Some inequalities for symmetric functions and an application to orthogonal polynomials

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Abstract
We present some sharp inequalities for symmetric functions and give an application to orthogonal polynomials.
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1. Introduction

Symmetric functions are important in several branches of mathematics, especially in approximation theory, probability theory, combinatorics and algebra, and they have many applications in different areas (see [5, Chapter 1] for details about symmetric functions).

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Let $Q(x)$ be a polynomial of degree $n (\in \mathbb{N})$ with zeros $\lambda_\nu, \nu = 1, \ldots, n$, i.e.,

$$Q(x) = C \prod_{k=1}^{n} (x - \lambda_\nu), \quad C \neq 0. \quad (1.1)$$

It is well known that the coefficients of the polynomial (1.1) can be represented, using symmetric functions, in the following form:

$$Q(x) = C\left(x^n - \sigma_{n,1}x^{n-1} + \sigma_{n,2}x^{n-2} - \cdots + (-1)^n \sigma_{n,n}\right),$$

where $\sigma_{n,k}, k=1,\ldots,n$, are the so-called elementary symmetric functions,

$$\sigma_{n,k} = \sum_{(i_1,\ldots,i_k)} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n,$$

and where the summation is performed over all combinations $(i_1,\ldots,i_k)$ of the basic set \{1,\ldots,n\}. Thus,

$$\sigma_{n,1} = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \quad \sigma_{n,2} = \lambda_1\lambda_2 + \cdots + \lambda_{n-1}\lambda_n,$$

$$\sigma_{n,n} = \lambda_1\lambda_2\cdots\lambda_n.$$ 

For the convenience we put $\sigma_{n,0} = 1$ and $\sigma_{n,k} = 0$, $k > n$ or $k < 0$. When we want to refer to the all elementary symmetric functions, we use notation $\sigma_n = (\sigma_{n,0}, \ldots, \sigma_{n,n})$, where $\sigma_n$ represents a vector with $n+1$ components.

There are several classical inequalities with symmetric functions (cf. [3,6,8,12,13,15–17]). For some recent results see [1,4,7,11,14]. For example, some general results on the positivity of symmetric functions have been recently obtained by Timofte [14].

In this paper we present the positivity for a special family of symmetric polynomials $p^n_k(\sigma_n)$ and give some applications to orthogonal polynomials. The paper is organized as follows. The main inequality $p^n_k(\sigma_n) > 0$ (Theorem 2.3) is stated in Section 2 and its proof is given in Section 3. A determinant representation of $p^n_k(\sigma_n)$ is presented in Section 4 and some special cases are analyzed in Section 5. Finally, Section 6 is devoted to some applications to linear functionals and orthogonal polynomials.

2. Inequalities

In this paper we assume that all zeros of the polynomial (1.1) are positive, i.e.,

$$\lambda_\nu > 0, \quad \nu = 1, \ldots, n.$$

Let the derivatives of $Q(x)$ at the point zero be denoted by $Q^{(k)}(0)$, i.e.,

$$Q^{(k)}(0) = \frac{d^k Q(x)}{dx^k} \bigg|_{x=0}, \quad k \in \mathbb{N}_0.$$ 

Obviously, we have

$$Q^{(k)}(0) = (-1)^{n-k}k!C\sigma_{n,n-k}, \quad k = 0, 1, \ldots, n. \quad (2.1)$$
We also define the sequence
\[ Q_k = \frac{1}{k!} \frac{d^k}{dx^k} \frac{1}{Q(x)} \bigg|_{x=0}, \quad k \in \mathbb{N}_0. \]

**Lemma 2.1.** The sequence \( Q_k, k \in \mathbb{N}_0, \) satisfies the following recurrence relation:
\[
\sigma_{n,n} Q_k = \left( -1 \right)^n \delta_{k,0} + \left( -1 \right)^{k-1} \sum_{v=\max\{0,k-n\}}^{k-1} (-1)^v \sigma_{n,n-k+v} Q_v, \quad k \in \mathbb{N}_0. \tag{2.2}
\]

If the sum is empty, we consider it to be zero.

