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ORIGINAL RESEARCH PAPER

A class of polynomials and connections with Bernoulli's numbers

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Abstract
Motivated by certain problems connected with the stochastic analysis of the recursively defined time series, in this paper, we define and study some polynomial sequences. Beside computation of these polynomials and their connection to the Euler–Apostol numbers, we provesome basic properties and give an interesting connection of these polynomials with the well-known Bernoulli numbers, as well as some new summation formulas for Bernoulli's numbers. Finally, we prove that zeros of these polynomials are simple, real and symmetrically distributed in [0,1].

Keywords Generating function · Algebraic polynomial · Real zeros · Recurrence relation · Bernoulli numbers

Mathematics Subject Classication 11B68 · 11B83 · 26C05 · 26C10

1 Introduction and motivation

Our main goal in this paper is to consider a class of polynomials with all real zeros, which appears in certain problems connected with stochastic analysis of recursively defined time series. In order to explain this connection let us suppose,

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for instance, that $\{\Delta_t\}_{t\in\mathbb{Z}}$ be, so-called *the autoregressive (AR) time series*, i.e., the sequence of the random variables (RVs), which satisfies the recurrence relation $\Delta_t = a\,\Delta_{t-1} + \eta_t$. Here, $0 < |a| < 1$ and, for a given probability $p \in (0,1)$, RVs η_t have a distribution which is p-mixture of the Gaussian distribution $\mathcal{N}(0,\delta^2)$ and the discrete-type distribution, concentrated at zero. Therefore, the sequence $\{\eta_t\}_{t\in\mathbb{Z}}$ has a singular-type distribution, usually called the *Contaminated Gaussian Distribution (CGD)*, which generates a non-Gaussian sequence $\{\Delta_t\}$. In [10, 15] it was proved that the characteristic function of the RVs Δ_t is

$$\varphi_\Delta(u) = \prod_{j=0}^{\infty}\left[1 - p\left(\mathrm{e}^{-\frac{1}{2}a^{2j}u^2\delta^2} - 1\right)\right].$$

Moreover, the function $\varphi_\Delta(u)$ satisfies the following recurrence

$$A_k = \sum_{j=1}^{k}\binom{2k-1}{2j-1}A_{k-j}\,Z_j, \quad k = 1,2,\ldots,$$

where $A_k = \varphi_\Delta^{(2k)}(0)$, and

$$Z_k = \begin{cases} -\dfrac{\delta^2 m_c}{1-a^2}, & k=1,\\ -\dfrac{(2k-1)!!\delta^{2k}p(p-1)}{1-a^{2k}}\,P_{k-2}(p), & k=2,3,\ldots. \end{cases}$$

Here $P_k(x)$, $k = 0,1,\ldots$, are algebraic polynomials of degree k,

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= 2x - 1,\\
P_2(x) &= 6x^2 - 6x + 1,\\
P_3(x) &= 24x^3 - 36x^2 + 14x - 1,\\
P_4(x) &= 120x^4 - 240x^3 + 150x^2 - 30x + 1,\\
P_5(x) &= 720x^5 - 1800x^4 + 1560x^3 - 540x^2 + 62x - 1,\\
P_6(x) &= 5040x^6 - 15120x^5 + 16800x^4 - 8400x^3 + 1806x^2 - 126x + 1,
\end{aligned}$$

etc. In the following, let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of real numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

On the other hand, let us consider time series $\{X_t\}_{t\in\mathbb{Z}}$ defined as follows $X_t = \varepsilon_t - \sum_{j=1}^{m}\alpha_j\,\eta_{t-j}$, $t \in \mathbb{Z}$, where $\varepsilon_t : \mathcal{N}(0,\delta^2)$ and $\alpha_1,\ldots,\alpha_m \in \mathbb{R}$. Then, the sequence $\{X_t\}$ is usually called *the Moving Average (MA)* time series, with the "permanent" noise $\{\varepsilon_t\}$, and the "optional" CGD–noise (η_t), defined as above. In [14], it is provided that RVs X_t have the characteristic function

$$\varphi_X(u) = \mathrm{e}^{-\delta^2 u^2/2} \prod_{j=1}^{m} \left[1 + p\left(\mathrm{e}^{-\alpha_j^2 \delta^2 u^2/2} - 1\right)\right],$$

which satisfies the relation

$$U_k + \sum_{j=1}^{k} \binom{2k-1}{2j-1} U_{k-j} V_j = 0, \quad k = 1, 2, \ldots .$$

Here, $U_k = \varphi_X^{(2k)}(0)$, $V_k = (2k-1)!!\, \delta^{2k}\, W_k(p)$, and, by using the induction method, it can be proved that

$$W_k(b_c) = \begin{cases} 1 + p \displaystyle\sum_{j=1}^{m} \alpha_j^2, \; k = 1, \\ L_k(p) \displaystyle\sum_{j=1}^{p} \alpha_j^{2k}, \; k = 2, 3, \ldots, \end{cases}$$

where $\{L_k(x\}$ are algebraic polynomials, which can be expressed as

$$L_{k+2}(x) = x(x-1)P_k(x), \quad k \geq 0. \tag{1.1}$$

This enables us to obtain the moments $E(\Delta_t^n) = \mathrm{i}^{-n} \varphi_\Delta^{(n)}(0)$ and $E(X_t^n) = \mathrm{i}^{-n} \varphi_X^{(n)}(0)$, as well as several others stochastic properties of the series $\{\Delta_t\}$ and $\{X_t\}$.

