $(C[0,1], \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the supremum norm on $C[0,1]$ defined by $\|\cdot\|_\infty = \{\sup|x(t)| : t \in [0,1]\}$. Consider the following first order initial value problem (IVP)

$$x'(t) = \frac{1}{2}x(t) - t, \quad x(0) = 0. \tag{2.3}$$

The existence of solutions of IVP (2.3) is equivalent to finding a fixed point of an integral operator $T : C[0,1] \to C[0,1]$ defined by

$$T(x(t)) = x(0) + \int_0^t \left[\frac{1}{2}x(\tau) - \tau\right] \mathrm{d}\tau. \tag{2.4}$$

Observe that the operator T in (2.4) is a contraction with $\delta = 1/2$ and thus, it has a unique fixed point $x_*(t) = 2t - 4\mathrm{e}^{t/2} + 4$ in $C[0,1]$. Hence, by Theorem 1 and Remark 1, the iterative algorithm (1.4) generated by T in (2.4) converges to $x_*(t)$ for the initial function $x_0 = 0$ and any choice of real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$.

We consider now the following nine cases (presented in Figure 1):

Case 1. If $\alpha_n = \dfrac{1}{n+1}$, $\beta_n = \dfrac{1}{n+1}$, $\gamma_n = \dfrac{1}{n+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$;

Case 2. If $\alpha_n = \dfrac{1}{n+1}$, $\beta_n = \dfrac{1}{n^2+1}$, $\gamma_n = \dfrac{1}{n+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$;

Case 3. If $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{\sqrt[3]{n}+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n = \infty$;

Case 4. If $\alpha_n = \dfrac{n}{n+1}$, $\beta_n = \dfrac{1}{n^3+1}$, $\gamma_n = \dfrac{1}{n^2+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$;

Case 5. If $\alpha_n = \beta_n = \dfrac{1}{n+1}$, $\gamma_n = \dfrac{1}{n^2+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$;

Case 6. If $\alpha_0 = \beta_0 = 0$, and $\alpha_n = \beta_n = \dfrac{1}{n^3}$, $\gamma_n = \dfrac{1}{3}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$;

Case 7. If $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{n^4+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$;

Case 8. If $\alpha_n = \gamma_n = \dfrac{1}{n^2+3}$, $\beta_n = \dfrac{n}{n+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$;

Case 9. If $\alpha_n = \dfrac{1}{n^2+3}$, $\beta_n = \gamma_n = \dfrac{n}{n+1}$, then $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$.

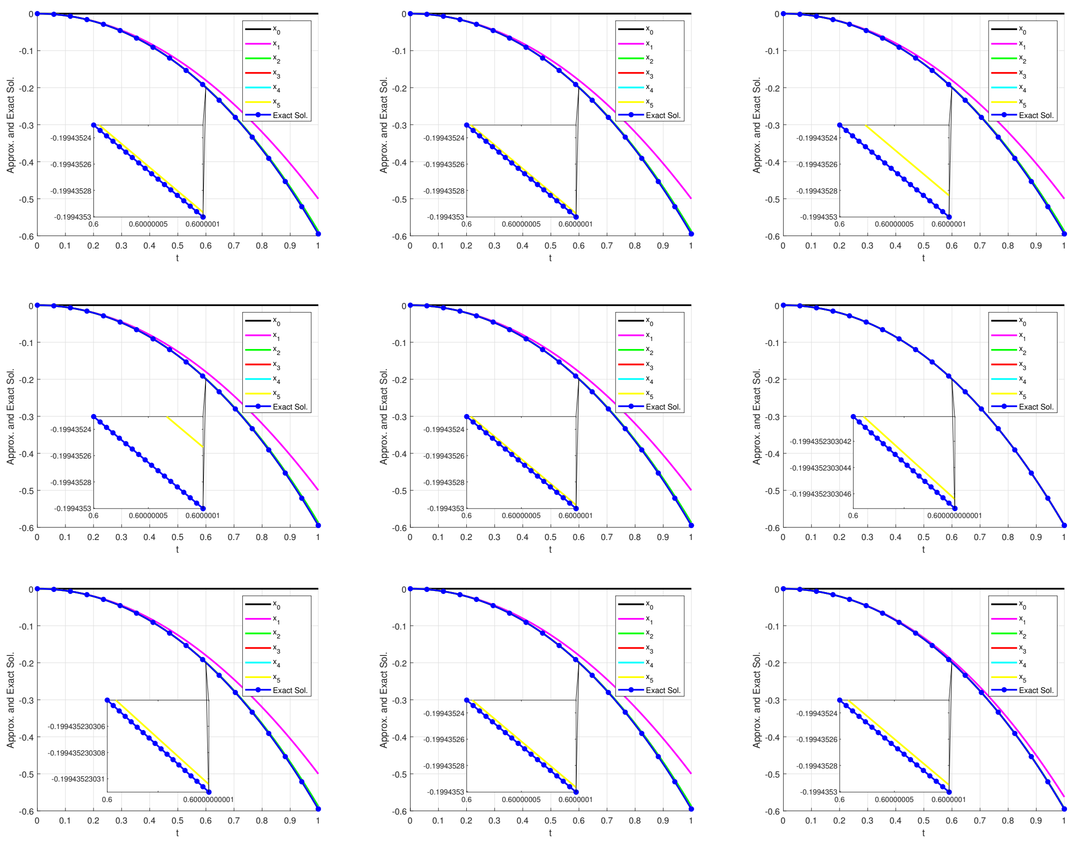

FIGURE 1. *Cases 1, 2, 3* (first line), *4, 5, 6* (second line), and *7, 8, 9* (third line) in Example 1

As we can see the iterative algorithm (1.4) generated by T in (2.4) and the initial function $x_0 = 0$ converges to $x_*(t) = 2t - 4e^{t/2} + 4$ for all the above *Cases 1–9*. The corresponding errors in these nine cases are presented in Figure 2.

Kumam et al. [1] proved the following theorem:

Theorem 2 ([1])**.** *Let C be a nonempty, closed subset of a Banach space K, and $T : C \to C$ be an operator in $D(a, b, c, d, e)$ with $0 \leq a, b, c, d, e \leq 1$, $a + 2b + c + 2d + 3e < 1$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (1.3) with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \beta_n < \infty$, converges to the unique fixed point of T, say p.*

Remark 2. The following inequality

$$\|x_{n+1} - p\| \leq \left(\frac{a + b + d + 2e}{1 - b - e}\right)\left[1 - \alpha_n\beta_n\left(1 - \frac{a + b + d + 2e}{1 - b - e}\right)\right]\|x_n - p\| \quad (2.5)$$

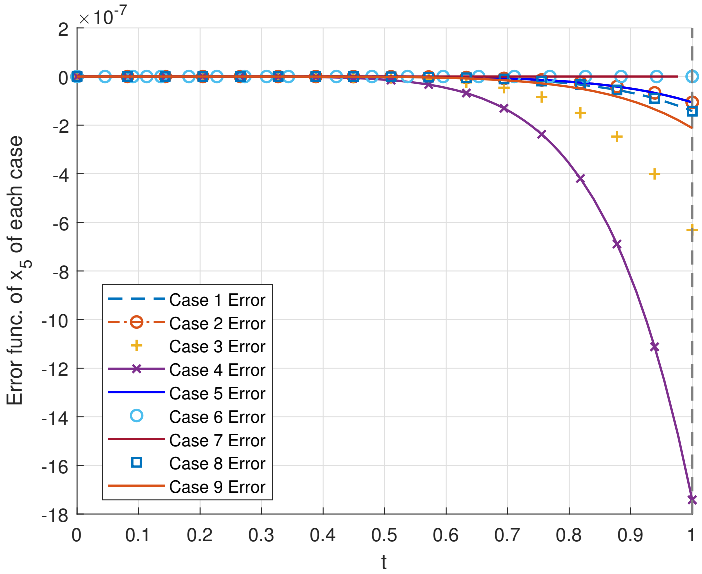

FIGURE 2. The errors in *Cases 1–9* in Example 1

where $0 \leq a, b, c, d, e \leq 1$, $a + 2b + c + 2d + 3e < 1$ and $\alpha_n, \beta_n \in (0, 1)$ for all n, was obtained in the proof of Theorem 3.1 in [1]. Since $1 - \alpha_n\beta_n \left(1 - \frac{a+b+d+2e}{1-b-e}\right) < 1$ for all $n \in \mathbb{N}$, the above inequality becomes

$$\|x_{n+1} - p\| \leq \left(\frac{a + b + d + 2e}{1 - b - e}\right) \|x_n - p\|,$$

which implies that

$$\|x_{n+1} - p\| \leq \Theta^{(n+1)} \|x_0 - p\|, \tag{2.6}$$

where $\Theta = (a + b + d + 2e)/(1 - b - e) < 1$ as $a + 2b + c + 2d + 3e < 1$. By passing to the limit in (2.6), we obtain $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.

Therefore, the conditions $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \beta_n < \infty$ on the real sequences $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ in Theorem 2 are redundant.

The following example shows that convergence of iterative algorithm (1.3) is independent of any choice of the real sequences $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.

Example 2. Let K be the Banach space $(C[0, 1], \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the supremum norm on $C[0, 1]$ defined by $\|x\|_\infty = \{\sup |x(t)| : t \in [0, 1]\}$. Define an operator

T on K by

$$T(x(t)) = \begin{cases} \gamma x(t) + \int_0^t \beta \mathrm{e}^{-\alpha(t-s)} x(s)\, \mathrm{d}s, & 0 \leq x(t) \leq A, \\ \rho x(t) + \int_0^t \beta \mathrm{e}^{-\alpha(t-s)} x(s)\, \mathrm{d}s, & A < x(t), \end{cases} \tag{2.7}$$

where $s \in [0, t]$, $\alpha > \beta > 0$, and $0 < \rho < \gamma < 1$ (see also [20]). The operator T in (2.7) belongs to the class

$$D\left(\frac{\beta}{\alpha(1-2\gamma)}, 0, \frac{\gamma}{1-2\gamma}, \frac{\gamma}{1-2\gamma}, 0\right).$$

Indeed,

Case A. If $0 \leq x(t)$, $y(t) \leq A$, then we have

$$x(t) - T(x(t)) = (1-\gamma)\, x(t) - \int_0^t \beta \mathrm{e}^{-\alpha(t-s)} x(s)\, \mathrm{d}s$$

and

$$T(x(t)) = \frac{\gamma}{1-\gamma}\left[x(t) - T(x(t)) + \frac{1}{\gamma}\int_0^t \beta \mathrm{e}^{-\alpha(t-s)} x(s)\, \mathrm{d}s\right],$$

which implies

$$\begin{aligned} \|Tx - Ty\|_\infty &\leq \frac{\gamma}{1-\gamma}\|x - Tx\|_\infty + \frac{\gamma}{1-\gamma}\|y - Ty\|_\infty \\ &\quad + \frac{1}{1-\gamma}\left(\int_0^t \beta e^{-\alpha(t-s)} \mathrm{d}s\right)\|x - y\|_\infty \\ &\leq \frac{\gamma}{1-\gamma}\|x - Tx\|_\infty + \frac{\gamma}{1-\gamma}\|y - Ty\|_\infty + \frac{\beta}{\alpha(1-\gamma)}\|x - y\|_\infty, \end{aligned}$$

which further gives

$$\|Tx - Ty\|_\infty \leq \frac{\beta}{\alpha(1-2\gamma)}\|x - y\|_\infty + \frac{\gamma}{1-2\gamma}\|y - Ty\|_\infty + \frac{\gamma}{1-2\gamma}\|x - Ty\|_\infty.$$

Hence,

$$T \in D\left(\frac{\beta}{\alpha(1-2\gamma)}, 0, \frac{\gamma}{1-2\gamma}, \frac{\gamma}{1-2\gamma}, 0\right).$$

Case B. If $0 \leq x(t) \leq A < y(t)$, then we have

$$\begin{aligned} \|Tx - Ty\|_\infty &\leq \frac{\gamma}{1-\gamma}\|x - Tx\|_\infty + \frac{\rho}{1-\rho}\|y - Ty\|_\infty \\ &\quad + \max\left\{\frac{1}{1-\gamma}, \frac{1}{1-\rho}\right\}\left(\int_0^t \beta \mathrm{e}^{-\alpha(t-s)} \mathrm{d}s\right)\|x - y\|_\infty \end{aligned}$$

$$\leq \frac{\gamma}{1-\gamma}\left\|x-Tx\right\|_\infty+\frac{\gamma}{1-\gamma}\left\|y-Ty\right\|_\infty+\frac{\beta}{\alpha\left(1-\gamma\right)}\left\|x-y\right\|_\infty,$$

as $0<\rho<\gamma<1$. Similar to *Case A*, we obtain

$$\left\|Tx-Ty\right\|_\infty \leq \frac{\beta}{\alpha\left(1-2\gamma\right)}\left\|x-y\right\|_\infty+\frac{\gamma}{1-2\gamma}\left\|y-Ty\right\|_\infty+\frac{\gamma}{1-2\gamma}\left\|x-Ty\right\|_\infty,$$

which implies that

$$T\in D\left(\frac{\beta}{\alpha\left(1-2\gamma\right)},0,\frac{\gamma}{1-2\gamma},\frac{\gamma}{1-2\gamma},0\right).$$

Case C. If $A<x(t)$, $y(t)$, then following the same lines as in *Cases A* and *B*, we obtain that

$$T\in D\left(\frac{\beta}{\alpha\left(1-2\gamma\right)},0,\frac{\gamma}{1-2\gamma},\frac{\gamma}{1-2\gamma},0\right).$$

Now, the operator T in (2.7) belongs to the class $D\left(\frac{\beta}{\alpha(1-2\gamma)},0,\frac{\gamma}{1-2\gamma},\frac{\gamma}{1-2\gamma},0\right)$ and by [1, Theorem 2.4], it has a unique fixed point, say $x_*(t)$ if $\gamma<\frac{1}{5}\left(1-\frac{\beta}{\alpha}\right)$. Hence, by Theorem 2 and Remark 2, the iterative algorithm (1.3) associated to T in (2.7) converges to $x_*(t)$ for an initial guess $x_0(t)$ and any choice of real sequences $\{\alpha_n\},\{\beta_n\}\subset(0,1)$. For instance, if we put $\alpha=2$, $\beta=1$, $\gamma=1/12$, and $\rho=1/15$ in (2.7) and consider the following choices of real sequences $\{\alpha_n\},\{\beta_n\}\subset(0,1)$:

Case 1. If $\alpha_n=\beta_n=\dfrac{2n}{2n+10}$, then we have $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\sum_{n=0}^{\infty}\beta_n=\infty$, which implies $\sum_{n=0}^{\infty}\alpha_n\beta_n=\infty$;

Case 2. If $\alpha_n=\dfrac{3n+5}{4n+7}$, $\beta_n=\dfrac{1}{n^3+5}$, then we have $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\sum_{n=0}^{\infty}\beta_n<\infty$, which implies $\sum_{n=0}^{\infty}\alpha_n\beta_n<\infty$;

Case 3. If $\alpha_n=\beta_n=\dfrac{1}{n^3+5}$, then we have $\sum_{n=0}^{\infty}\alpha_n<\infty$, $\sum_{n=0}^{\infty}\beta_n<\infty$, which implies $\sum_{n=0}^{\infty}\alpha_n\beta_n<\infty$;

Case 4. If $\alpha_n=\dfrac{1}{n^2+4}$, $\beta_n=\dfrac{n+1}{4n+1}$, then we have $\sum_{n=0}^{\infty}\alpha_n<\infty$, $\sum_{n=0}^{\infty}\beta_n=\infty$, which implies $\sum_{n=0}^{\infty}\alpha_n\beta_n<\infty$.