**Proof.** Put \( f(x) = Q(x) \) and \( g(x) = 1/Q(x) \), obviously we have \( fg = 1 \). If we apply the Leibnitz rule for the derivative of a product, we get
\[
\frac{d^k}{dx^k} (fg) \bigg|_{x=0} = \sum_{v=0}^{k} \binom{k}{v} f^{(k-v)}(0) g^{(v)}(0) = \delta_{k,0}.
\]
Substituting \( f^{(v)}(0) = Q^{(v)}(0), \ g^{(v)}(0) = Q_v, \) and using (2.1), the previous equation reduces to
\[
\sigma_{n,n} Q_k = \left( -1 \right)^n \delta_{k,0} + \left( -1 \right)^{k-1} \sum_{v=0}^{k-1} (-1)^v \sigma_{n,n-k+v} Q_v, \quad k \in \mathbb{N}_0.
\]
According to the fact that \( \sigma_{n,k} = 0 \) for \( k < 0 \), we can truncate the summation in the previous form and so we get (2.2).

Equality in (2.2) holds even for the choice \( k = 0 \), in which case it reduces to \( Q_0 = 1/Q(0) = (-1)^n/(C \sigma_{n,n}) \). \( \square \)

For \( k > 0 \), in (2.2) we have the homogenous difference equation (\( \delta_{k,0} = 0 \)), which generates the solution \( \sigma_{n,n}^k Q_k = p_k^n(\sigma_n) Q_0 \), where \( p_k^n(\sigma_n), \ k \in \mathbb{N}, \) is a polynomial in \( \sigma_{n,0}, \ldots, \sigma_{n,n} \).

Now, we can state the following result:

**Lemma 2.2.** The solution of the difference equation (2.2) admits a representation of the following form:
\[
\sigma_{n,n}^k Q_k = p_k^n(\sigma_n) Q_0, \quad k \in \mathbb{N},
\]
where \( p_k^n \) is a polynomial of degree \( k \) of the elementary symmetric functions \( \sigma_{n,v}, \ v = 0, 1, \ldots, n. \)

**Proof.** For \( k = 1 \) the statement is obvious, since
\[
\sigma_{n,n} Q_1 = \sigma_{n,n-1} Q_0 \quad \text{and} \quad p_1^n(\sigma_n) = \sigma_{n,n-1}.
\]
Assuming it is true for \( Q_1, \) we are able to prove the statement for \( Q_2, \) since
\[
\sigma_{n,n}^2 Q_2 = -\sum_{v=0}^{1} (-1)^v \sigma_{n,n-k+v} \sigma_{n,n} Q_v \\
= -\left(\sigma_{n,n} \sigma_{n,n-2} - \sigma_{n,n-1} p_1^0(\sigma_n)\right) Q_0 = p_2^0(\sigma_n) Q_0,
\]

where \( p_1^0(\sigma_n) = \sigma_{n,n-1}^2 - \sigma_{n,n-2}. \) Repeating the same arguments, we can prove that our statement holds for \( k \leq n. \) Starting from that point, we can apply the induction.

Assuming that statement holds for \( Q_{k-\nu}, \ldots, Q_{k-1}, \) we prove that it is true for \( Q_k, \) since

\[
\sigma_{n,n}^k Q_k = (-1)^{k-1} \sum_{v=k-n}^{k-1} (-1)^v \sigma_{n,n}^{k-v-1} \sigma_{n,n-k+v} \sigma_{n,n}^v Q_v \\
= (-1)^{k-1} Q_0 \sum_{v=k-n}^{k-1} (-1)^v \sigma_{n,n}^{k-v-1} \sigma_{n,n-k+v} p_0^n(\sigma_n) = p_k^n(\sigma_n) Q_0,
\]

where obviously we have

\[
p_k^n(\sigma_n) = (-1)^k \sum_{v=k-n}^{k-1} (-1)^v \sigma_{n,n}^{k-v-1} \sigma_{n,n-k+v} p_0^n(\sigma_n), \quad k > n. \quad \square
\]

Adopting \( p_0^0(\sigma_n) = 1 \) and \( p_k^n(\sigma_n) = 0, \) \( k < 0, \) we rewrite the recurrence (2.2) for the sequence \( Q_k, k \in \mathbb{N}_0, \) into the recurrence for the sequence \( p_k^n(\sigma_n), k \in \mathbb{N}_0. \) Thus, we have

\[
p_k^n(\sigma_n) = \delta_{k,0} + (-1)^{k-1} \sum_{v=\max\{0,k-n\}}^{k-1} (-1)^v \sigma_{n,n}^{k-v-1} \sigma_{n,n-k+v} p_0^n(\sigma_n), \quad k \in \mathbb{N}_0.
\]

Using the previously defined quantities, we can state our main result.