In the sequel, we introduce two sequences of polynomials $\{M_k(x)\}_{k=0}^{\infty}$ and $\{L_k(x)\}_{k=1}^{\infty}$. Some collections of special classes of polynomials can be found in [9] and [3].

Definition 1.1 The sequences of algebraic polynomials $\{M_k(x)\}_{k=0}^{\infty}$ and $\{L_k(x)\}_{k=1}^{\infty}$ are defined by the generating functions

$$G(x,t) = \frac{1}{(1-x)\mathrm{e}^t + x} \quad \text{and} \quad F(x,t) = \log G(x,t), \tag{1.2}$$

respectively, i.e.,

$$G(x,t) = \sum_{k=0}^{\infty} M_k(x) \frac{t^k}{k!} \quad \text{and} \quad F(x,t) = \sum_{k=1}^{\infty} L_k(x) \frac{t^k}{k!}. \tag{1.3}$$

A first few of $M_k(x)$ are

$$
\begin{aligned}
M_0(x) &= 1,\\
M_1(x) &= x - 1,\\
M_2(x) &= 2x^2 - 3x + 1,\\
M_3(x) &= 6x^3 - 12x^2 + 7x - 1,\\
M_4(x) &= 24x^4 - 60x^3 + 50x^2 - 15x + 1,\\
M_5(x) &= 120x^5 - 360x^4 + 390x^3 - 180x^2 + 31x - 1,\\
M_6(x) &= 720x^6 - 2520x^5 + 3360x^4 - 2100x^3 + 602x^2 - 63x + 1.
\end{aligned}
$$

In this paper we study these sequences of polynomials, as well as the polynomials $P_k(x)$ given by (1.1). The paper is organized as follows. Computation of polynomials $M_k(x)$ and their connection to the Euler–Apostol numbers are given in Sect. 2. Basic properties of polynomials $M_k(x)$ (and $L_k(x)$) are presented in Sect. 3. An interesting connection of these polynomials with the well-known Bernoulli numbers, as well as some new summation formulas for Bernoulli's numbers, are also given in Sect. 4. Finally, some properties of the polynomials $P_k(x)$ and especially a distribution of their zeros are given in Sect. 5.

2 Recurrence relation and computation of the polynomials $M_k(x)$

Using the Apostol–Euler numbers $E_k(\lambda)$ defined by the following generating function (cf. [2, 5, 11–13])

$$
\frac{2}{\lambda \mathrm{e}^t + 1} = \sum_{k=0}^{\infty} E_k(\lambda) \frac{t^k}{k!}, \tag{2.1}
$$

we can compute the polynomials $M_k(x)$. From (2.1) it is easy to see that

$$
E_0(\lambda) = \frac{2}{1+\lambda}.
$$

Also, the following well-known result

$$
E_k(\lambda) + \lambda \sum_{j=0}^{k} \binom{k}{j} E_j(\lambda) = 0, \tag{2.2}
$$

holds.

Comparing the function $G(x, t)$ from (1.2) and (1.3) with the generating function of the Apostol–Euler numbers (2.1), we find

$$M_k(x) = \frac{1}{2x} E_k\left(\frac{1-x}{x}\right).$$

Also, using (2.2) we obtain the recurrence relation for the polynomials $M_k(x)$

$$xM_k(x) + (1-x)\sum_{j=0}^{k}\binom{k}{j}M_j(x) = 0.$$

Or, equivalently,

$$M_k(x) = (x-1)\sum_{j=0}^{k-1}\binom{k}{j}M_j(x), k = 1, 2, \dots, \tag{2.3}$$

where

$$M_0(x) = \frac{1}{2x}E_0\left(\frac{1-x}{x}\right) = \frac{1}{2x}\frac{2}{1+\frac{1-x}{x}} = 1.$$

The recurrence relation (2.3) enables us to generate the sequence of polynomials $\{M_k(x)\}_{k=0}^{\infty}$.

3 Properties of polynomials $M_k(x)$ and $L_k(x)$

Let the polynomials $M_k(x)$ and $L_k(x)$ be defined by (1.2) and (1.3).

Lemma 3.1 *The polynomials $L_k(x)$, $k \geq 2$, are symmetric with respect to $x = 1/2$,* i.e.,

$$L_k(x) = (-1)^k L_k(1-x). \tag{3.1}$$

In particular, $L_k(0) = L_k(1) = 0$ for each $k \geq 2$, and $L_1(x) = x - 1$.