Then Figure 3 shows that the iterative algorithm (1.3) generated by T in (2.7), with $\alpha = 2$, $\beta = 1$, $\gamma = 1/12$, and $\rho = 1/15$ and the initial function $x_0 = t$ converges to the unique fixed point $x_*(t) = 0$ of T for all the above *Cases 1–4*. The

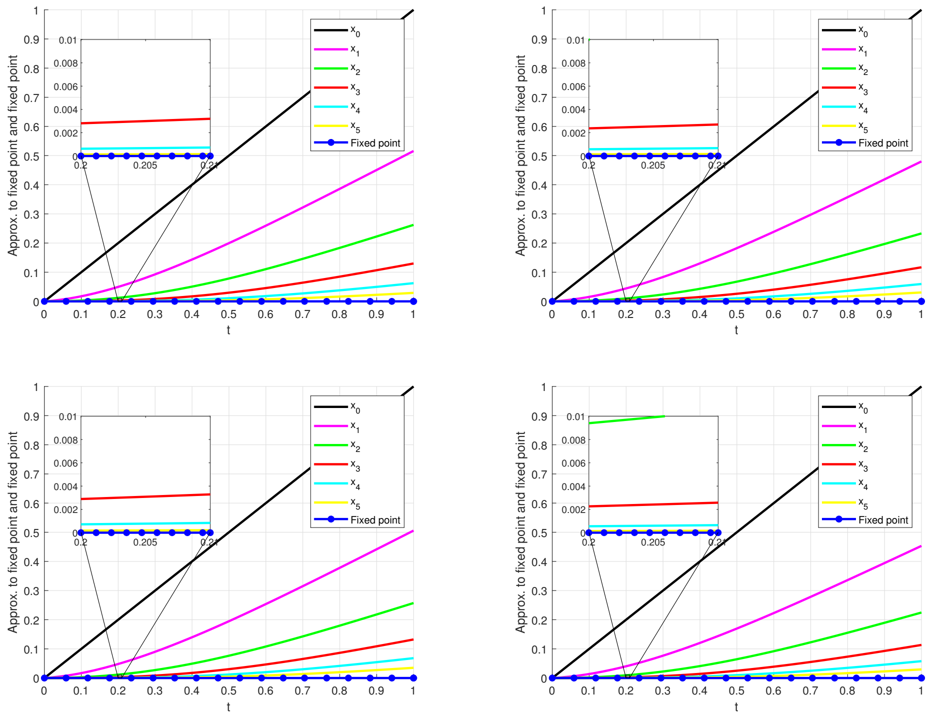

FIGURE 3. *Cases 1, 2* (first line), and *3, 4* (second line) in Example 2

corresponding errors in these four cases are presented in Figure 4.

Remark 3. Nova [20] has defined a class of discontinuous operators as follows: Let C be a subset of a Banach space K. An operator $T : C \to C$ belongs to the class $D(a, b)$ if

$$\|Tx - Ty\| \leq a\,\|x - y\| + b\,[\|x - Tx\| + \|y - Ty\|] \tag{2.8}$$

for all $x, y \in C$, and $0 \leq a, b \leq 1$. This class of operators turned out to be a versatile tool in the study of fixed points (cf. [21]–[22]).

The operators satisfying condition (2.8) was actually introduced independently by Ćirić [23], Reich [24] and Rus [25], and it was sometimes referred to under the generic name Ćirić-Reich-Rus operators. It is also claimed in [20] that the operator T

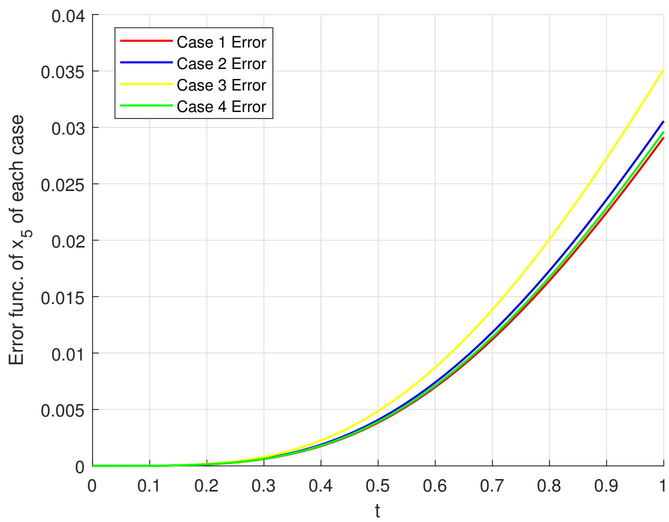

FIGURE 4. The errors in *Cases 1–4* in Example 2

defined by (2.7), where $s \in [0,t]$, $\alpha, \beta > 0$, and $0 < \rho < \gamma < 1$, is in $D\left(\frac{\beta}{\alpha(1-\gamma)}, \frac{\gamma}{1-\gamma}\right)$ and it has a unique fixed point if

$$0 < \gamma < \min\left\{\frac{1}{2}, \frac{1}{3}\left(1 - \frac{\beta}{\alpha}\right)\right\}. \tag{2.9}$$

However, we observe that the condition (2.9) must be replaced by

$$0 < \gamma < \frac{1}{3}\left(1 - \frac{\beta}{\alpha}\right),$$

as always $\frac{1}{3}\left(1 - \frac{\beta}{\alpha}\right) < \frac{1}{2}$. Otherwise, let us say $\frac{1}{3}\left(1 - \frac{\beta}{\alpha}\right) \geq \frac{1}{2}$, then we have $-\frac{1}{2} \geq \frac{\beta}{\alpha}$ which contradicts the assumption $\alpha, \beta > 0$.

3. Stability Results

In this section, we reconsider the stability result in [18] for the contraction mappings.

Let us recall some known concepts and results.

Definition 1 ([26], [27])**.** Let $T : C \to C$ be a mapping. Define an iteration algorithm by

$$x_{n+1} = f\left(T, y_n\right) \tag{3.1}$$

such that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in C. Set

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$$

for $n \geq 1$. The iterative algorithm (3.1) is said to be T-stable or stable w.r.t. T if the following condition is satisfied:

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ if and only if } \lim_{n\to\infty} y_n = p.$$

Lemma 1 ([28]). *Let θ be a real number in $[0,1]$ and $(s_n)_{n\in\mathbb{N}}$ a sequence of positive numbers such that $\lim_{n\to\infty} s_n = 0$. Then, for any sequence of positive numbers $(s_n)_{n\in\mathbb{N}}$ satisfying*

$$b_{n+1} \leq \theta b_n + s_n,$$

one has

$$\lim_{n\to\infty} b_n = 0.$$

Maniu [18] obtained the following stability result for iteration algorithm (1.4):

Theorem 3 ([18]). *Let K, C and T be as in Theorem 1. Consider the sequence generated by (1.4), $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $(0,1)$, satisfying $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n = \infty$. Then the iterative algorithm (1.4) is T-stable.*

In the following theorem we show that the stability of the iterative algorithm (1.4) w.r.t. contraction mappings is independent of any choice of sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $(0,1)$:

Theorem 4. *Let K, C and T be as in Theorem 1. Then the iterative sequence $\{x_n\}$ generated by iterative algorithm (1.4) with $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ is T-stable.*

Proof. Let $\{u_n\}$ be an arbitrary sequence in C. Define a sequence $\{\varepsilon_n\}$ in $\mathbb{R}^+$ by

$$\varepsilon_n = \|u_{n+1} - (1-\alpha_n) T u_n - \alpha_n T v_n\|, \tag{3.2}$$

where $v_n = (1-\beta_n) w_n + \beta_n T w_n$ and $w_n = (1-\gamma_n) u_n + \gamma_n T u_n$ for all $n \in \mathbb{N}$. Let $\lim_{n\to\infty} \varepsilon_n = 0$. We prove that $\lim_{n\to\infty} y_n = p$.

It follows from (1.1), (1.4), (3.2) and the fact that $1 - \alpha_n\beta_n\gamma_n (1-\delta)^2 < 1$ for all $n \in \mathbb{N}$ that

$$\|u_{n+1} - p\| \leq \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - p\|$$

$$
\begin{aligned}
&\leq \left\|u_{n+1}-(1-\alpha_n)Tu_n-\alpha_n Tv_n\right\|+(1-\alpha_n)\left\|Tx_n-Tu_n\right\| \\
&\qquad +\alpha_n\left\|Ty_n-Tv_n\right\|+\left\|x_{n+1}-p\right\| \\
&\leq \varepsilon_n+(1-\alpha_n)\,\delta\left\|x_n-u_n\right\|+\alpha_n\delta\left\|y_n-v_n\right\|+\left\|x_{n+1}-p\right\| \\
&\leq \varepsilon_n+\delta\left\{1-\alpha_n+\alpha_n\left[1-\beta_n(1-\delta)\right]\left[1-\gamma_n(1-\delta)\right]\right\}\left\|x_n-u_n\right\| \\
&\qquad +\left\|x_{n+1}-p\right\| \\
&\leq \varepsilon_n+\delta\Big\{1-\alpha_n\beta_n(1-\delta)-\alpha_n\gamma_n(1-\delta) \\
&\qquad +\alpha_n\beta_n\gamma_n(1-\delta)^2\Big\}\left\|x_n-u_n\right\|+\left\|x_{n+1}-p\right\| \\
&= \varepsilon_n+\delta\left[1-\alpha_n\beta_n\gamma_n(1-\delta)^2\right]\left\|x_n-u_n\right\|+\left\|x_{n+1}-p\right\| \\
&\leq \delta\left\|u_n-p\right\|+\varepsilon_n+\delta\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\| \qquad (3.3)
\end{aligned}
$$

for all $n \in \mathbb{N}$.

Now, we put $b_n = \|u_n - p\|$, $\theta = \delta \in [0,1)$ and

$$s_n = \varepsilon_n + \delta\left\|x_n-p\right\| + \left\|x_{n+1}-p\right\|$$

for all $n \in \mathbb{N}$.

By Theorem 1 and Remark 1, we have

$$\lim_{n\to\infty}\left\|x_n-p\right\| = \lim_{n\to\infty}\left\|x_{n+1}-p\right\| = 0.$$

Using this fact together with the assumption $\lim_{n\to\infty}\varepsilon_n = 0$, we obtain that $\lim_{n\to\infty} s_n = 0$. Now it can be easily checked that inequality (3.3) fulfills all the conditions of Lemma 1 and so by its conclusion, we have $\lim_{n\to\infty} u_n = p$.

Conversely, assume that $\lim_{n\to\infty} u_n = p$. By (1.1), (1.4) and (3.2), we obtain that

$$
\begin{aligned}
\varepsilon_n &= \left\|u_{n+1}-(1-\alpha_n)Tu_n-\alpha_n Tv_n\right\| \\
&\leq \left\|u_{n+1}-x_{n+1}\right\|+\left\|x_{n+1}-(1-\alpha_n)Tu_n-\alpha_n Tv_n\right\| \\
&\leq \left\|u_{n+1}-x_{n+1}\right\|+\left\|(1-\alpha_n)Tx_n+\alpha_n Ty_n-(1-\alpha_n)Tu_n-\alpha_n Tv_n\right\| \\
&\leq \left\|u_{n+1}-x_{n+1}\right\|+(1-\alpha_n)\left\|Tx_n-Tu_n\right\|+\alpha_n\left\|Ty_n-Tv_n\right\| \\
&\leq \left\|u_{n+1}-x_{n+1}\right\|+(1-\alpha_n)\,\delta\left\|x_n-u_n\right\|+\alpha_n\delta\left\|y_n-v_n\right\| \\
&\leq \left\|u_{n+1}-x_{n+1}\right\|+\Gamma_n\left\|x_n-u_n\right\|, \qquad (3.4)
\end{aligned}
$$

where $\Gamma_n = \delta \{1 - \alpha_n (1 - [1 - \beta_n (1 - \delta)] [1 - \gamma_n (1 - \delta)])\}$ for all $n \in \mathbb{N}$. Since $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $\delta \in [0, 1)$,

$$0 \leq \Gamma_n < 1 \quad \text{for all} \quad n \in \mathbb{N},$$

that is, Γ_n is a bounded sequence of positive numbers. On the other hand, we have

$$\begin{aligned} &0 \leq \|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - p\| + \|p - x_{n+1}\|, \\ &0 \leq \Gamma_n \|x_n - u_n\| \leq \Gamma_n \|x_n - p\| + \Gamma_n \|p - u_n\| \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.5}$$

By Theorem 1, Remark 1 and the assumption $\lim_{n\to\infty} u_n = p$, the inequalities in (3.5) lead to $\lim_{n\to\infty} \|u_{n+1} - x_{n+1}\| = \lim_{n\to\infty} \Gamma_n \|x_n - u_n\| = 0$. Now, taking the limit on both sides of (3.4) leads to $\lim_{n\to\infty} \epsilon_n = 0$. Thus $\{x_n\}$ is T-stable. □

Remark 4. (i) The technique of the proof of Theorem 4 is slightly different than that of Theorem 3.

(ii) The proof of Theorem 4 shows that the condition $\sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \infty$ on the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ in Theorem 3 is superfluous.