**Theorem 2.3.** Provided all zeros \( \lambda_{\nu}, \nu = 1, \ldots, n, \) counting multiplicities, of the polynomial \( Q \) are positive, we have

\[
p_k^n(\sigma_n) > 0, \quad k \in \mathbb{N}_0.
\]

As an illustration, we give values of the polynomials \( p_k^n, k \in \mathbb{N}, \) for the case when the polynomial \( Q \) has only two zeros. Thus, we have the following statement:

**Theorem 2.4.** Suppose that the polynomial \( Q \) is of the second degree, then

\[
p_k^2(\sigma_2) = \sum_{v=0}^{[k/2]} a_{2,v}^k \sigma_{2,1}^{k-2v} (\sigma_{2,0}\sigma_{2,2})^v, \quad k \in \mathbb{N}_0.
\]

where the coefficients \( a_{2,v}^k, v, k \in \mathbb{N}_0, \) satisfy the following recurrences:
\[ a_{2,v}^k = a_{2,v}^{k-1} - a_{2,v}^{k-2}, \quad v = 1, \ldots, [k/2] - 1, \quad a_{2,0}^k = 1, \quad k \in \mathbb{N}_0. \]
\[ a_{2,v}^{2v} = 0, \quad a_{2,v}^{2v+1} = 0, \quad v \in \mathbb{N}_0, \quad a_{2,v}^k = 0, \quad v \notin \{0, 1, \ldots, [k/2]\}. \]

(2.6)

Moreover,
\[ a_{2,v}^k = (-1)^v \binom{k}{v}, \quad k \geq 2v, \quad a_{2,v}^k = 0, \quad k < 2v. \]

(2.7)

3. Proof of Theorem 2.3

We assume that the polynomial \( Q \) has \( M \) distinct zeros, denoted by \( \mu_v, v = 1, \ldots, M \). Their multiplicities are denoted by \( M_v \), respectively, where

\[ \sum_{v=1}^{M} M_v = n. \]

Proof of Theorem 2.3. Obviously, in the case \( n = 1 \), the polynomial \( Q \) has only one simple zero. There is nothing to prove, since using (2.3), we can calculate

\[ p_1^k(\sigma_1) = 1 > 0 \]

and (2.4) holds.

In the sequel, we assume \( n > 1 \). First, we assume that for the zero \( \mu_1 \) we have multiplicity \( M_1 \geq 2 \). Consider now the following polynomials:

\[ P_k(x) = 1 + \int_0^x \frac{Q(t)}{\prod_{v=1}^{M}(t-\mu_v)} \frac{q(t)}{t-\mu_1} t^k \, dt, \quad q \in \mathcal{P}_{M-1}, \ k \in \mathbb{N}. \]

(3.1)

For different polynomials \( q \in \mathcal{P}_{M-1} \) we have different polynomials \( P_k \). For example, taking the special case \( q \equiv 0 \), we have \( P_k \equiv 1 \).

For a polynomial \( q \in \mathcal{P}_{M-1} \) which is not identically zero, since

\[ P_k'(x) = \frac{Q(x)}{\prod_{v=1}^{M}(x-\mu_v)} \frac{q(x)}{x-\mu_1} x^k, \]

we conclude that \( P_k' \) has zeros at \( \mu_v, v = 1, \ldots, M \), of the multiplicities \( M_v - 1 - \delta_{v,1} \), \( v = 1, \ldots, M \), respectively, and a zero at the point zero of the multiplicity \( k \). It is also easy to verify that the degree of \( P_k \) is \( n + k - 1 \).