Proof Putting $x = 1$ in (1.2) and (1.3), we obtain

$$\sum_{k=1}^{\infty} L_k(1)\frac{t^k}{k!} = F(1,t) = \log G(1,t) = 0,$$

i.e., $L_k(1) = 0$ for each $k \geq 1$. On the other hand, for $x = 0$, we get

$$\sum_{k=1}^{\infty} L_k(0)\frac{t^k}{k!} = F(0,t) = \log\frac{1}{\mathrm{e}^t} = -t,$$

i.e.,

$$L_1(0) = -1, \quad L_k(0) = 0, \quad k \geq 2.$$

Thus, these polynomials for each $k \geq 2$ have zeros at ± 1. Using these values at $x = 0$ and $x = 1$, for $k = 1$ we can see that $L_1(x) = x - 1$.

In order to prove the first part of this lemma, we put $(1 - x, -t)$ instead of (x, t) in (1.2) and (1.3). Since

$$F(1-x,-t) = \log \frac{\mathrm{e}^t}{x + (1-x)\mathrm{e}^t} = t + F(x,t),$$

we get

$$\sum_{k=1}^{\infty} L_k(1-x)(-1)^k \frac{t^k}{k!} \equiv t + \sum_{k=1}^{\infty} L_k(x) \frac{t^k}{k!},$$

i.e.,

$$-L_1(1-x) = 1 + L_1(x) \quad \text{and} \quad (-1)^k L_k(1-x) = L_k(x), \quad k \geq 2.$$

Since $L_1(x) = x - 1$, the first part in the previous formula is satisfied and proof is finished. □

In a similar way, we can prove the following result.

Lemma 3.2 *For polynomials $M_k(x)$, $k \geq 0$, we have*

$$M_k(0) = (-1)^k \quad (k \geq 0), \quad M_0(1) = 1, \quad M_k(1) = 0 \quad (k \geq 1).$$

Some connections between the polynomials $L_k(x)$ and $M_k(x)$ can be established.

Theorem 3.1 *The polynomials $L_k(x)$ and $M_k(x)$ satisfy the following difference-differential relations*

$$\frac{L_{k+1}(x)}{x-1} - L_k'(x) = M_k(x), \quad k \geq 1, \tag{3.2}$$

$$L_{k+1}(x) = x(x-1)L_k'(x), \quad k \geq 1, \tag{3.3}$$

$$L_{k+1}(x) = xM_k(x), \quad k \geq 1. \tag{3.4}$$

Proof Using partial derivatives of the generating function $F(x, t)$,

$$\frac{\partial F}{\partial x} = \frac{\mathrm{e}^t - 1}{(1-x)\mathrm{e}^t + x}, \quad \frac{\partial F}{\partial t} = \frac{(x-1)\mathrm{e}^t}{(1-x)\mathrm{e}^t + x},$$

we obtain

$$\frac{1}{x-1}\frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} = G(x,t), \tag{3.5}$$

where $G(x, t)$ is given by (1.2).

On the other hand,

$$\frac{\partial F}{\partial x} = \sum_{k=1}^{\infty} L_k'(x)\frac{t^k}{k!} \quad \text{and} \quad \frac{\partial F}{\partial t} = \sum_{k=1}^{\infty} L_k(x)\frac{t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} L_{k+1}(x)\frac{t^k}{k!}.$$

Putting these expressions into (3.5) we get

$$\frac{1}{x-1}\sum_{k=0}^{\infty} L_{k+1}(x)\frac{t^k}{k!} - \sum_{k=1}^{\infty} L_k'(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty} M_k(x)\frac{t^k}{k!},$$

from which we directly obtain (3.2), as well as $L_1(x)/(x-1) = M_0(x) = 1$.

In order to prove (3.3), we start with

$$\frac{\partial F}{\partial t} = L_1(x) + \sum_{k=1}^{\infty} L_{k+1}(x)\frac{t^k}{k!}.$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty} L_{k+1}(x)\frac{t^k}{k!} &= \frac{(x-1)\mathrm{e}^t}{(1-x)\mathrm{e}^t + x} - (x-1)\\ &= \frac{x(x-1)(\mathrm{e}^t-1)}{(1-x)\mathrm{e}^t + x}\\ &= x(x-1)(\mathrm{e}^t-1)G(x,t)\\ &= x(x-1)\frac{\partial F}{\partial t},\end{aligned}$$

i.e.,

$$\sum_{k=1}^{\infty} L_{k+1}(x)\frac{t^k}{k!} = x(x-1)\sum_{k=1}^{\infty} L_k'(x)\frac{t^k}{k!},$$

from which we conclude that (3.3) holds.

Finally, combining (3.2) and (3.3), we obtain (3.4). □

The following theorem gives a connection of polynomials M_k and L_k, with the well known Bernoulli numbers B_k defined by (cf. [1])

$$\varepsilon(t) = \frac{t}{\mathrm{e}^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{3.6}$$

For example, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, etc. The function $\varepsilon(t)$ and the corresponding quadrature formulas on $(0, +\infty)$ with respect to this (weight) function [4] are widely used in solid state physics, e.g., the total energy of thermal vibration of a crystal lattice can be expressed in the form $\int_0^\infty f(t)\varepsilon(t)\,\mathrm{d}t$, where $f(t)$ is related to the phonon density of states. Also, integrals of the previous type can be used for summation of slowly convergent series (see [4, 6–8]).