Example 3. Let K be the Banach space $(C^1 [0, 1], \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the supremum norm on $C^1 [0, 1]$ defined by $\|x\|_\infty = \{\sup |x(t)| : t \in [0, 1]\}$. Consider the following boundary value problem (BVP)

$$x''(t) + tx(t) = t^3 + 2, \quad t \in [0, 1] \tag{3.6}$$

subject to

$$x(0) = 0, \quad x(1) = 1. \tag{3.7}$$

The exact solution of BVP defined in (3.6)–(3.7) is given by $x(t) = t^2$. The existence of solutions of BVP (3.6)–(3.7) is equivalent to finding a continuous solution of an integral equation given by

$$T(x(t)) = x(t) + \int_0^1 G(t, s) f(s, x, x') \, \mathrm{d}s, \tag{3.8}$$

where

$$f(t, x, x') = x''(t) + tx(t) - t^3 - 2, \tag{3.9}$$

and

$$G(t, s) = \begin{cases} s(1 - t), & 0 \leq s \leq t, \\ t(1 - s), & t \leq s \leq 1. \end{cases} \tag{3.10}$$

The mapping $T : C^1[0,1] \to C[0,1]$ defined by (3.8)–(3.10) is a contraction mapping with the contractivity factor

$$\frac{1}{4\sqrt{3}} \sup_{[0,1]\times\mathbb{R}^3} \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| < 1,$$

Then, by Theorem 1 and Remark 1, the iteration algorithm (1.4) associated to T in (3.8) converges to the exact solution $x(t) = t^2$ of T in (3.8). Furthermore, by Theorem 4, the iterative algorithm (1.4) associated to T in (3.8) is $T-$stable independent of any choice of real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$. Indeed, we have the following cases:

Case 1. Let $y_n(t) = t^2 + t/(n^{10}+1)$ for all $n \in \mathbb{N}$, $t \in [0,1]$. It is easy to see that $y_n(t) \to t^2 = x(t) = T(x(t))$ as $n \to \infty$. Let $\alpha_n = 1/(n+2)$, $\beta_n = 1/(n+1)$, $\gamma_n = 1/(n+1)$ for any $n \in \mathbb{N}$, so that

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \gamma_n = \infty, \quad \sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n < \infty$$

and set $\varepsilon_n = \|y_{n+1} - F(T, y_n)\|_\infty$, where F stands for the iterative algorithm (1.4). Then, Table 1 and Figure 5 show that $\varepsilon_n \to 0$, when $n \to \infty$. Numbers in parentheses indicate the decimal exponents, e.g., 2.234(−5) means 2.234×10^{-5}.

TABLE 1. Values of $\varepsilon_n(t) = |y_{n+1} - F(T, y_n)|$ for $n = 25$ and $n = 50(50)250$

t	$\varepsilon_{25}(t)$	$\varepsilon_{50}(t)$	$\varepsilon_{100}(t)$	$\varepsilon_{150}(t)$	$\epsilon_{200}(t)$	$\epsilon_{250}(t)$
0.1	4.2750(−16)	2.6922(−19)	1.7796(−22)	2.5585(−24)	1.2881(−25)	1.2833(−26)
0.2	8.5379(−16)	5.3724(−19)	3.5476(−22)	5.0968(−24)	2.5648(−25)	2.5543(−26)
0.3	1.2757(−15)	8.0099(−19)	5.2739(−22)	7.5628(−24)	3.8008(−25)	3.7817(−26)
0.4	1.6880(−15)	1.0554(−18)	6.9086(−22)	9.8699(−24)	4.9474(−25)	4.9130(−26)
0.5	2.0833(−15)	1.2932(−18)	8.3815(−22)	1.1897(−23)	5.9360(−25)	5.8747(−26)
0.6	2.4523(−15)	1.5052(−18)	9.6028(−22)	1.3487(−23)	6.6788(−25)	6.5725(−26)
0.7	2.7833(−15)	1.6802(−18)	1.0462(−21)	1.4450(−23)	7.0685(−25)	6.8911(−26)
0.8	3.0627(−15)	1.8049(−18)	1.0830(−21)	1.4560(−23)	6.9780(−25)	6.6942(−26)
0.9	3.2749(−15)	1.8638(−18)	1.0557(−21)	1.3558(−23)	6.2609(−25)	5.8244(−26)
1.0	3.4020(−15)	1.8396(−18)	9.4713(−22)	1.1148(−23)	4.7512(−25)	4.1035(−26)

Case 2. Let $y_n(t) = nt^2/(n+1)$ for all $n \in \mathbb{N}$, $t \in [0,1]$. Observe that $y_n(t) \to t^2 = x(t) = T(x(t))$ as $n \to \infty$. Let $\alpha_n = (n+1)/(n+2)$, $\beta_n = (n+2)/(n+3)$,

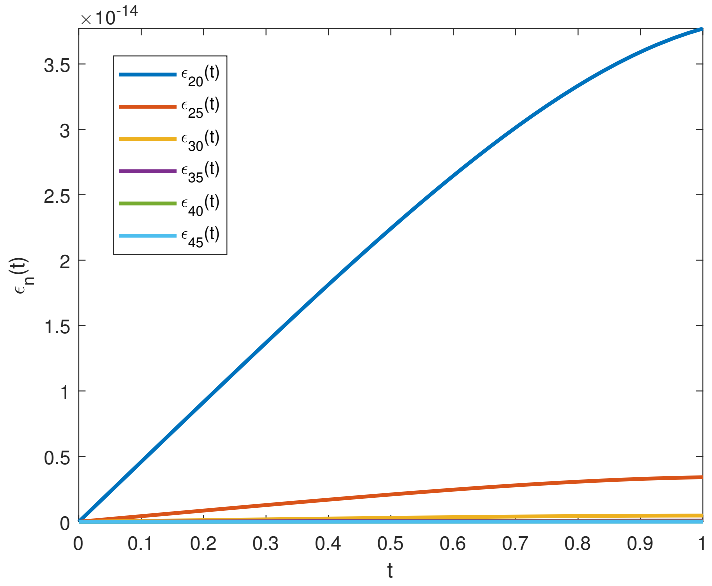

FIGURE 5. Convergence behavior of $\varepsilon_n(t) = |y_{n+1} - F(T, y_n)|$ for $n = 20(5)45$

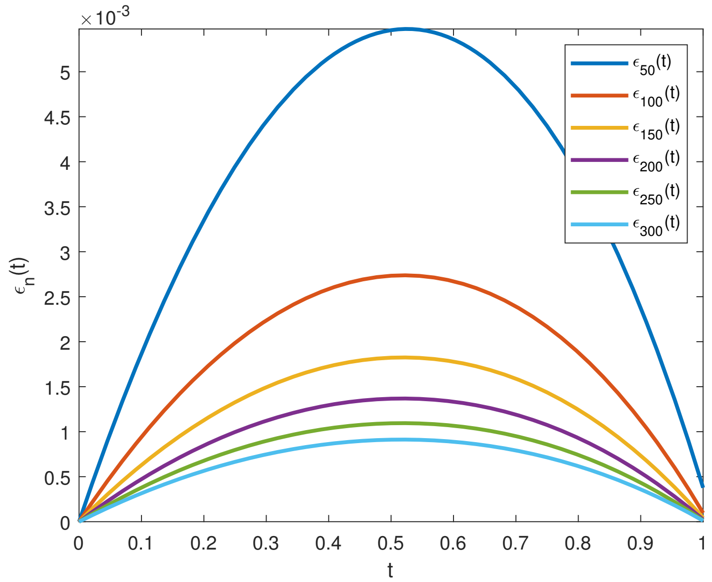

FIGURE 6. Convergence behavior of $\varepsilon_n(t) = |y_{n+1} - F(T, y_n)|$ for $n = 50(50)300$

$\gamma_n = 1/(n+1)$ for all $n \in \mathbb{N}$, so that

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \gamma_n = \infty, \quad \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \infty$$

and set $\varepsilon_n = \|y_{n+1} - F(T, y_n)\|_\infty$ where F stands for the iterative algorithm (1.4). Graphics of $t \mapsto \varepsilon_n(t) = |y_{n+1} - F(T, y_n)|$ for $t \in [0,1]$ and $n = 50(50)300$ are presented in Figure 6.

As we can see the convergence $\varepsilon_n \to 0$ in this case is very slowly regarding the previous case; for example, the values of $\varepsilon_{400}(t)$ are of order 10^{-4}.

4. Data Dependency Results

In this section, we reconsider the data dependence results in [1] and [18].

Let us recall some known concepts and results.

Definition 2 ([28])**.** Let $T, \widetilde{T} : C \to C$ be two mappings. Then $\widetilde{T}$ is said to be an approximate operator of T if there exists $\varepsilon > 0$ such that

$$\left\|Tx - \widetilde{T}x\right\| \leq \varepsilon \tag{4.1}$$

for all $x \in C$.

Maniu [18] obtained the following data dependence result:

Theorem 5 ([18])**.** *Let K, C, and T be as in Theorem 1. Let $\widetilde{T}$ be an approximate mapping of contraction mapping T, with maximum admissible error ε. Let $\{x_n\}$ be an iterative sequence generated by (1.4) and define an iterative sequence $\{\widetilde{x}_n\}$ as follows:*

$$\begin{cases} \widetilde{x}_0 \in C, \\ \widetilde{x}_{n+1} = (1-\alpha_n)\widetilde{T}\widetilde{x}_n + \alpha_n \widetilde{T}\widetilde{y}_n, \\ \widetilde{y}_n = (1-\beta_n)\widetilde{z}_n + \beta_n \widetilde{T}\widetilde{z}_n, \\ \widetilde{z}_n = (1-\gamma_n)\widetilde{x}_n + \gamma_n \widetilde{T}\widetilde{x}_n, \end{cases} \tag{4.2}$$

with real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $[0,1]$, satisfying $\alpha_n\beta_n\gamma_n \geq \frac{1}{2}$, $\beta_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. If $Tp = p$ and $\widetilde{T}\widetilde{p} = \widetilde{p}$, such that $\lim_{n\to\infty} \widetilde{x}_n = \widetilde{p}$, then

$$\|p - \widetilde{p}\| \leq \frac{7\varepsilon}{(1-\delta)^2}. \tag{4.3}$$

Remark 5. At a first glance in the hypotheses of Theorem 5, one can immediately see that the condition $\alpha_n\beta_n\gamma_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$ implies the condition $\beta_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$ as $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $(0,1)$ for all $n \in \mathbb{N}$. Hence, the condition $\beta_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 5 is redundant.

We now improve Theorem 5 as follows.

Theorem 6. *Let K, C, and T be as in Theorem 1 and $\widetilde{T}$ be an approximate mapping of T, with maximum admissible error $\varepsilon > 0$. Let $\{x_n\}_{n=0}^{\infty}$ and $\{\widetilde{x}_n\}_{n=0}^{\infty}$ be the iterative sequences generated by (1.4) and (4.2) with real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $(0,1)$ for all $n \in \mathbb{N}$, respectively. If $Tp = p$ and $\widetilde{T}\widetilde{p} = \widetilde{p}$, such that $\lim_{n\to\infty} \widetilde{x}_n = \widetilde{p}$, then*

$$\|p - \widetilde{p}\| \leq \left(\frac{1+2\delta}{1-\delta}\right)\varepsilon. \tag{4.4}$$

Proof. It follows from (1.1), (1.4), and (4.2) that

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}_{n+1}\| &= \left\|(1-\alpha_n)Tx_n + \alpha_n T y_n - (1-\alpha_n)\widetilde{T}\widetilde{x}_n - \alpha_n \widetilde{T}\widetilde{y}_n\right\| \\
&\leq (1-\alpha_n)\left\|Tx_n - \widetilde{T}\widetilde{x}_n\right\| + \alpha_n\left\|Ty_n - \widetilde{T}\widetilde{y}_n\right\| \\
&\leq (1-\alpha_n)\|Tx_n - T\widetilde{x}_n\| + (1-\alpha_n)\left\|T\widetilde{x}_n - \widetilde{T}\widetilde{x}_n\right\| \\
&\quad + \alpha_n\|Ty_n - T\widetilde{y}_n\| + \alpha_n\left\|T\widetilde{y}_n - \widetilde{T}\widetilde{y}_n\right\| \\
&\leq (1-\alpha_n)\delta\|x_n - \widetilde{x}_n\| + (1-\alpha_n)\varepsilon + \alpha_n\varepsilon + \alpha_n\delta\|y_n - \widetilde{y}_n\| \\
&\leq (1-\alpha_n)\delta\|x_n - \widetilde{x}_n\| + (1-\alpha_n)\varepsilon + \alpha_n\varepsilon \\
&\quad + \alpha_n\delta\{(1-\beta_n(1-\delta))\|z_n - \widetilde{z}_n\| + \beta_n\varepsilon\} \\
&\leq (1-\alpha_n)\delta\|x_n - \widetilde{x}_n\| + (1-\alpha_n)\varepsilon + \alpha_n\varepsilon \\
&\quad + \alpha_n\delta\{(1-\beta_n(1-\delta))\{(1-\gamma_n(1-\delta))\|x_n - \widetilde{x}_n\| + \gamma_n\varepsilon\} + \beta_n\varepsilon\} \\
&= \delta\{1-\alpha_n + \alpha_n(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))\}\|x_n - \widetilde{x}_n\| \\
&\quad + \{1-\alpha_n+\alpha_n+\alpha_n\delta(1-\beta_n(1-\delta))\gamma_n + \alpha_n\delta\beta_n\}\varepsilon \\
&\leq \delta[1-\alpha_n\beta_n(1-\delta)]\|x_n - \widetilde{x}_n\| \\
&\quad + \{1+\alpha_n\delta(1-\beta_n(1-\delta))\gamma_n + \alpha_n\delta\beta_n\}\varepsilon.
\end{aligned} \tag{4.5}$$

Since $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ in $(0,1)$ for all $n \in \mathbb{N}$ and $\delta \in [0,1)$,

$$1 - \beta_n(1-\delta) < 1,$$

$$1-\alpha_n\beta_n\left(1-\delta\right)<1 \quad \text{for all } n\in\mathbb{N}. \tag{4.6}$$

Applying inequalities in (4.6) to (4.5), we obtain that

$$\left\|x_{n+1}-\widetilde{x}_{n+1}\right\|\leq\delta\left\|x_n-\widetilde{x}_n\right\|+\left(1+2\delta\right)\varepsilon \quad \text{for all } n\in\mathbb{N}. \tag{4.7}$$

By Theorem 1 and Remark 1, we have $\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}x_n=p$. By passing to the limit in (4.7) and then using the facts $\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}x_n=p$ and the continuity of the norm, as well as the assumption $\lim_{n\to\infty}\widetilde{x}_{n+1}=\lim_{n\to\infty}\widetilde{x}_n=\widetilde{p}$, we get that

$$\lim_{n\to\infty}\left\|x_{n+1}-\widetilde{x}_{n+1}\right\|\leq\delta\lim_{n\to\infty}\left\|x_n-\widetilde{x}_n\right\|+\left(1+2\delta\right)\varepsilon,$$

which implies that

$$\left\|p-\widetilde{p}\right\|\leq\left(\frac{1+2\delta}{1-\delta}\right)\varepsilon.$$

□

Remark 6. (i) The technique of the proof of Theorem 6 is slightly different than that of Theorem 5.

(ii) The proof of Theorem 6 shows that the conditions $\alpha_n\beta_n\gamma_n\geq\frac{1}{2},\beta_n\geq\frac{1}{2}$ for all $n\in\mathbb{N}$ on the real sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1)$ in Theorem 5 are superfluous.

(iii) Since $\delta\in[0,1)$ and $\varepsilon>0$,

$$\left(1+2\delta\right)\left(1-\delta\right)\leq\frac{9}{8}<7,$$

which implies that

$$\left(\frac{1+2\delta}{1-\delta}\right)\varepsilon<\frac{7\epsilon}{\left(1-\delta\right)^2}.$$

Hence, the estimate

$$\left\|p-\widetilde{p}\right\|\leq\left(\frac{1+2\delta}{1-\delta}\right)\varepsilon$$

obtained in Theorem 6 is sharper than the estimate $\left\|p-\widetilde{p}\right\|\leq 7\varepsilon/\left(1-\delta\right)^2$ in Theorem 5.