We show that the system of equations

\[ P_k(\mu_v) = 1 + \int_0^{\mu_v} \frac{Q(t)}{\prod_{v=1}^{M}(t-\mu_v)} \frac{q(t)}{t-\mu_1} t^k \, dt = 0, \quad v = 1, \ldots, n, \]

(3.2)

has a solution \( q \in \mathcal{P}_{M-1} \).

\[
\begin{align*}
\frac{a_{2,v}^k}{a_{2,v}^{k-1} - a_{2,v}^{k-2}}, & \quad v = 1, \ldots, [k/2] - 1, \\
\frac{a_{2,v}^{2v}}{a_{2,v}^{2v+1}}, & \quad v \in \mathbb{N}_0, \\
\frac{a_{2,v}^k}{a_{2,v}^{k-1} - a_{2,v}^{k-2}}, & \quad v \notin \{0, 1, \ldots, [k/2]\}.
\end{align*}
\]

(2.6)

Moreover,
\[
\frac{a_{2,v}^k}{(-1)^v \binom{k}{v}}, \quad k \geq 2v, \quad a_{2,v}^k = 0, \quad k < 2v.
\]

(2.7)
First, note that we can rewrite this system of equations in the form

$$\int_{\mu_v}^{\mu_{v+1}} \frac{Q(t) - q(t)}{\prod_{v=1}^{M}(t - \mu_v)} \frac{t^k \, dt}{t - \mu_1} = -\delta_{v,0}, \quad v = 0, 1, \ldots, M - 1, \tag{3.3}$$

where we use the convention $\mu_0 = 0$. To prove that the system (3.3) has a unique solution, it is enough to prove that the corresponding homogeneous system

$$\int_{\mu_v}^{\mu_{v+1}} \frac{Q(t) - q(t)}{\prod_{v=1}^{M}(t - \mu_v)} \frac{t^k \, dt}{t - \mu_1} = 0, \quad v = 0, 1, \ldots, M - 1, \tag{3.4}$$

has only the trivial solution $q \equiv 0$ in $\mathcal{P}_{M-1}$. Note that the polynomial

$$\frac{Q(t) - q(t)}{\prod_{v=1}^{M}(t - \mu_v)} \frac{t^k \, dt}{t - \mu_1},$$

has a constant sign on the intervals $(\mu_v, \mu_{v+1})$, $v = 0, 1, \ldots, M - 1$, since it has no zeros in these intervals. Therefore, the homogenous equations (3.4) imply that polynomial $q$ must have at least one zero in each of the intervals $(\mu_v, \mu_{v+1})$, $v = 0, 1, \ldots, M - 1$. This means that the polynomial $q$ must have at least $M$ zeros, the only polynomial from $\mathcal{P}_{M-1}$ satisfying this condition is, of course a polynomial which is identically zero.

This means that the system of equations (3.2) has a unique solution $q \in \mathcal{P}_{M-1}$. We denote that solution $q^\ast$. So that there exists (uniquely) polynomial $P_k$ of the form (3.1), which has zeros at $\mu_v$, $v = 1, \ldots, M$, of the order $M_v - \delta_{v,1}$, with $v = 1, \ldots, M$, denoted here $P_k^\ast$.

For the polynomial $(P_k^\ast)'$, we know that it has zeros of the order $M_v - 1 - \delta_{v,1}$ at the points $\mu_v$, $v = 1, \ldots, M$, and a zero of degree $k$ at the point zero. Since it is of degree $k + n - 2$, there are $M - 1$ more zeros those are zeros of $q^\ast$. Using Role’s theorem, we know that $(P_k^\ast)'$ must have at least one zero in each interval $(\mu_v, \mu_{v+1})$, $v = 1, \ldots, M - 1$, since the polynomial $P_k^\ast$ has zeros at the points $\mu_v$, $v = 1, \ldots, M$. There are $M - 1$ such zeros, so that the zeros of $q^\ast \zeta_v$, $v = 1, \ldots, M - 1$, are simple and belong to the intervals $\zeta_v \in (\mu_v, \mu_{v+1})$, $v = 1, \ldots, M - 1$. Since the polynomial

$$\frac{Q(x) - x^k \, dt}{\prod_{v=1}^{M}(t - \mu_v)} \frac{q(x)}{x - \mu_1},$$

does not have any zeros in the interval $(0, \mu_1)$, it is of a constant sign there, $P_k^\ast$ is also of the positive sign on the interval $(0, \mu_1)$. If it is not the case, then since $P_k^\ast(0) = 1$, there is at least one zero of the polynomial $P_k^\ast$ in the interval $(0, \mu_1)$ suppose it is the point $\zeta$. Then, according to the Role’s theorem, there must be at least one zero of the polynomial $(P_k^\ast)'$ in the interval $(\zeta, \mu_1)$, but this is a contradiction. Thus, the polynomial $P_k^\ast$ is of the positive sign on the interval $(0, \mu_1)$.