4 Connections with Bernoulli numbers

Theorem 4.1 *Let polynomials* $\{L_k\}_{k=1}^{\infty}$ *and* $\{M_k\}_{k=0}^{\infty}$ *be given in* (1.2) *and* (1.3), *respectively. Then*

$$\int_0^1 M_k(x)\,\mathrm{d}x = B_k, \quad k = 0, 1, 2, \ldots, \tag{4.1}$$

and

$$\int_0^1 L_1(x)\,\mathrm{d}x = -(1 + B_1), \quad \int_0^1 L_k(x)\,\mathrm{d}x = -B_k, \quad k = 2, 3, \ldots, \tag{4.2}$$

where B_k *are Bernoulli's numbers.*

Proof Integrating the first equality in (1.3) with respect to x over [0,1], we obtain

$$\int_0^1 G(x,t)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}x}{(1-x)\mathrm{e}^t + x} = \sum_{k=0}^{\infty}\left(\int_0^1 M_k(x)\,\mathrm{d}x\right)\frac{t^k}{k!},$$

i.e.,

$$\varepsilon(t) = \frac{t}{\mathrm{e}^t - 1} = \sum_{k=0}^{\infty}\left(\int_0^1 M_k(x)\,\mathrm{d}x\right)\frac{t^k}{k!}.$$

Comparing this with (3.6) we conclude that (3.6) holds for each $k = 0, 1, \ldots$.

Similarly, by an integration of the second equality in (1.3) we get

$$\int_0^1 F(x,t)\,\mathrm{d}x = \int_0^1 \log\frac{1}{(1-x)\mathrm{e}^t + x}\,\mathrm{d}x = 1 - t - \varepsilon(t),$$

and then using (4.2), we obtain (4.2). □

In the sequel we consider more general integrals (moments)

$$\mu_k^{(n)} = \int_0^1 x^n M_k(x)\mathrm{d}x \quad (n \in \mathbb{N}_0; k = 0,1,2,\ldots). \tag{4.3}$$

Theorem 4.2 *For each $n \in \mathbb{N}$ for the moments (4.3) the following relations*

$$\mu_0^{(n)} = -\frac{1}{n}\left(1 - \mu_0^{(n-1)} - \mu_1^{(n-1)}\right), \tag{4.4}$$

$$\mu_k^{(n)} = -\frac{1}{n}\sum_{j=0}^{k+1}\binom{k+1}{j}\mu_j^{(n-1)} \quad (k = 1,2,\ldots) \tag{4.5}$$

hold.

Proof As in the proof of Theorem 4.1 we start now from the weighted integral (with respect to the power function $x \mapsto x^n$)

$$\mathcal{M}_n = \mathcal{M}_n(t) = \int_0^1 x^n G(x,t)\,\mathrm{d}x = \int_0^1 \frac{x^n}{(1-x)\mathrm{e}^t + x}\,\mathrm{d}x = \sum_{k=0}^{\infty}\mu_k^{(n)}\frac{t^k}{k!}, \tag{4.6}$$

where $\mu_k^{(n)}$ are given by (4.3). Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_n(t) = -\int_0^1 \frac{x^n(1-x)\mathrm{e}^t}{[(1-x)\mathrm{e}^t + x]^2}\mathrm{d}x = \sum_{k=1}^{\infty}\mu_k^{(n)}\frac{t^{k-1}}{(k-1)!},$$

i.e.,

$$-\int_0^1 \frac{x^n}{(1-x)\mathrm{e}^t + x}\,\mathrm{d}x + \int_0^1 \frac{x^{n+1}}{[(1-x)\mathrm{e}^t + x]^2}\,\mathrm{d}x = \sum_{k=0}^{\infty}\mu_{k+1}^{(n)}\frac{t^k}{k!},$$

we conclude that

$$\int_0^1 \frac{x^{n+1}}{[(1-x)\mathrm{e}^t + x]^2}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\mu_{k+1}^{(n)} + \mu_k^{(n)}\right)\frac{t^k}{k!}$$

On the other hand, by an integration by parts we reduce the integral $\mathcal{M}_{n+1}$ to the integral on the left side in the previous equality. Namely, we have

$$\mathcal{M}_{n+1} = \int_0^1 \frac{x^{n+1}}{(1-x)\mathrm{e}^t + x}\,\mathrm{d}x = \frac{1}{n+1} - \frac{\mathrm{e}^t}{n+1}\int_0^1 \frac{x^{n+1}}{[(1-x)\mathrm{e}^t + x]^2}\,\mathrm{d}x,$$

so that

$$\begin{aligned}(n+1)\sum_{k=0}^{\infty}\mu_k^{(n+1)}\frac{t^k}{k!} &= 1 - \mathrm{e}^t\sum_{k=0}^{\infty}\left(\mu_{k+1}^{(n)} + \mu_k^{(n)}\right)\frac{t^k}{k!}\\ &= 1 - \left(\sum_{i=0}^{\infty}\frac{t^i}{i!}\right)\left(\sum_{j=0}^{\infty}\left(\mu_{j+1}^{(n)} + \mu_j^{(n)}\right)\frac{t^j}{j!}\right)\\ &= 1 - \sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}\binom{k}{j}\left(\mu_{j+1}^{(n)} + \mu_j^{(n)}\right)\right\}\frac{t^k}{k!}.\end{aligned}$$