(iv) By imposing an additional condition $\lim_{n\to\infty}\alpha_n=0$ on the real sequence $\{\alpha_n\}\subset(0,1)$ in Theorem 6, a much better estimate for upper bound for the error in approximating $\widetilde{p}$ by p can be obtained as follows:

$$\left\|p-\widetilde{p}\right\|\leq\frac{\varepsilon}{1-\delta}.$$

Example 4. Let K be the Banach space $(C^1[0,1], \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the supremum norm on $C^1[0,1]$ defined by $\|x\|_\infty = \{\sup|x(t)| : t \in [0,1]\}$. Consider BVP (3.6)–(3.7) given in Example 3 and the following BVP

$$x''(t) + tx(t) = t^3 + 2 + \frac{t}{k}, \quad t \in [0,1], \tag{4.8}$$

subject to

$$x(0) = 0, \ x(1) = 1. \tag{4.9}$$

By Example 3, we know that the exact solution of BVP (3.6)–(3.7) is the function $x_*(t) = t^2$, the existence of solutions of BVP (3.6)–(3.7) is equivalent to finding a continuous solution of the integral operator $T : C^1[0,1] \to C^1[0,1]$ defined by (3.8)–(3.10) which is a contraction mapping with the contractivity factor

$$\delta = \frac{1}{4\sqrt{3}} \sup_{[0,1]\times\mathbb{R}^3} \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| = \frac{1}{4\sqrt{3}} < 1.$$

On the other hand, the existence of solutions of BVP (4.8)–(4.9) is equivalent to finding a continuous solution of an integral operator $\widetilde{T} : C^1[0,1] \to C^1[0,1]$ defined by

$$\widetilde{T}(x(t)) = x(t) + \int_0^1 G(t,s)\,\widetilde{f}(s,x,x')\,\mathrm{d}s, \tag{4.10}$$

where

$$\widetilde{f}(t,x,x') = x''(t) + tx(t) - t^3 - 2 - \frac{t}{k} \tag{4.11}$$

and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t, \\ t(1-s), & t \le s \le 1. \end{cases} \tag{4.12}$$

Observe that $\widetilde{T} \to T$ as $k \to \infty$. For $k = 1$, we have

$$\begin{aligned} \|Tx - \widetilde{T}x\|_\infty &= \left\|\int_0^1 G(t,s)\,f(s,x,x')\,\mathrm{d}s - \int_0^1 G(t,s)\,\widetilde{f}(s,x,x')\,\mathrm{d}s\right\|_\infty \\ &= \left\|\int_0^t s(1-t)s\,\mathrm{d}s + \int_t^1 t(1-s)s\,\mathrm{d}s\right\|_\infty \\ &= \left\|\frac{t(2t+1)(t-1)^2}{6} - \frac{t^3(t-1)}{3}\right\|_\infty \\ &\cong 0.06415, \end{aligned}$$

which implies

$$\left\| T\left(x(t)\right) - \widetilde{T}\left(x\left(t\right)\right) \right\|_{\infty} \leq \varepsilon \quad \text{for all } t \in [0,1],$$

where $\varepsilon = 0.066$. Thus, we consider the operator $\widetilde{T}$ as an approximate operator of T in the sense of Definition 2. Let $\widetilde{x}_*(t)$ be an exact solution of BVP (4.8)–(4.9).

Consider the following cases for the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$:

Case 1. Let

$$\alpha_n = \frac{1}{n+2}, \quad \beta_n = \frac{1}{n+1}, \quad \gamma_n = \frac{1}{n+1}$$

for all $n \in \mathbb{N}$, so that

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \gamma_n = \infty, \quad \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n < \infty.$$

Case 2. Let

$$\alpha_n = \frac{n+1}{n+2}, \quad \beta_n = \frac{n+2}{n+3}, \quad \gamma_n = \frac{1}{n+1}$$

for all $n \in \mathbb{N}$, so that

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \gamma_n = \infty, \quad \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \infty.$$

For *Cases 1–2*, the iterative algorithm (1.4) associated to the operator T in (3.8)–(3.10) and the iterative algorithm (4.2) of the operator $\widetilde{T}$ in (4.10)–(4.12) with initial functions $x_0 = \widetilde{x}_0 = 0$, converges to $x_*(t)$ and $\widetilde{x}_*(t)$, respectively, as shown in Table 2 and Figure 7.

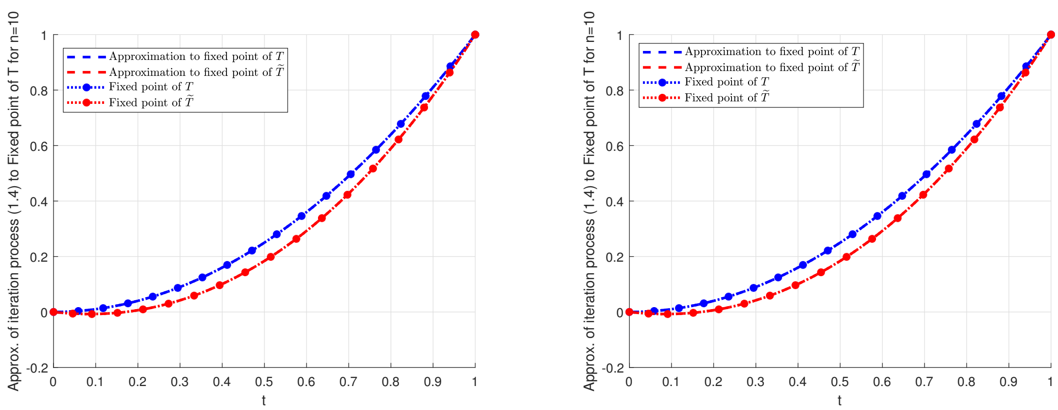

FIGURE 7. *Case 1* (left) and *Case 2* (right)

TABLE 2. The difference of $x_{10}-\widetilde{x}_{10}$, where x_{10} and $\widetilde{x}_{10}$ are 10th step of sequences generated by the iteration algorithm (1.4) of T and the iterative algorithm (4.2) of $\widetilde{T}$ with initial functions $x_0 = \widetilde{x}_0 = 0$, respectively.

	Case 1		*Case 2*	
t	$x_{10}-x_*$	$\widetilde{x}_{10}-\widetilde{x}_*$	$x_{10}-x_*$	$\widetilde{x}_{10}-\widetilde{x}_*$
0.1	−1.0137(−22)	−1.2966(−22)	−1.2127(−16)	−1.5513(−16)
0.2	−2.0050(−22)	−2.5647(−22)	−2.3987(−16)	−3.0683(−16)
0.3	−2.9175(−22)	−3.7319(−22)	−3.4904(−16)	−4.4648(−16)
0.4	−3.6622(−22)	−4.6845(−22)	−4.3814(−16)	−5.6045(−16)
0.5	−4.1291(−22)	−5.2817(−22)	−4.9400(−16)	−6.3189(−16)
0.6	−4.2068(−22)	−5.3811(−22)	−5.0329(−16)	−6.4378(−16)
0.7	−3.8110(−22)	−4.8748(−22)	−4.5594(−16)	−5.8321(−16)
0.8	−2.9171(−22)	−3.7314(−22)	−3.4900(−16)	−4.4642(−16)
0.9	−1.5899(−22)	−2.0338(−22)	−1.9022(−16)	−2.4332(−16)

Consequently, $\|x_*(t)-\widetilde{x}_*(t)\|_\infty = 0.06781805245\ldots$. As a matter of fact, without knowing the fixed point of the operator $\widetilde{T}$, that is, the exact solution of BVP (4.8)–(4.9) and without computing it, we can find the following upper bound for the error in approximating $\widetilde{x}_*(t)$ by $x_*(t)$ by using the result of Theorem 6:

$$\|x_*(t)-\widetilde{x}_*(t)\|_\infty \le \left(\frac{1+2\delta}{1-\delta}\right)\varepsilon = \left(\frac{1+\frac{1}{2\sqrt{3}}}{1-\frac{1}{4\sqrt{3}}}\right)0.066 = 0.0993997. \tag{4.13}$$

The following estimate has been obtained in [18, Theorem 4.7]:

$$\|x_*(t)-\widetilde{x}_*(t)\|_\infty \le \frac{7\varepsilon}{(1-\delta)^2}. \tag{4.14}$$

Now, using the above arguments, inequality (4.14) leads to

$$\|x_*(t)-\widetilde{x}_*(t)\|_\infty \le \frac{7\times 0.066}{\left(1-\frac{1}{4\sqrt{3}}\right)^2} = 0.631011. \tag{4.15}$$

Thus, from (4.13) and (4.15), we conclude that the estimate given by Theorem 6 is better than the estimate (4.14).

Kumam et al. [1] obtained the following data dependence result:

Theorem 7 ([1]). *Let $T: C\to C$ be as in Theorem 2 with a unique fixed point p and $\widetilde{T}: C\to C$ an approximate mapping of T. Let $\{x_n\}_{n=0}^\infty$ be the S-iterative*

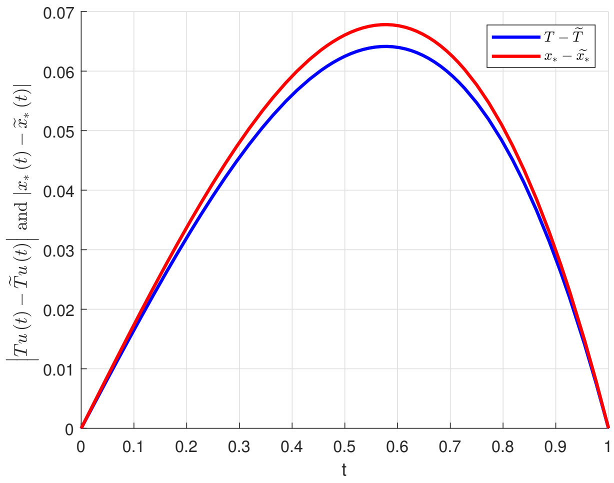

FIGURE 8. The graphics of $\left|Tu\left(t\right)-\widetilde{T}u\left(t\right)\right|$ and $\left|x_*\left(t\right)-\widetilde{x}_*\left(t\right)\right|$.

scheme (1.3) *with* $(1-c-d)/(2-a-3c-4d-e)\leq\alpha_n\beta_n$ *for all* $n\in\mathbb{N}$ *and define an iterative sequence* $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *generated by*

$$\begin{cases}\widetilde{x}_0\in C,\\ \widetilde{x}_{n+1}=(1-\alpha_n)\widetilde{T}\widetilde{x}_n+\alpha_n\widetilde{T}\widetilde{y}_n,\\ \widetilde{y}_n=(1-\beta_n)\widetilde{x}_n+\beta_n\widetilde{T}\widetilde{x}_n,\end{cases}\tag{4.16}$$

for all $n\in\mathbb{N}$.

If $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *converges to* $\widetilde{p}=\widetilde{T}\widetilde{p}$, *then*

$$\|p-\widetilde{p}\|\leq\frac{2\left(1-c-d\right)\varepsilon}{1-\left(a+2c+3d+e\right)}.$$

In the following, we have obtained that the data dependence result which is independent of any choice of sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$ for discontinuous operator by using S-iterative algorithm (1.3):

Theorem 8. *Let* K *and* C *be as in Theorem* 2, $T:C\to C$ *be a mapping satisfying* (1.2) *with* $0\leq a,b,c,d,e\leq1$, $a+2b+2c+3d+3e<1$ *and* $\widetilde{T}:C\to C$ *an approximate mapping of* T, *with maximum admissible error* ε. *Let* $\{x_n\}_{n=0}^{\infty}$ *and* $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *be two iterative sequences defined by* S*-iterative algorithm* (1.3) *and the iterative algorithm* (4.16) *with real sequences* $\{\alpha_n\},\{\beta_n\}$ *in* $(0,1)$, *respectively. If* $Tp=p$ *and* $\widetilde{T}\widetilde{p}=\widetilde{p}$, *such that* $\lim_{n\to\infty}\widetilde{x}_n = \widetilde{p}$, *then*

$$\|p-\widetilde{p}\| \le \left(\frac{1+\theta}{1-\theta}\right)\varepsilon,$$

where $\theta = (a+c+2d+e)/(1-c-d)$.

Proof. Theorem 2.4 from [1] guarantees existence of a unique fixed point $p = Tp$. Using (1.2), (1.3), (4.16) and following the same lines as those in the proof from [1, Theorem 4.1], we get

$$\begin{aligned}
\|x_{n+1}-\widetilde{x}_{n+1}\| &\le (1-\alpha_n)\left\|Tx_n-\widetilde{T}\widetilde{x}_n\right\| + \alpha_n\left\|Ty_n-\widetilde{T}\widetilde{y}_n\right\| \\
&\le (1-\alpha_n)(a+d+e)\|x_n-\widetilde{x}_n\| + (1-\alpha_n)(b+e)\|x_n-Tx_n\| \\
&\quad + (1-\alpha_n)(c+d)\|\widetilde{x}_n-T\widetilde{x}_n\| + (1-\alpha_n)\varepsilon \\
&\quad + \alpha_n(a+d+e)\|y_n-\widetilde{y}_n\| + \alpha_n(b+e)\|y_n-Ty_n\| \\
&\quad + \alpha_n(c+d)\|\widetilde{y}_n-T\widetilde{y}_n\| + \alpha_n\varepsilon \\
&\le \theta\left[1-\alpha_n\beta_n(1-\theta)\right]\|x_n-\widetilde{x}_n\| + \Gamma_n\|x_n-p\| + (1+\alpha_n\beta_n\theta)\varepsilon,
\end{aligned} \tag{4.17}$$

for all $n \in \mathbb{N}$, where

$$\theta = \frac{a+c+2d+e}{1-c-d}, \quad \lambda = \frac{a+b+d+2e}{1-b-e},$$

$$\Gamma_n = \left\{1-\alpha_n(1-\beta_n\theta) + \alpha_n\left[1-\beta_n(1-\lambda)\right]\right\}\tau$$

for all $n \in \mathbb{N}$, and

$$\tau = (1+a+d+e)\left(\frac{b+e}{1-b-e} + \frac{c+d}{1-c-d}\right).$$

Since $a+2b+2c+3d+3e<1$ and $\alpha_n, \beta_n \in (0,1)$ for all $n \in \mathbb{N}$,

$$\theta < 1, \quad \lambda < 1, \quad \tau < 4 \tag{4.18}$$

and

$$1-\alpha_n\beta_n(1-\theta) < 1, \quad 1-\alpha_n(1-\beta_n\theta) < 1, \quad \alpha_n\left[1-\beta_n(1-\lambda)\right] < 1 \tag{4.19}$$

for all $n \in \mathbb{N}$.

Obviously, inequalities in (4.18) and (4.19) imply that Γ_n is a bounded sequence of positive numbers. Since $\alpha_n, \beta_n \in [0,1]$ and $1 - \alpha_n\beta_n(1-\theta) < 1$ for all $n \in \mathbb{N}$, then inequality in (4.17) becomes

$$\|x_{n+1} - \widetilde{x}_{n+1}\| \leq \theta \|x_n - \widetilde{x}_n\| + \Gamma_n \|x_n - p\| + (1+\theta)\varepsilon \tag{4.20}$$

for all $n \in \mathbb{N}$.