Comparing the coefficients of $t^k/k!$ on the left and right side in the previous equality and putting $n-1$ instead of n, we obtain $n\mu_0^{(n)} = 1 - \mu_1^{(n-1)} - \mu_0^{(n-1)}$, i.e., (4.4), and

$$n\mu_k^{(n)} = -\sum_{j=0}^{k}\binom{k}{j}\left(\mu_{j+1}^{(n-1)} + \mu_j^{(n-1)}\right), \quad k \in \mathbb{N},$$

which can be written in the form

$$\begin{aligned}n\mu_k^{(n)} &= -\sum_{j=1}^{k+1}\binom{k}{j-1}\mu_j^{(n-1)} - \sum_{j=0}^{k}\binom{k}{j}\mu_j^{(n-1)}\\ &= -\left\{\binom{k}{0}\mu_0^{(n-1)} + \sum_{j=1}^{k}\left[\binom{k}{j-1} + \binom{k}{j}\right]\mu_j^{(n-1)} + \binom{k}{k}\mu_{k+1}^{(n-1)}\right\}\\ &= -\sum_{j=0}^{k+1}\binom{k+1}{j}\mu_j^{(n-1)},\end{aligned}$$

i.e., (4.5). □

The result of Theorem 4.2 can be expressed in terms of Bernoulli numbers. In addition, some new summation formulas for Bernoulli numbers can be derived.

Define a sequence of algebraic polynomials $\{\mathbb{Q}_n\}_{n\in\mathbb{N}_0}$ by

$$\mathbb{Q}_0(r) = 1, \quad \mathbb{Q}_{n+1}(r) = \mathbb{Q}_n(r)(r-n), \quad n \in \mathbb{N}_0. \tag{4.7}$$

Thus, $\mathbb{Q}_n(r) = r(r-1)\cdots(r-n+1)$ has real zeros at the points $0, 1, \dots, n-1$. Also, we use here the standard linear difference operators $\mathbf{\Delta}$ (the forward-difference operator), $\mathbf{E}$ (the shifting operator), and $\mathbf{I}$ (the identity operator), defined by

$$\mathbf{\Delta} a_k = a_{k+1} - a_k, \quad \mathbf{E} a_k = a_{k+1}, \quad \text{and} \quad \mathbf{I} a_k = a_k,$$

respectively. For example,

$$\mathbb{Q}_4(\mathbf{E}) = \mathbf{E}(\mathbf{E}-\mathbf{I})(\mathbf{E}-2\mathbf{I})(\mathbf{E}-3\mathbf{I}) = \mathbf{E}^4 - 6\mathbf{E}^3 + 11\mathbf{E}^2 - 6\mathbf{E},$$

i.e.,

$$\mathbb{Q}_4(\mathbf{E})a_k = a_{k+4} - 6a_{k+3} + 11a_{k+2} - 6a_{k+1}.$$

Theorem 4.3 *For Bernoulli's numbers B_k defined by* (3.6), *and for each $n \in \mathbb{N}_0$, the following identities*

$$\sum_{j=0}^{k+1} \binom{k+1}{j} \mathbb{Q}_n(\mathbf{E}) B_j = \begin{cases} (\mathbf{I}+\mathbf{E})\mathbb{Q}_n(\mathbf{E})B_0, & k = 0, \\ \mathbb{Q}_{n+1}(\mathbf{E})B_k, & k = 1, 2, \dots\ , \end{cases} \tag{4.8}$$

hold, where $\mathbb{Q}_n$ are polynomials defined by (4.7).

Proof We prove this result by induction on n.

For $n = 0$ the identity (4.8) is true, because it reduces to the following well known identity (cf. [1], p. 241)

$$\sum_{j=0}^{k+1} \binom{k+1}{j} B_j = \begin{cases} \dfrac{1}{2}, & k = 0, \\ B_{k+1}, & k = 1, 2, \dots\ . \end{cases} \tag{4.9}$$

Suppose now that (4.8) hold for some $n = m \in \mathbb{N}$, i.e., let

$$\sum_{j=0}^{k+2} \binom{k+2}{j} \mathbb{Q}_m(\mathbf{E}) B_j = \mathbb{Q}_{m+1}(\mathbf{E}) B_{k+1}, \quad k = 0, 1, \dots\ . \tag{4.10}$$

Then, for $k \geq 1$, we have

$$
\begin{aligned}
\sum_{j=0}^{k+2} \binom{k+2}{j} \mathbb{Q}_m(\mathbf{E}) B_j = {} & \mathbb{Q}_m(\mathbf{E}) B_0 + \mathbb{Q}_m(\mathbf{E}) B_{k+2} \\
& + \sum_{j=1}^{k+1} \left[\binom{k+1}{j} + \binom{k+1}{j-1} \right] \mathbb{Q}_m(\mathbf{E}) B_j \\
& = \sum_{j=0}^{k+1} \binom{k+1}{j} \mathbb{Q}_m(\mathbf{E}) B_j + \sum_{j=0}^{k+1} \binom{k+1}{j} \mathbb{Q}_m(\mathbf{E}) B_{j+1} \\
& = \sum_{j=0}^{k+1} \binom{k+1}{j} \mathbb{Q}_m(\mathbf{E})(\mathbf{I} + \mathbf{E}) B_j,
\end{aligned}
$$

i.e.,

$$
\sum_{j=0}^{k+1} \binom{k+1}{j} \mathbb{Q}_m(\mathbf{E})(\mathbf{I} + \mathbf{E}) B_j = \mathbb{Q}_{m+1}(\mathbf{E}) B_{k+1}.
$$

By multiplying the induction hypothesis (4.10) (written k for $k-1$) by $m+1$ and its substructing from the previous equality, we get

$$
\sum_{j=0}^{k+1} \binom{k+1}{j} \mathbb{Q}_m(\mathbf{E})[\mathbf{I} + \mathbf{E} - (m+1)\mathbf{I}] B_j = \mathbb{Q}_{m+1}(\mathbf{E}) B_{k+1} - (m+1) \mathbb{Q}_{m+1}(\mathbf{E}) B_k,
$$

i.e.,

$$
\begin{aligned}
\sum_{j=0}^{k+1} \binom{k+1}{j} \mathbb{Q}_{m+1}(\mathbf{E}) B_j &= \mathbb{Q}_{m+1}(\mathbf{E})[\mathbf{E} - (m+1)\mathbf{I}] B_k \\
&= \mathbb{Q}_{m+2}(\mathbf{E}) B_k,
\end{aligned}
$$

due to (4.7).

This shows that the identities (4.8) are true for $n = m+1$ and $k \geq 1$. The proof for $k = 0$ is trivial. □

Now we return to the moments (4.3).

Theorem 4.4 *For each $n \in \mathbb{N}$ and $k \geq 1$ the moments (4.3) can be expressed in the following form*

$$
\mu_k^{(n)} = \frac{(-1)^n}{n!} \mathbb{Q}_n(\mathbf{E}) B_k, \tag{4.11}
$$

where $\mathbb{Q}_n$ *are polynomials defined by* (4.7).

According to (4.11), the moments (4.3) for $n \le 6$ are:

$$\mu_k^{(0)} = B_k,$$
$$\mu_k^{(1)} = -B_{k+1},$$
$$\mu_k^{(2)} = \frac{1}{2}\big(B_{k+2} - B_{k+1}\big),$$
$$\mu_k^{(3)} = -\frac{1}{6}\big(B_{k+3} - 3B_{k+2} + 2B_{k+1}\big),$$
$$\mu_k^{(4)} = \frac{1}{24}\big(B_{k+4} - 6B_{k+3} + 11B_{k+2} - 6B_{k+1}\big),$$
$$\mu_k^{(5)} = -\frac{1}{120}\big(B_{k+5} - 10B_{k+4} + 35B_{k+3} - 50B_{k+2} + 24B_{k+1}\big),$$
$$\mu_k^{(6)} = \frac{1}{720}\big(B_{k+6} - 15B_{k+5} + 85B_{k+4} - 225B_{k+3} + 274B_{k+2} - 120B_{k+1}\big).$$

Remark 4.1 Using the forward-difference operator $\mathbf{\Delta}$ $(= \mathbf{E} - \mathbf{I})$, the equality (4.11) can be expressed in an alternative form,

$$\mu_k^{(n)} = \frac{(-1)^n}{n!}\mathbb{Q}_{n-1}(\mathbf{\Delta})B_{k+1}.$$

For example,

$$\mu_k^{(4)} = \frac{1}{24}\big(\mathbf{\Delta}^3 - 3\mathbf{\Delta}^2 + 2\mathbf{\Delta}\big)B_{k+1} = \frac{1}{24}\big(\mathbf{\Delta}^3 B_{k+1} - 3\mathbf{\Delta}^2 B_{k+1} + 2\mathbf{\Delta}B_{k+1}\big).$$

Starting from the generating function of the Bernoulli numbers (3.6), we see

$$\mathbf{D}^\nu \varepsilon(t) = \frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu}\left\{\frac{t}{\mathrm{e}^t - 1}\right\} = \sum_{k=0}^{\infty} B_{k+\nu}\frac{t^k}{k!} \quad (\nu \in \mathbb{N}_0),$$

where $\mathbf{D}$ is the differentiation operator. This relation can be expressed in the form

$$\mathbf{D}^\nu \varepsilon(t) = \sum_{k=0}^{\infty}\left\{\mathbf{E}^\nu B_k\right\}\frac{t^k}{k!} \quad (\nu \in \mathbb{N}_0).$$

Also, in view of linearity, we have for each $n \in \mathbb{N}_0$,

$$\mathbb{Q}_n(\mathbf{D})\varepsilon(t) = \sum_{k=0}^{\infty}\left\{\mathbb{Q}_n(\mathbf{E})B_k\right\}\frac{t^k}{k!}. \tag{4.12}$$