By Theorem 2 and Remark 2, we have $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = p$. By passing to the limit in (4.7) and then using the facts that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = p$ and the continuity of the norm, as well as the assumption $\lim_{n\to\infty} \widetilde{x}_{n+1} = \lim_{n\to\infty} \widetilde{x}_n = \widetilde{p}$, we get that

$$\lim_{n\to\infty} \|x_{n+1} - \widetilde{x}_{n+1}\| \leq \theta \lim_{n\to\infty} \|x_n - \widetilde{x}_n\| + \lim_{n\to\infty} \Gamma_n \|x_n - p\| + (1+\theta)\varepsilon,$$

which gives

$$\|p - \widetilde{p}\| \leq \left(\frac{1+\theta}{1-\theta}\right)\varepsilon.$$

□

Remark 7. (i) The technique of the proof of Theorem 8 is slightly different than that of Theorem 7.

(ii) The proof of Theorem 8 shows that the condition

$$\frac{1-c-d}{2-a-3c-4d-e} \leq \alpha_n\beta_n$$

for all $n \in \mathbb{N}$ on the real sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ in Theorem 7 is redundant.

(iii) From the assumption $a + 2b + 2c + 3d + 3e < 1$ in Theorem 8, we have

$$0 < 1 - (a + 2c + 3d + e), \tag{4.21}$$

and

$$\begin{aligned} a + 2c + 3d + e &< 1 \\ &\Rightarrow 1 + a + 2c + 3d + e < 2 \\ &\Rightarrow 1 + a + d + e < 2(1-c-d). \end{aligned} \tag{4.22}$$

By (4.21) and (4.22), we get that

$$\frac{1+a+d+e}{1-(a+2c+3d+e)} < \frac{2(1-c-d)}{1-(a+2c+3d+e)},$$

which leads to

$$\left(\frac{1+\theta}{1-\theta}\right)\varepsilon < \frac{2\left(1-c-d\right)\varepsilon}{1-\left(a+2c+3d+e\right)},$$

where $\theta = (a+c+2d+e)/(1-c-d)$. Hence, the estimate

$$\|p-\widetilde{p}\| \leq \left(\frac{1+\theta}{1-\theta}\right)\varepsilon$$

obtained in Theorem 6 is sharper than the estimate

$$\|p-\widetilde{p}\| \leq \frac{2\left(1-c-d\right)\varepsilon}{1-\left(a+2c+3d+e\right)}$$

in Theorem 7.

(iv) By imposing an additional condition $\lim_{n\to\infty}\alpha_n = 0$ on the real sequence $\{\alpha_n\} \subset (0,1)$ in Theorem 8, a much better estimate for upper bound for the error in approximating $\widetilde{p}$ by p can be obtained as follows:

$$\|p-\widetilde{p}\| \leq \frac{\varepsilon}{1-\theta},$$

where $\theta = (a+c+2d+e)/(1-c-d)$.

Example 5. Let X be the Banach space $(C\left[0,1\right], \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the supremum norm on $C\left[0,1\right]$ defined by $\|x\|_\infty = \{\sup|x\left(t\right)| : t \in [0,1]\}$. Define two operators $T, \widetilde{T} : C\left[0,1\right] \to C\left[0,1\right]$ by

$$T\left(x(t)\right) = \begin{cases} \dfrac{1}{20}x(t) + \displaystyle\int_0^t \mathrm{e}^{-2(t-s)}x\left(s\right)\mathrm{d}s, & 0 \leq x(t) \leq 1, \\ \dfrac{1}{25}x(t) + \displaystyle\int_0^t \mathrm{e}^{-2(t-s)}x\left(s\right)\mathrm{d}s, & 1 < x(t), \end{cases} \tag{4.23}$$

and

$$\widetilde{T}(x(t)) = \begin{cases} \dfrac{1}{20}\left(x(t) + \dfrac{t}{100}\right) + \displaystyle\int_0^t \mathrm{e}^{-2(t-s)}x\left(s\right)\mathrm{d}s, & 0 \leq x(t) \leq 1, \\ \dfrac{1}{25}\left(x(t) + \dfrac{t}{100}\right) + \displaystyle\int_0^t \mathrm{e}^{-2(t-s)}x\left(s\right)\mathrm{d}s, & 1 < x(t), \end{cases} \tag{4.24}$$

respectively. By Example 2, if we let $\gamma = 1/20$, $\rho = 1/25$, $\alpha = 2$, $\beta = 1$, then $T \in D\left(3/5,\, 0,\, 0.0\overline{5},\, 0.0\overline{5},\, 0\right)$ and the fixed point of T is $x_*(t) = 0$. Now, we have the following estimates:

(i) If $0 \leq x(t) \leq 1$, then

$$\left\|T\left(x(t)\right)-\widetilde{T}\left(x(t)\right)\right\|_{\infty}=\left\|\frac{t}{2000}\right\|_{\infty}=0.0005;$$

(ii) If $1 < x(t)$, then

$$\left\|T\left(x(t)\right)-\widetilde{T}\left(x(t)\right)\right\|_{\infty}=\left\|\frac{t}{2500}\right\|_{\infty}=0.0004.$$

This leads to

$$\left\|T\left(x(t)\right)-\widetilde{T}\left(x(t)\right)\right\|_{\infty}\leq\varepsilon \quad \text{for all } t\in[0,1],$$

where $\varepsilon = 0.0005$. Thus, we consider the operator $\widetilde{T}$ as an approximate operator of T in the sense of Definition 2. Let $\widetilde{x}_*(t)$ be a fixed point of $\widetilde{T}$. Consider the following cases for the real sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$:

Case 1. Let $\alpha_n = 1/(n+2)$ and $\beta_n = 1/(n+1)$ for all $n \in \mathbb{N}$, so that

$$\sum_{n=0}^{\infty}\alpha_n=\infty,\quad \sum_{n=0}^{\infty}\beta_n=\infty,\quad \sum_{n=0}^{\infty}\alpha_n\beta_n<\infty.$$

Case 2. Let $\alpha_n = (n+1)/(n+2)$ and $\beta_n = 1/(n+1)$ for all $n \in \mathbb{N}$, so that

$$\sum_{n=0}^{\infty}\alpha_n=\infty,\quad \sum_{n=0}^{\infty}\beta_n=\infty,\quad \sum_{n=0}^{\infty}\alpha_n\beta_n=\infty.$$

For *Cases 1–2*, the iterative algorithm (1.3) associated to the operator T in (4.23) and the iterative algorithm (4.16) of the operator $\widetilde{T}$ in (4.24) with initial functions $x_0(t) = \widetilde{x}_0(t) = t$, converges to $x_*(t)$ and $\widetilde{x}_*(t)$, respectively, as shown in Table 3 and Figure 9.

Consequently, $\|x_*(t) - \widetilde{x}_*(t)\|_\infty = 0.0009658....$ As a matter of fact, without knowing the fixed point of the operator $\widetilde{T}$ and without computing it, we can find the following upper bound for the error in approximating $\widetilde{x}_*(t)$ by $x_*(t)$ by using the conclusion of Theorem 6:

$$\|x_*(t)-\widetilde{x}_*(t)\|_\infty \leq \left(\frac{1+\theta}{1-\theta}\right)\varepsilon = 0.0038\overline{6}, \tag{4.25}$$

where $\theta = (a + c + 2d + e)/(1 - c - d) = 0.8125$.

The following estimate has been obtained in [1, Theorem 4.1]:

$$\|x_*(t)-\widetilde{x}_*(t)\|_\infty \leq \frac{2\,(1-c-d)\,\varepsilon}{1-(a+2c+3d+e)}. \tag{4.26}$$

Table 3. The differences of $x_{10} - x_*$, and $\widetilde{x}_{10} - \widetilde{x}_*$ where x_{10} and $\widetilde{x}_{10}$ are 10th step of sequences generated by iteration algorithm (1.3) of T and the iterative algorithm (4.16) of $\widetilde{T}$ with initial functions $x_0 = \widetilde{x}_0 = x$, respectively.

	Case 1		*Case 2*	
t	$x_{10} - x_*$	$\widetilde{x}_{10} - \widetilde{x}_*$	$x_{10} - x_*$	$\widetilde{x}_{10} - \widetilde{x}_*$
0.1	$-4.6948(-11)$	$-3.5771(-12)$	$-6.0915(-11)$	$-5.3429(-12)$
0.2	$-6.9916(-9)$	$-1.2118(-9)$	$-7.4315(-9)$	$-1.3883(-9)$
0.3	$-1.5082(-7)$	$-3.7502(-8)$	$-1.4571(-7)$	$-3.7969(-8)$
0.4	$-1.2442(-6)$	$-3.7776(-7)$	$-1.1425(-6)$	$-3.5809(-7)$
0.5	$-5.7968(-6)$	$-1.9941(-6)$	$-5.1737(-6)$	$-1.8203(-6)$
0.6	$-1.8583(-5)$	$-6.9523(-6)$	$-1.6314(-5)$	$-6.2050(-6)$
0.7	$-4.5929(-5)$	$-1.8234(-5)$	$-3.9934(-5)$	$-1.6049(-5)$
0.8	$-9.4018(-5)$	$-3.8981(-5)$	$-8.1278(-5)$	$-3.4012(-5)$
0.9	$-1.6719(-4)$	$-7.1612(-5)$	$-1.4405(-4)$	$-6.2134(-5)$
1.0	$-2.6702(-4)$	$-1.1725(-4)$	$-2.2962(-4)$	$-1.0136(-4)$

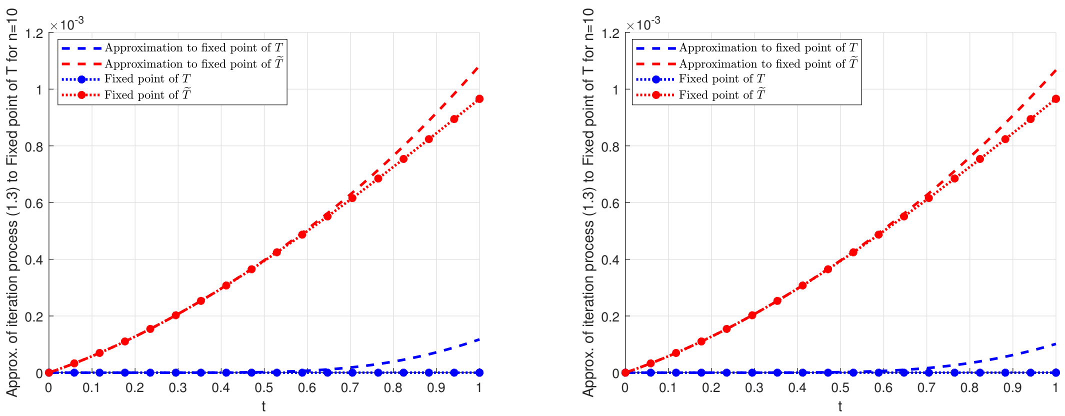

Figure 9. *Case 1* (left) and *Case 2* (right)

Now, using the above arguments, inequality (4.26) leads to

$$\|x_*(t) - \widetilde{x}_*(t)\|_\infty \leq 0.0042\overline{6}. \tag{4.27}$$

Thus, from (4.25) and (4.27), we conclude that the estimate given by Theorem 6 is better than the estimate (4.26).

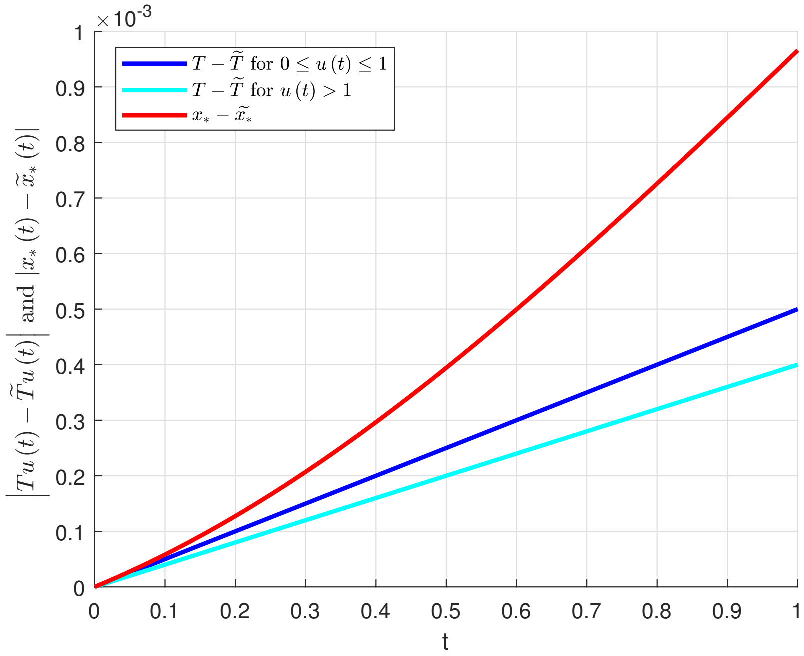

FIGURE 10. The graphs of $\left|Tu(t) - \widetilde{T}u(t)\right|$ and $|x_*(t) - \widetilde{x}_*(t)|$

5. Applications

5.1. **Application to Two-Point Second Order BVP.** In this section, we propose a new method based on the iterative algorithm (1.4) to solve two-point second order boundary value problem (in short TPSO–BVP). We present some test examples to demonstrate the superiority of our method in terms of convergence, accuracy, and computational time against another method introduced by Bello et al. [19].

Bello et al. [19] considered the following TPSO–BVP:

$$x'' = f(t, x, x'), \quad a \leq t \leq b, \tag{5.1}$$

$$\begin{cases} \lambda_0 x(a) + \beta_0 x'(a) = \gamma_0, \\ \lambda_1 x(b) + \beta_1 x'(b) = \gamma_1, \end{cases} \tag{5.2}$$

where $\lambda_i, \beta_i, \gamma_i \in \mathbb{R}$, with $\lambda_i^2 + \beta_i^2 > 0$ for $i = 0, 1$. It has been observed in [19] that $x(t)$ is a solution of TPSO–BV (5.1)–(5.2) if and only if $x(t)$ is a solution of the equivalent integral equation

$$x(t)=\int_a^b G\left(t,s\right)f\left(s,x\left(s\right),x'\left(s\right)\right)\mathrm{d}s+w(t)$$

on $[a,b]$, where

$$G\left(t,s\right)=\begin{cases}\dfrac{\left(t-a\right)\left(s-b\right)}{b-a}, & a\leq t\leq s,\\[2mm] \dfrac{\left(t-b\right)\left(s-a\right)}{b-a}, & s\leq t\leq b,\end{cases}\tag{5.3}$$

is the Green function associated to the TPSO–BVP

$$x''=0,\quad a\leq t\leq b,\tag{5.4}$$

$$\begin{cases}\lambda_0 x(a)+\beta_0 x'(a)=\gamma_0,\\ \lambda_1 x(b)+\beta_1 x'(b)=\gamma_1,\end{cases}\tag{5.5}$$

and $w(t)$ is the solution of TPSO-BVP (5.4)–(5.5).