According to (4.6) we conclude that $t \mapsto \mathcal{M}_n(t)$ is a generating function for numbers (moments) $\{\mu_k^{(n)}\}_{k=0}^{\infty}$. It can be expressed in terms of the hypergeometric function ${}_2F_1$ in the form

$$\mathcal{M}_n(t) = \int_0^1 x^n G(x,t)\,\mathrm{d}x = \frac{\mathrm{e}^{-t}}{n+1}\,{}_2F_1\big(1, n+1; n+2; 1-\mathrm{e}^{-t}\big). \tag{4.13}$$

In particular,

$$\mathcal{M}_1(t) = \frac{(t-1)\mathrm{e}^t + 1}{(\mathrm{e}^t-1)^2}, \quad \mathcal{M}_2(t) = \frac{1}{2}\frac{(2t-3)\mathrm{e}^{2t} + 4\mathrm{e}^t - 1}{(\mathrm{e}^t-1)^3},$$
$$\mathcal{M}_3(t) = \frac{1}{6}\frac{(6t-11)\mathrm{e}^{3t} + 18\mathrm{e}^{2t} - 9\mathrm{e}^t + 2}{(\mathrm{e}^t-1)^4},$$
$$\mathcal{M}_4(t) = \frac{1}{12}\frac{(12t-25)\mathrm{e}^{4t} + 48\mathrm{e}^{3t} - 36\mathrm{e}^{2t} + 16\mathrm{e}^t - 3}{(\mathrm{e}^t-1)^5},$$
$$\mathcal{M}_5(t) = \frac{1}{60}\frac{(60t-137)\mathrm{e}^{5t} + 300\mathrm{e}^{4t} - 300\mathrm{e}^{3t} + 200\mathrm{e}^{2t} - 75\mathrm{e}^t + 12}{(\mathrm{e}^t-1)^6}, \quad \text{etc.}$$

Using Theorem 4.4, as well as the relation (4.12), for the representation (4.13) of $\mathcal{M}_n(t)$ we obtain the following result.

Theorem 4.5 *For each $n \in \mathbb{N}$ we have*

$$\mathcal{M}_n(t) = \frac{\mathrm{e}^{-t}}{n+1}\,{}_2F_1\big(1, n+1; n+2; 1-\mathrm{e}^{-t}\big) = \frac{(-1)^n}{n!}\mathbb{Q}_n(\mathbf{D})\varepsilon(t).$$

For example, for $n \le 6$ we have

$$\mathcal{M}_0(t) = \varepsilon(t),$$
$$\mathcal{M}_1(t) = -\varepsilon'(t),$$
$$\mathcal{M}_2(t) = \frac{1}{2}\big(\varepsilon''(t) - \varepsilon'(t)\big),$$
$$\mathcal{M}_3(t) = -\frac{1}{6}\big(\varepsilon'''(t) - 3\varepsilon''(t) + 2\varepsilon'(t)\big),$$
$$\mathcal{M}_4(t) = \frac{1}{24}\big(\varepsilon^{iv}(t) - 6\varepsilon'''(t) + 11\varepsilon''(t) - 6\varepsilon'(t)\big),$$
$$\mathcal{M}_5(t) = -\frac{1}{120}\big(\varepsilon^{v}(t) - 10\varepsilon^{iv}(t) + 35\varepsilon'''(t) - 50\varepsilon''(t) + 24\varepsilon'(t)\big),$$
$$\mathcal{M}_6(t) = \frac{1}{720}\big(\varepsilon^{vi}(t) - 15\varepsilon^{v}(t) + 85\varepsilon^{iv}(t) - 225\varepsilon'''(t) + 274\varepsilon''(t) - 120\varepsilon'(t)\big).$$

At the end of this section we give a simple summation formula for Bernoulli's numbers:

Theorem 4.6 *For each* $n \in \mathbb{N}_0$ *we have*

$$\sum_{j=0}^{k+1} \binom{k+1}{j} B_{j+n} = \begin{cases} B_n + B_{n+1}, & k = 0, \\ \mathbf{\Delta}^n B_{k+1}, & k \geq 1. \end{cases} \tag{4.14}$$

Proof For $n = 0$ the equality (4.14) reduces to the well known identity (4.9) (cf. [1], p. 241). In fact, it is a special case of Theorem 4.3 for $n = 0$. Also, for $n = 1$ (4.8) becomes

$$\sum_{j=0}^{k+1} \binom{k+1}{j} \mathbf{E} B_j = \begin{cases} (\mathbf{E} + \mathbf{E}^2) B_0 = B_1 + B_2, & k = 0, \\ (\mathbf{E}^2 - \mathbf{E}) B_k = B_{k+2} - B_{k+1}, & k = 1, 2, \dots \end{cases},$$

which is equivalent to (4.14) for $n = 1$. Thus, we should prove (4.14) for $n \geq 2$. The case $k = 0$ is trivial. Therefore, we suppose $k \geq 1$, and as in the proof of Theorem 4.3, we apply induction on n.