Bello et al. [19] proposed the following method based on the Mann iterative algorithm [3] to solve TPSO–BVP (5.1)–(5.2):

$$\begin{cases}x''_{n+1}=(1-\alpha_n)x''_n+\alpha_n f\left(t,x_n,x'_n\right)\\ \lambda_0 x_{n+1}\left(a\right)+\beta_0 x'_{n+1}\left(a\right)=\gamma_0\\ \lambda_1 x_{n+1}\left(b\right)+\beta_1 x'_{n+1}\left(b\right)\ =\gamma_1\end{cases}\tag{5.6}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\lambda_i,\beta_i,\gamma_i\in\mathbb{R}$ with $\lambda_i^2+\beta_i^2>0$ for $i=0,1$, and $x_0\left(t\right)$ is an initial function satisfying the boundary conditions in (5.2). More precisely they proved the following theorem:

Theorem 9. *Let $\{\alpha_n\}$ be a sequence of real numbers that satisfies the following conditions*

$$\text{(i)}\quad 0\leq\alpha_n\leq 1,\qquad\qquad \text{(ii)}\quad \sum_{n=0}^{\infty}\alpha_n=\infty\tag{5.7}$$

and let T be the Lipschitzian operator with Lipschitz constant $L\in(0,1)$ defined by

$$T\left[x(t)\right]=\int_a^b G(t,s)f(s,x(s),x'(s))\mathrm{d}s+w\left(t\right).\tag{5.8}$$

Let $\{x_n\}$ be an iterative sequence in $C^1\left[a,b\right]$, generated by the Mann iterative algorithm of T in (5.8),

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T\left(x_n\right),$$

and an affine function x_0 *in* $C^1[a,b]$ *that satisfies* $x''=0$ *as well as the boundary condition* (5.2)*. Then,* $\{x_n\}$ *converges to a unique solution* x^* *in* $C^1[a,b]$ *of* TPSO–BVP (5.1) – (5.2).

We propose a new method based on the iterative algorithm (1.4) to solve TPSO–BVP (5.1)–(5.2) as under:

$$\begin{cases} x''_{n+1}=(1-\alpha_n)\,f(t,x_n,x'_n)+\alpha_n f(t,y_n,y'_n),\\ \lambda_0 x_{n+1}(a)+\beta_0 x'_{n+1}(a)=\gamma_0,\ \ \lambda_1 x_{n+1}(b)+\beta_1 x'_{n+1}(b)=\gamma_1;\\ y''_n=(1-\eta_n)\,z''_n+\eta_n f(t,z_n,z'_n),\\ \lambda_0 y_n(a)+\beta_0 y'_n(a)=\gamma_0,\ \ \lambda_1 y_n(b)+\beta_1 y'_n(b)=\gamma_1;\\ z''_n=(1-\delta_n)\,x''_n+\delta_n f(t,x_n,x'_n),\\ \lambda_0 z_n(a)+\beta_0 z'_n(a)=\gamma_0,\ \ \lambda_1 z_n(b)+\beta_1 z'_n(b)=\gamma_1, \end{cases} \tag{5.9}$$

where $\{\alpha_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ are real sequences in $[0,1]$, $\lambda_i,\beta_i,\gamma_i\in\mathbb{R}$ with $\lambda_i^2+\beta_i^2>0$ for $i=0,1$, and $x_0(t)$ is a initial function satisfying the boundary conditions in (5.2).

Theorem 10. *Let* $f(t,x(t),x'(t))$ *be a function whose derivative is bounded w.r.t.* x *and* $\{x_n\}$ *be an iterative sequence in* $C^1[a,b]$ *generated by the method in* (5.9) *with real sequences* $\{\alpha_n\},\{\eta_n\},\{\delta_n\}\subseteq[0,1]$ *and any initial function* $x_0(t)$ *in* $C^1[a,b]$ *that satisfies* $x''=0$*, as well as the boundary condition* (5.2)*. Assume that*

$$\mu=\frac{3}{8}\left|a-b\right|^2\lambda_C<1,\qquad \lambda_C=\max_{[a,b]\times\mathbb{R}^2}\left|\frac{\partial f}{\partial x}\right|.$$

Then, TPSO–BVP (5.1) – (5.2) *has a unique exact solution* $x_*(t)$ *in* $C^1[a,b]$ *and the iterative sequence* $\{x_n\}$ *converges uniformly to* $x_*(t)$*.*

Proof. As it is mentioned above, the existence of the exact solution of TPSO–BVP (5.1)–(5.2), where $\lambda_i,\beta_i,\gamma_i\in\mathbb{R}$ with $\lambda_i^2+\beta_i^2>0$ for $i=0,1$ is equivalent to finding a continuous solution of the integral equation (5.8), where $G(t,s)$ is Green's function in (5.3) associated to the TPSO–BVP (5.4)–(**??**) and $w(t)$ is the solution of TPSO–BVP (5.4)–(5.5). We prove now that the hypotheses of this theorem guarantee that $T[x(t)]$ in (5.8) is a contraction mapping which is a sufficient condition for the convergence, according to the Banach contraction principle.

For all $x(t)\neq y(t)\in C^1[a,b]$, we consider

$$|T[x(t)]-T[y(t)]|=\left|\int_a^b G(t,s)\,f(s,x(s),x'(s))\,\mathrm{d}s+w(t)\right.$$

$$- \int_a^b G(t,s) f(s, y(s), y'(s))\,\mathrm{d}s - w(t) \Big|$$

$$= \left| \int_a^b G(t,s) \left[f(s, x(s), x'(s)) - f(s, y(s), y'(s)) \right] \mathrm{d}s \right|$$

$$\leq \int_a^b |G(t,s)| \, |f(s, x(s), x'(s)) - f(s, y(s), y'(s))| \, \mathrm{d}s.$$

Using the definition of $G(t,s)$ in (5.3) and making some simple calculations, we get

$$\int_a^b G(t,s)\,\mathrm{d}s = \left(\frac{t-a}{b-a}\right)\left(\frac{t^2}{2} - tb - \frac{a^2}{2} + ab\right) + \left(\frac{t-b}{b-a}\right)\left(\frac{b^2}{2} - ba - \frac{t^2}{2} + ta\right).$$

By some routine calculations, we obtain the absolute maximum value of the function $\int_a^b G(t,s)\,\mathrm{d}s$ at the point $t = (a+b)/2 \in [a,b]$ as

$$\left| \int_a^b G(t,s)\,\mathrm{d}s \right| \leq \frac{3}{8} |a-b|^2.$$

Hence,

$$\left| T[x(t)] - T[y(t)] \right| \leq \frac{3}{8} |a-b|^2 \int_a^b |f(s, x(s), x'(s)) - f(s, y(s), y'(s))| \, \mathrm{d}s.$$

Applying the mean value theorem for f, we obtain

$$\begin{aligned} |T[x(t)] - T[y(t)]| &\leq \frac{3}{8} |a-b|^2 \max_{s\in[a,b]} |f(s, x(s), x'(s)) - f(s, y(s), y'(s))| \\ &\leq \mu \|x-y\|, \end{aligned}$$

where

$$\|x-y\| = \max_{s\in[a,b]} |x(s) - y(s)|, \quad \mu = \frac{3}{8}|a-b|^2 \lambda_C, \quad \lambda_C = \max_{[a,b]\times\mathbb{R}^2} \left| \frac{\partial f}{\partial x} \right|.$$

It follows from the assumption $\mu = \frac{3}{8}|a-b|^2 \lambda_C < 1$ that $T[x(t)]$ is a contraction mapping and hence, by the Banach contraction principle, it has a unique fixed point $x_*(t)$ in $C^1[a,b]$. Therefore, by Theorem 1 and Remark 1, the iterative sequence $\{x_n\}$ converges uniformly to $x_*(t)$. □

Remark 8. In Theorem 9, Bello et al. [19] imposed condition $\sum_{n=0}^{\infty} \alpha_n = \infty$ on real sequence $\{\alpha_n\} \subseteq [0,1]$ to guarantee the convergence of iterative algorithm in (5.6). But, we do not use such a condition for the iterative algorithm (5.6) employed in Theorem 10.

We now present some test examples to illustrate the performance and efficiency of the method given in (5.9). The numerical results verify the fast convergence and high accuracy of this method.

Example 6. Let $C^1[a,b] = C^1[0,1]$ and consider the following TPSO–BVP (see [19, Example 5.1]):

$$x''(t) + x'(t) = 1, \quad x(0) = 1, \quad x(1) = 0. \tag{5.10}$$

If we take $f(t, x, x') = 1 - x'$, then we have

$$\lambda_C = \max_{[0,1]\times\mathbb{R}^2} \left| \frac{\partial f}{\partial x} \right| = 0,$$

which implies that the derivative of $f(t, x, x')$ w.r.t. x is bounded and

$$\mu = 0 = \frac{3}{8} |0 - 1|^2 \lambda_C < 1.$$

Moreover, if we choose $x_0(t) = 1 - t$ as a initial function, then we have $x_0''(t) = 0$ and the boundary conditions $x_0(0) = 1$ and $x_0(1) = 0$.

Now, all the requirements in Theorem 10 are satisfied and so by its conclusion, TPSO–BVP (5.10) has a unique exact solution

$$x_*(t) = -\frac{2\mathrm{e}^{-t}}{\mathrm{e}^{-1} - 1} + t + \frac{1 + \mathrm{e}^{-1}}{\mathrm{e}^{-1} - 1} \in C^1[0,1]$$

and the iterative sequence $\{x_n\}$ generated by (5.9), with $\alpha_n = \eta_n = \delta_n = 2^{-n}$ for all $n \in \mathbb{N}$ converges uniformly to $x_*(t)$. Table 4 and Figure 11 show that the method in (5.9) converges to the unique exact solution $x_*(t)$ of the problem (5.10) faster than the method in (5.6). Approximative solutions obtained by the methods (5.6) and (5.9) are denoted by $x^{(5.6)}(t)$ and $x^{(5.9)}(t)$, respectively. The corresponding absolute errors are

$$\mathrm{Err}^{(5.6)}(t) = \left| x^{(5.6)}(t) - x_*(t) \right| \quad \text{and} \quad \mathrm{Err}^{(5.9)}(t) = \left| x^{(5.9)}(t) - x_*(t) \right|.$$

Example 7. Let $C^1[a,b] = C^1[0,1]$ and consider the following TPSO–BVP (see [19, Example 5.2]):

$$\begin{cases} x''(t) + t x(t) = t^3 + 2, \\ x'(0) - x(0) = 0, \\ x'(1) + x(1) = 3. \end{cases} \tag{5.11}$$

TABLE 4. Comparison of results obtained by the methods (5.9) and (5.6) in Example 6 after 10 iterations

t	Exact solution $x_*(t)$	Method in (5.9) $x^{(5.9)}(t)$	Method in (5.9) $\mathrm{Err}^{(5.9)}(t)$	Method in (5.6) $x^{(5.6)}(t)$	Method in (5.6) $\mathrm{Err}^{(5.6)}(t)$
0.1	0.798910024	0.798910024	6.8399(−25)	0.798910022	1.78679(−9)
0.2	0.626472547	0.626472547	4.0724(−25)	0.626472546	1.03854(−9)
0.3	0.479960925	0.479960925	2.8569(−24)	0.479960927	1.96251(−9)
0.4	0.356907984	0.356907984	5.7292(−24)	0.356907990	6.05669(−9)
0.5	0.255081338	0.255081338	7.9272(−24)	0.255081347	9.67019(−9)
0.6	0.172461036	0.172461036	8.6112(−24)	0.172461047	1.14288(−8)
0.7	0.107219353	0.107219353	7.5199(−24)	0.107219364	1.06736(−8)
0.8	0.057702496	0.057702496	5.0703(−24)	0.057702504	7.69644(−9)
0.9	0.022414049	0.022414049	2.1979(−24)	0.022414053	3.62379(−9)

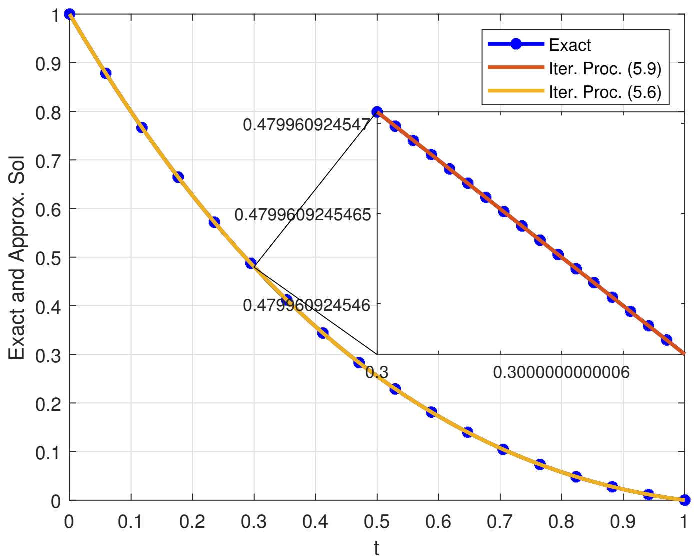

FIGURE 11. Comparison among the exact solution and the approximate solutions by the iteration algorithms (5.6) and (5.9) in Example 6 after 10 iterations

If we take $f(t, x, x') = t^3 + 2 - tx'$, then we have

$$\lambda_C = \max_{[0,1]\times\mathbb{R}^2} \left| \frac{\partial f}{\partial x} \right| = 1,$$

which implies that the derivative of $f(t,x,x')$ w.r.t. x is bounded and

$$\mu = \frac{3}{8}\left|0-1\right|^2 \lambda_C < 1.$$

Moreover, if we choose $x_0(t) = 1+t$ as an initial function, then we have $x_0''(t) = 0$ and $x_0(0) - x_0'(0) = 0$, $x_0(1) + x_0'(1) = 3$,

since $x_0'(t) = 1$.

Now, all the requirements in Theorem 10 are satisfied and so by its conclusion, TPSO–BVP (5.11) has a unique exact solution $x_*(t) = t^2$ in $C^1[0,1]$ and the iterative sequence $\{x_n\}$ generated by (5.9) with $\alpha_n = \eta_n = \delta_n = n/(n+1)$ for all $n \in \mathbb{N}$ converges uniformly to $x_*(t)$. Table 5 and Figure 12 show that the method in (5.9) converges to the unique exact solution $x_*(t)$ of the problem (5.11) faster than the method in (5.6).

Table 5. Comparison of results obtained by the methods (5.9) and (5.6) in Example 7 after 10 iterations

t	$x_*(t)$	Method in (5.9)		Method in (5.6)	
		$x^{(5.9)}(t)$	$\mathrm{Err}^{(5.9)}(t)$	$x^{(5.6)}(t)$	$\mathrm{Err}^{(5.6)}(t)$
0.0	0.	0.000000078	7.833(−8)	0.000220378	2.20378(−4)
0.1	0.01	0.010000086	8.612(−8)	0.010242289	2.42289(−4)
0.2	0.04	0.040000094	9.363(−8)	0.040263393	2.63393(−4)
0.3	0.09	0.090000101	1.005(−7)	0.090282757	2.82757(−4)
0.4	0.16	0.160000106	1.064(−7)	0.160299328	2.99328(−4)
0.5	0.25	0.250000111	1.109(−7)	0.250311966	3.11966(−4)
0.6	0.36	0.360000114	1.136(−7)	0.360319486	3.19486(−4)
0.7	0.49	0.490000114	1.140(−7)	0.490320723	3.20723(−4)
0.8	0.64	0.640000112	1.118(−7)	0.640314609	3.14609(−4)
0.9	0.81	0.810000107	1.067(−7)	0.810300261	3.00261(−4)
1.0	1.	1.000000098	9.850(−8)	1.000277079	2.77079(−4)

Example 8. Let $C^1[a,b] = C^1[0,1]$ and consider the following TPSO–BVP (see [19, Example 5.4]):

$$\begin{cases} x''(t) - x(t) = -2t^2, & 0 \le t \le 1, \\ x(0) = 1, \quad x(1) = 6. \end{cases} \tag{5.12}$$

If we take $f(t,x,x') = x - 2t^2$, then we have $\lambda_C = \max\limits_{[0,1]\times\mathbb{R}^2} \left|\frac{\partial f}{\partial x}\right| = 1$, which implies that the derivative of $f(t,x,x')$ w.r.t. x is bounded and $\mu = \frac{3}{8}\left|0-1\right|^2 \lambda_C < 1$.

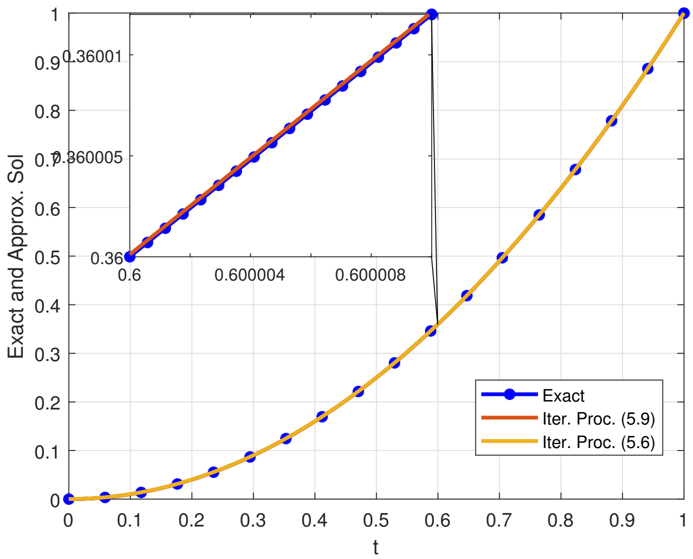

FIGURE 12. Comparison among the exact solution and the approximate solutions by the iteration algorithms (5.6) and (5.9) for Example 7 after 10 iterations

Moreover, if we choose $x_0(t) = 6t$ as a initial function, then we have $x_0''(t) = 0$ and $x_0(0) = 1$, $x_0(1) = 6$.

Now, all the requirements in Theorem 10 are satisfied and so by its conclusion, TPSO–BVP (5.12) has a unique exact solution

$$x_*(t) = \frac{2(t^2 \mathrm{e}^2 - t^2 + 2\mathrm{e}^t - 2\mathrm{e}^{2-t} + 2\mathrm{e}^2 - 2)}{\mathrm{e}^2 - 1} \in C^1\,[0,1]$$

and the iterative sequence $\{x_n\}$ generated by (5.9) with $\alpha_n = \eta_n = \delta_n = n/(n+1)$ for all $n \in \mathbb{N}$ converges uniformly to $x_*(t)$. Table 6 and Figure 13 show that the method in (5.9) converges to the unique exact solution $x_*(t)$ of the problem (5.12) faster than the method in (5.6).

Example 9. Let $C^1[a,b] = C^1[0,\pi/4]$ and consider the following TPSO–BVP:

$$\begin{cases} x''(t) - 2x'(t) + 2x(t) = 0, & 0 \le t \le \pi/4, \\ x(0) = 0, & x(\pi/4) = 1. \end{cases} \tag{5.13}$$

If we take $f(t,x,x') = 2x' - 2x$, then we have $\lambda_C = \max\limits_{[0,1]\times\mathbb{R}^2} \left|\frac{\partial f}{\partial x}\right| = 2$ which implies that the derivative of $f(t,x,x')$ w.r.t. x is bounded and $\mu = \frac{3}{8}\left|0 - \frac{\pi}{4}\right|^2 \lambda_C < 1$.

TABLE 6. Comparison of results obtained by the methods (5.9) and (5.6) in Example 8 after 5 iterations

		Method in (5.9)		Method in (5.6)	
t	$x_*(t)$	$x^{(5.9)}(t)$	$\mathrm{Err}^{(5.9)}(t)$	$x^{(5.6)}(t)$	$\mathrm{Err}^{(5.6)}(t)$
0.1	0.5260732366	0.5260732366	1.10268(−19)	0.5260733308	9.41942(−8)
0.2	1.0571780798	1.0571780798	2.09742(−19)	1.0571782590	1.79169(−7)
0.3	1.5980295053	1.5980295053	2.88686(−19)	1.5980297519	2.46609(−7)
0.4	2.1530397022	2.1530397022	3.39370(−19)	2.1530399921	2.89910(−7)
0.5	2.7263622321	2.7263622321	3.56835(−19)	2.7263625369	3.04835(−7)
0.6	3.3219335990	3.3219335990	3.39370(−19)	3.3219338890	2.89920(−7)
0.7	3.9435126476	3.9435126476	2.88686(−19)	3.9435128942	2.46624(−7)
0.8	4.5947181823	4.5947181823	2.09742(−19)	4.5947183615	1.79185(−7)
0.9	5.2790651862	5.2790651862	1.10268(−19)	5.2790652804	9.42040(−8)

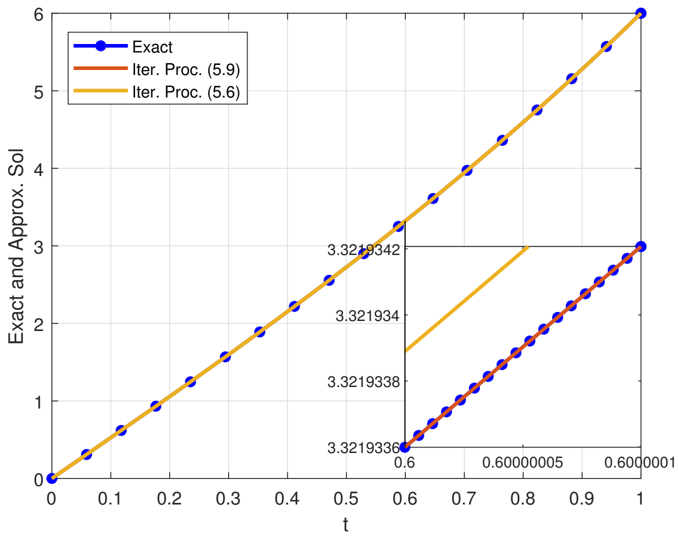

FIGURE 13. Comparison among the exact solution and the approximate solutions by iteration algorithms (5.6) and (5.9) for Example 8 after 5 iterations

Moreover, if we choose $x_0(t) = 4t/\pi$ as a initial function, then we have $x_0''(t) = 0$ and $x_0(0) = 0$, $x_0(\pi/4) = 1$.

Now, all the requirements in Theorem 10 are satisfied and so by its conclusion, TPSO-BVP (5.13) has a unique exact solution $x_*(t) = \sqrt{2}\,\mathrm{e}^{t-\pi/4}\sin t$ in $C^1\,[0, 1]$ and

TABLE 7. Comparison of results obtained by the methods (5.9) and (5.6) in Example 9 after 10 iterations

t	$x_*(t)$	Method in (5.9)		Method in (5.6)	
		$x^{(5.9)}(t)$	$\mathrm{Err}^{(5.9)}(t)$	$x^{(5.6)}(t)$	$\mathrm{Err}^{(5.6)}(t)$
$\pi/40$	0.054723469	0.054723469	1.19935(−10)	0.053851680	8.71788(−4)
$\pi/20$	0.118024500	0.118024500	1.88098(−10)	0.116659647	1.36485(−3)
$3\pi/40$	0.190517287	0.190517288	1.66975(−10)	0.189082843	1.43445(−3)
$\pi/10$	0.272797828	0.272797828	4.90440(−11)	0.271705074	1.09275(−3)
$\pi/8$	0.365432875	0.365432874	1.35047(−10)	0.365015826	4.17049(−4)
$3\pi/20$	0.468947395	0.468947394	3.22481(−10)	0.469390917	4.43522(−4)
$7\pi/40$	0.583810462	0.583810461	4.38374(−10)	0.585073191	1.26273(−3)
$\pi/5$	0.710419486	0.710419486	4.24209(−10)	0.712153454	1.73397(−3)
$9\pi/40$	0.849082725	0.849082725	2.64415(−10)	0.850551867	1.46914(−3)

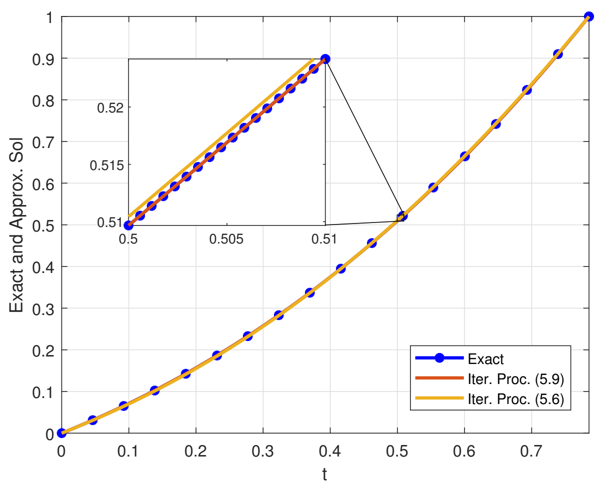

FIGURE 14. Comparison among the exact solution and the approximate solutions obtained by iteration algorithms (5.6) and (5.9) in Example 9 after 10 iterations

the iterative sequence $\{x_n\}$ generated by (5.9) with $\alpha_n = \eta_n = \delta_n = n/(n+1)$ for all $n \in \mathbb{N}$ converges uniformly to $x_*(t)$.

Table 7 and Figure 14 show that the method in (5.9) converges to the unique exact solution $x_*(t)$ of the problem (5.13) faster than the method in (5.6). In this example we take a set of equidistant points $\nu\pi/40$, $\nu = 0, 1, \ldots, 10$. i.e.,

$$\left\{0, \frac{\pi}{40}, \frac{\pi}{20}, \frac{3\pi}{40}, \frac{\pi}{10}, \frac{\pi}{8}, \frac{3\pi}{20}, \frac{7\pi}{40}, \frac{\pi}{5}, \frac{9\pi}{40}, \frac{\pi}{4}\right\}.$$

5.2. **Application to Supervised Learning.** A wide range of real-world problems arising in different branches of medicine, economics, engineering, social sciences, and science in general can be handled within the supervised learning analysis framework. The main idea behind the supervised learning is to find the minimum of the sum of two convex functions. In this section, our aim is to tackle this problem under the LASSO framework which concerns in minimization of the squared loss function and the non-smooth l_1 norm as a regularizer, and to apply algorithms (1.3) and (1.4) to solve it. We present the detailed comparative analysis for three algorithms of gradient projection type on six real-world benchmark datasets. Our experimental results reveal that one of the algorithms (algorithm (5.17)) employed in this section outperforms overall among others (algorithms (5.15) and (5.16)) in terms of convergence, accuracy and computational time.

Let us consider the dataset with m samples and d attributes denoted as $X \in \mathbb{R}^{m\times d}$ and the set of outcomes (labels) $Y \in \mathbb{R}^m$. We concern the following minimization problem

$$\min F(x) = \min_{w\in\mathbb{R}^d} \frac{1}{2} \|Xw - Y\|_2^2 + \delta \|w\|_1 . \tag{5.14}$$

In [29], this problem was solved via the following proximal gradient algorithm

$$w_{n+1} = T(w_n) = \operatorname{prox}_{\delta\eta_n\|\cdot\|_1}\left(w_n - \delta\eta_n X^t (Xw_n - Y)\right), \tag{5.15}$$

where

$$\operatorname{prox}_{\delta\eta_n\|\cdot\|_1}(w) = \left(\left|w^i\right| - \delta\eta_n\right)_+ \operatorname{sgn}\left(w^i\right)$$

and $\operatorname{sgn}(\cdot)$ is signum function.

To solve the problem (5.14) more effectively, we propose the following gradient projection type algorithms

$$\begin{cases} w_{n+1} = (1-\alpha_n) Tw_n + \alpha_n Tv_n, \\ v_n = (1-\beta_n) w_n + \beta_n Tw_n \end{cases} \tag{5.16}$$

and

$$\begin{cases} w_{n+1} = (1-\alpha_n)\, Tw_n + \alpha_n T v_n, \\ v_n = (1-\beta_n)\, u_n + \beta_n T u_n, \\ u_n = (1-\gamma_n)\, w_n + \gamma_n T w_n, \end{cases} \tag{5.17}$$

where $n \in \mathbb{N}$ and $Tw = \operatorname{prox}_{\delta\eta_n \|\cdot\|_1}(w)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $(0,1)$ for all $n \in \mathbb{N}$.

The datasets we tested in numerical experiments are listed as follows:

- **+:** Adults: The data set that classifies people with some attributes whether they have an annual income over 50K or less in a year.[1]
- **+:** Parkinson's Disease Classification: The data were gathered from 188 patients with PD (107 men and 81 women) at the Department of Neurology in Cerrahpasa Faculty of Medicine, Istanbul University. The control group consists of 64 healthy individuals (23 men and 41 women). The data set includes 756 instances with 754 attributes.[1]
- **+:** Heart Disease (Cleveland): The Cleveland database is the only one that has been used by ML researchers to this date. The "goal" field refers to the presence of heart disease in the patient. The Data set includes 303 instances with 14 attributes.[1]
- **+:** Iris: The database contains 3 classes of 50 instances where each class refers to a type of iris plant. The data set includes 150 instances with 4 attributes.[1]
- **+:** German Credit Data: The dataset classifies people by a set of attributes as good or bad credit risks. The data set includes 1000 instances with 20 attributes.[1]
- **+:** Wine Quality (Red): The dataset related classifies red wine quality. The data set includes 4898 instances with 12 attributes.[1]

Preparation of the datasets and presentation of the experimental results are implemented in MATLAB environment. The optimal value of η (η_n) is calculated by a backtracking algorithm.