Suppose that (4.14) is true for some $n \in \mathbb{N}$, i.e.,

$$\sum_{j=0}^{k+1} \binom{k+1}{j} B_{j+n} = \mathbf{\Delta}^n B_{k+1}, \quad k \geq 1.$$

Then, from this equality (written for $k + 2$ and $k + 1$) we get

$$\sum_{j=0}^{k+2} \binom{k+2}{j} B_{j+n} - \sum_{j=0}^{k+1} \binom{k+1}{j} B_{j+n} = \mathbf{\Delta}^n B_{k+2} - \mathbf{\Delta}^n B_{k+1},$$

i.e.,

$$\sum_{j=1}^{k+1} \binom{k+1}{j-1} B_{j+n} + B_{k+2+n} = \sum_{j=0}^{k+1} \binom{k+1}{j} B_{j+n+1} = \mathbf{\Delta}^{n+1} B_{k+1}, \quad k \geq 1.$$

This complete the proof. □

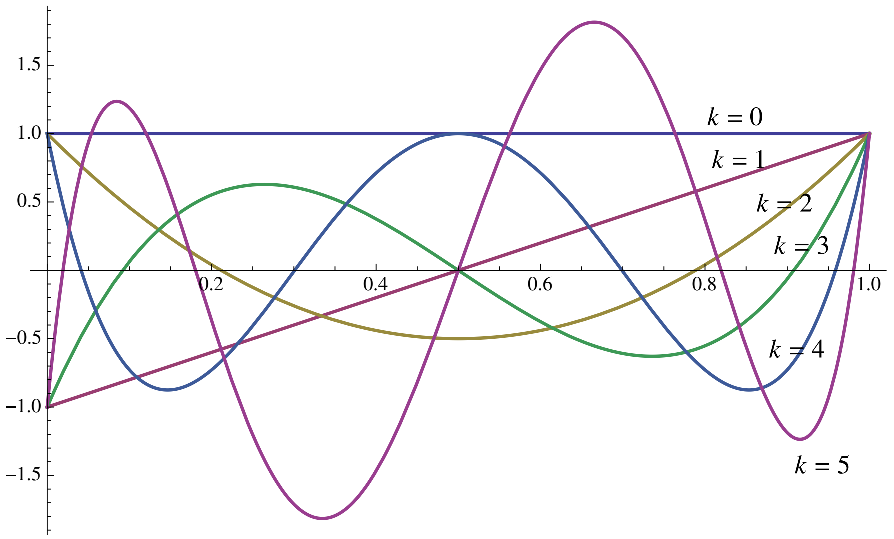

Fig. 1 Graphics of the polynomials $P_k(x)$, $0 \le k \le 5$, on (0, 1)

5 Properties of polynomials $P_k(x)$ and their zero distribution

In this section we investigate the polynomials $P_k(x)$ defined by (1.1), i.e.,

$$P_k(x) = \frac{M_{k+1}(x)}{x-1} = \frac{L_{k+2}(x)}{x(x-1)}, \quad k \ge 0, \tag{5.1}$$

which graphics for $0 \le k \le 5$ are presented in Fig. 1. These polynomials are symmetric with respect to $x = 1/2$, i.e., $P_k(1-x) = (-1)^k P_k(x)$. Another interesting property of polynomials $P_k(x)$, $k \ge 1$, is that all their zeros are real, simple and contained in (0, 1).

Theorem 5.1 *Polynomals $L_k(x)$ ($k \in \mathbb{N}$) have k simple real zeros symmetrically distributed in* [0,1]. *For $k \ge 2$ the endpoints 0 and 1 are zeros of $L_k(x)$.*

Proof We prove this statement by induction.

For $k = 2$, the zeros of $L_2(x) = x^2 - x$ are the endpoints 0 and 1. According to (3.3), these points are also zeros of $L_k(x)$ for each $k \ge 3$.

Suposse now that $k \ge 3$ and that the polynomial $L_k(x)$ has k simple zeros ξ_k in [0,1], such that $0 = \xi_1 < \xi_1 \cdots < \xi_{k-1} < \xi_k = 1$, i.e.,

$$L_k(x) = a_k(x-\xi_1)(x-\xi_2)\cdots(x-\xi_{k-1})(x-\xi_k), \quad a_k \ne 0.$$

According to Rolle's theorem, in each of intervals $(\xi_{\nu-1}, \xi_\nu)$, $\nu = 2, \ldots, k$, there is at least one zero $\eta_\nu \in (\xi_{\nu-1}, \xi_\nu)$ of $L'_k(x)$. Since the number of these intervals is $k-1$

($= \deg L_k'$), we conclude that these zeros are simple and the derivative $L_k'(x)$ can be expressed as

$$L_k'(x) = ka_k(x - \eta_2) \cdots (x - \eta_k). \tag{5.2}$$

Now, using (3.3) and (5.2), we conclude that

$$L_{k+1}(x) = a_{k+1}(x - \eta_1)(x - \eta_2) \cdots (x - \eta_k)(x - \eta_{k+1}), \quad a_{k+1} = ka_k \neq 0.$$

Thus, $L_{k+1}(x)$ has $k + 1$ simple zeros, such that $0 = \eta_1 < \eta_2 < \cdots < \eta_k < \eta_{k+1} = 1$.

Due to Lemma 3.1, this zero distribution is symmetric with respect to the point $x = 1/2$. □

Corollary 5.1 *Polynomials $P_k(x)$ ($k \in \mathbb{N}$), defined by* (5.1), *have k simple real zeros symmetrically distributed in* (0, 1).

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval Not applicable.

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