All data sets divide into %60−%40 training and testing samples. The difference between two successive function values (tolerance) set up to 10^{-4}, and the maximum number of iteration set up to 10^4 as stopping criteria. We measure the performance of algorithms on

[1] https://archive.ics.uci.edu/ml/index.php

TABLE 8. Comparison of the efficiency of the algorithms (5.15), (5.16), and (5.17)

		PG Algor. (5.15)	Algor. (5.16)	Algor. (5.17)
Adult	*Numb. of Iter.*	10000	10000	8667
	The Last Func. Val.	1747.470178	1739.949373	1739.08412
	The Last rMse (Training)	0.325599113	0.323673233	0.323094999
	rMse (Test)	0.267824272	0.266190023	0.265690208
	Time (s)	21.0541076	25.4807814	22.1767781
Parkinson Disease	*Numb. of Iter.*	737	405	286
	The Last Func. Val.	38.71598921	38.71156302	38.71000522
	The Last rMse (Training)	0.320002153	0.319983785	0.319977315
	rMse (Test)	0.296428766	0.296611822	0.296692434
	Time (s)	4.2732993	2.566806	1.9718091
Heart Disease (Cleveland)	*Numb. of Iter.*	3269	1914	1368
	The Last Func. Val.	143.2151919	143.1783967	143.1676167
	The Last rMse (Training)	0.963363224	0.963118954	0.96302718
	rMse (Test)	0.73677197	0.737153149	0.737268504
	Time (s)	0.7345463	0.7667583	0.6476523
Iris	*Numb. of Iter.*	1289	809	640
	The Last Func. Val.	3.8460299	3.823829721	3.812053187
	The Last rMse (Training)	0.198753687	0.198230529	0.198091975
	rMse (Test)	0.177315634	0.176153779	0.175363814
	Time (s)	0.233264	0.3036546	0.3300645
German Bank Credit	*Numb. of Iter.*	239	131	93
	The Last Func. Val.	51.10493668	51.1039319	51.10355766
	The Last rMse (Training)	0.451903587	0.451898937	0.451897192
	rMse (Test)	0.464758101	0.46479238	0.464807651
	Time (s)	0.0640537	0.0678401	0.0585505
Wine Quality (Red)	*Numb. of Iter.*	2008	1011	678
	The Last Func. Val.	306.9647375	306.964705	306.9646811
	The Last rMse (Training)	0.619388582	0.619388548	0.619388523
	rMse (Test)	0.519335719	0.51933475	0.51933406
	Time (s)	0.5841445	0.5415088	0.4283849

each dataset in terms of $F(w_n)$ and $\|F(w_n) - F(w^*)\|$, the calculation times, the prediction accuracy (rMSE) for training samples, as well as the prediction accuracy (rMSE) for testing samples. All graphics presented herein are in log scale. The function values $F(w_n)$ of the algorithms and their performances in terms of $\|F(w_n) - F(w^*)\|$ are presented in Figures 15 and 16, respectively. Finally, in Figure 17, we show performances of algorithms based on the prediction accuracy (rMSE-training).

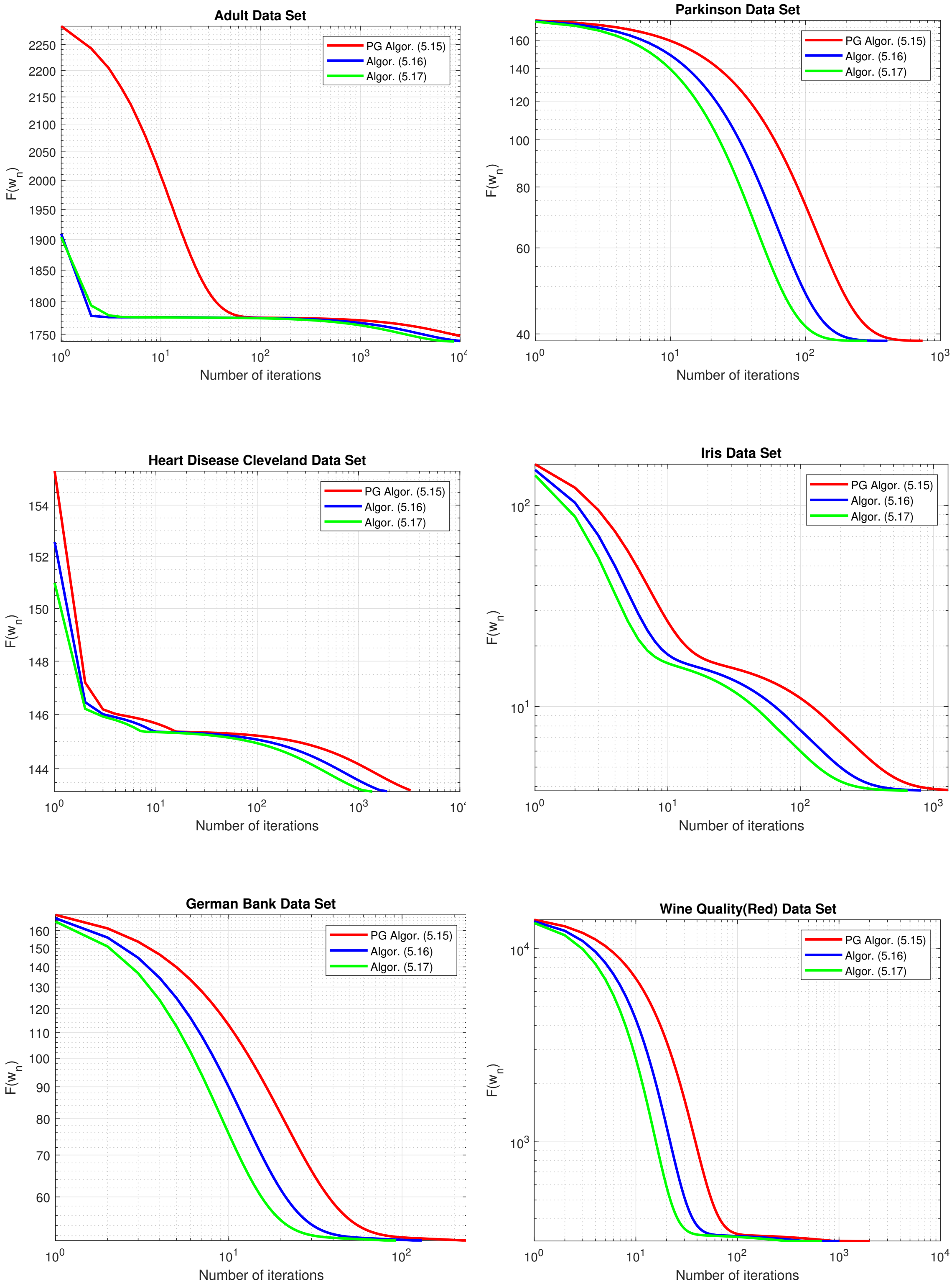

FIGURE 15. Comparison of the efficiency of algorithms (5.15), (5.16), and (5.17) based on reduction in function values $F(w_n)$ in each step

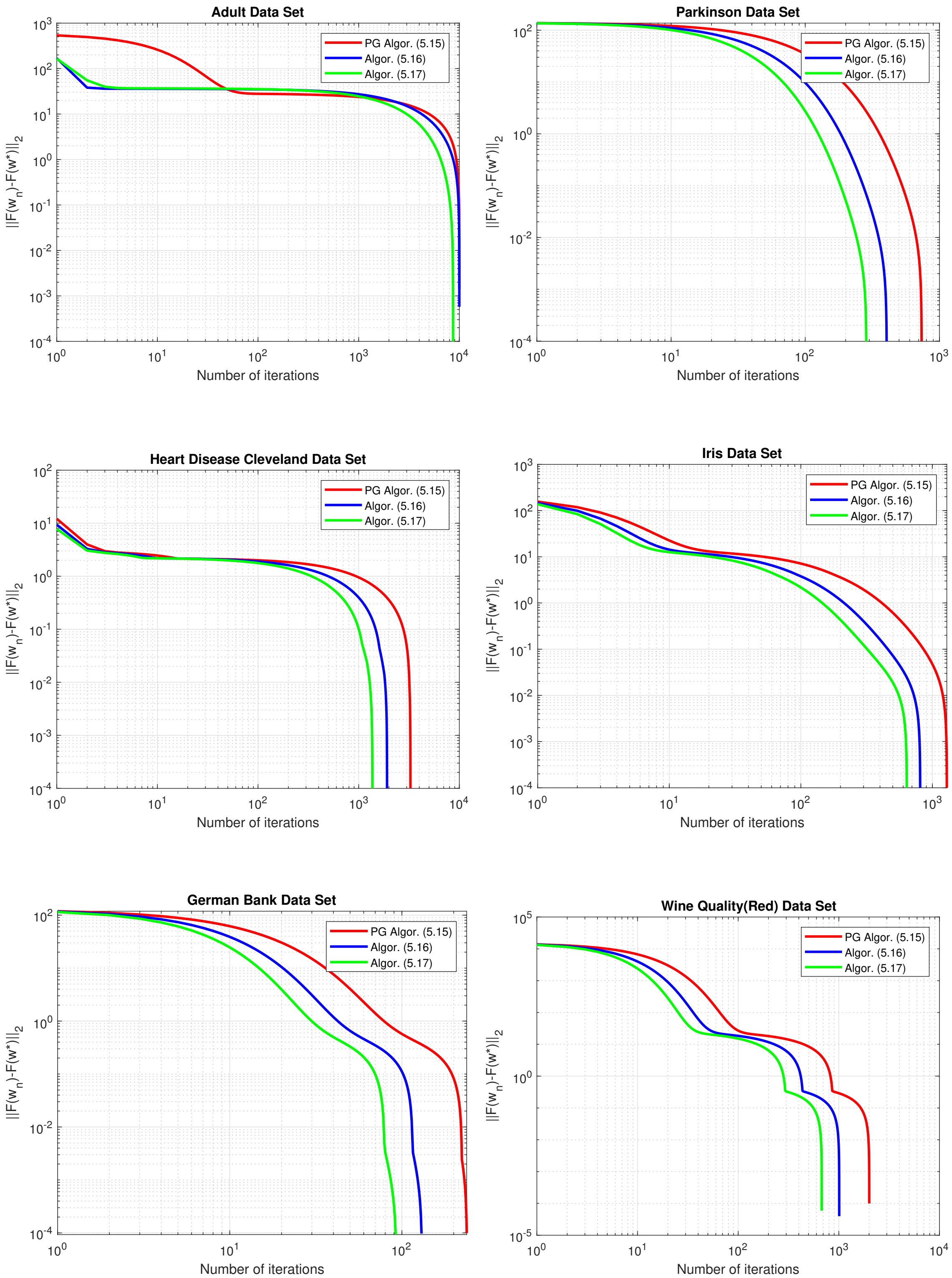

FIGURE 16. Comparison of the efficiency of algorithms (5.15), (5.16), and (5.17) based on $\|F(w_n) - F(w^*)\|$ in each step

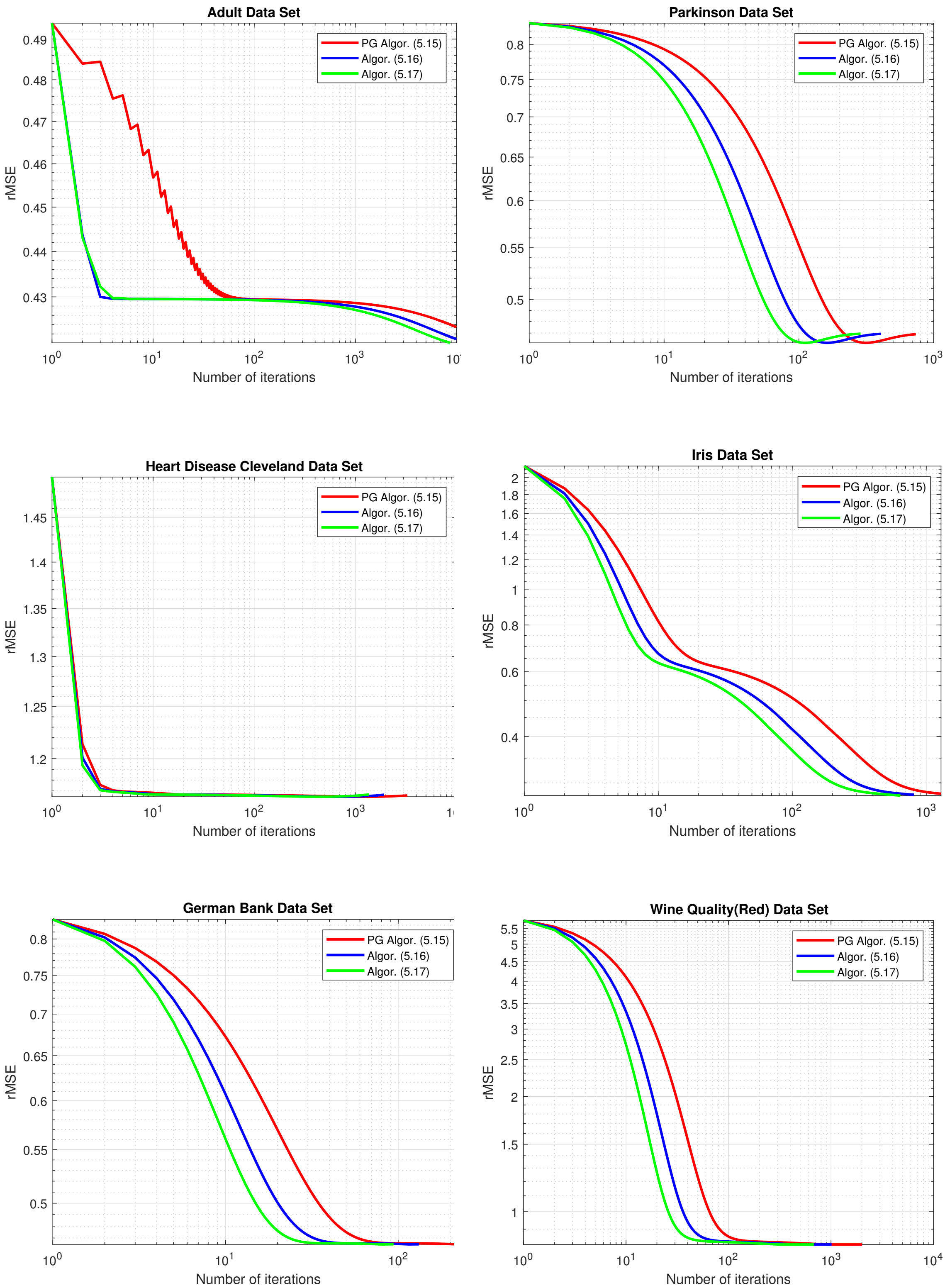

FIGURE 17. Comparison of the efficiency of algorithms (5.15), (5.16), and (5.17) based on rMSE in each step

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7. Declarations

7.1. Conflicts of Interest/ Competing Interests. The authors have no conflicts of interest to declare that are relevant to the content of this article.

7.2. Availability of Data and Material. The datasets used to generated Table 8 and Figures 15–17 are available at https://archive.ics.uci.edu/ml/index.php

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