Obviously, $P_k^\ast$ cannot have zero in the interval $(\mu_1, \mu_2)$, if it does then $(P_k^\ast)'$ must have two zeros in the interval $(\mu_1, \mu_2)$ and those must be zeros of $q^\ast$, which is a contradiction. This leads to an observation that the polynomial $P_k^\ast(x)/(x - \mu_1)^{M-1}$ has a constant sign on the interval $(0, \mu_2)$ and that sign is $(-1)^{M_1-1}$.

Consider now the rational function $P_k/(x^{k+1}Q)$. It has poles of order $k + 1$ at the point zero and of order $M_v$ at points $\mu_v$, $v = 1, \ldots, M$. When $x$ approaches the complex infinity,
we have $P_k/(x^{k+1}Q) = O(x^{-2})$. Applying the Cauchy residue theorem to the function $P_k/(x^{k+1}Q)$, over the contour which has in its interior $\{0, \mu_1, \ldots, \mu_M\}$, we have

$$\frac{1}{k!} \left( \frac{P_k}{Q} \right)^{(k)} \bigg|_{x=0} = -\sum_{\nu=1}^{M} \text{Res}_{x=\mu_\nu} \frac{P_k(x)}{x^{k+1}Q(x)}. \quad (3.5)$$

But since polynomial $P_k$ is of the form (3.1), we know that $P_k^{(v)}(0) = \delta_{v,0}, \nu = 0, 1, \ldots, k$. Using the Leibnitz rule, we get

$$\frac{1}{k!} \left( \frac{P_k}{Q} \right)^{(k)} \bigg|_{x=0} = -\sum_{\nu=1}^{k} \binom{k}{\nu} P_k^{(\nu)}(x) \frac{1}{Q(x)} \bigg|_{x=0} = Q_k. \quad (3.5)$$

Thus, for every polynomial $P_k$ of the form (3.1), we have

$$\frac{1}{k!} \left( \frac{P_k}{Q} \right)^{(k)} \bigg|_{x=0} = Q_k. \quad (3.6)$$

Now, choose $P_k = P_k^*$, using (3.5), we have

$$Q_k = -\frac{1}{\mu_1^{k+1}} \frac{(x - \mu_1)P_k^*(x)}{Q(x)} \bigg|_{x=\mu_1}. \quad (3.6)$$

This equation can be rewritten in the form

$$p_k^n(\sigma_n) = \frac{\sigma_n^{k+1} Q_k}{Q_0} = \frac{\sigma_n^{k+1}}{\mu_1^{k+1}} \frac{P_k^*(x)}{(x - \mu_1)^{M_1}Q(x)} \bigg|_{x=\mu_1} = \frac{1}{\mu_1^{k+1}} \frac{P_k^*(x)}{(x - \mu_1)^{M_1-1}Q(x)} \bigg|_{x=\mu_1} \prod_{\nu=2}^{M} (\mu_1 - \mu_\nu)^{M_\nu}. \quad (3.7)$$

Taking only the sign of the terms, from this equation we get

$$\text{sgn}(p_k^n(\sigma_n)) = \frac{P_k^*(x)}{(x - \mu_1)^{M_1-1}Q(x)} \bigg|_{x=\mu_1} \prod_{\nu=2}^{M} \text{sgn}(\mu_1 - \mu_\nu)^{M_\nu}.$$
Also, it is easy to show that all $M - 2$ zeros of $q^*$ are contained in the intervals $(\mu_v, \mu_{v+1})$, $v = 2, \ldots, M - 1$, and that the polynomial $P_k^*$ has the positive sign on the interval $(0, \mu_2)$. Using (3.5), we find

$$Q_k = -\frac{1}{\mu_{k+1}^{k+1}} \frac{P_k^*(\mu_1)}{Q'*(\mu_1)},$$

which gives

$$\text{sgn}(p_k^n(\sigma_n)) = (-1)^n \frac{1}{\prod_{v=2}^M \text{sgn}(\mu_1 - \mu_v)M_v} = (-1)^n + \sum_{v=1}^M M_v = 1,$$

where we used the fact that $M_1 = 1$. This proves (2.4), also for the case $M_1 = 1$. □

There is one simple generalization of Theorem 2.3. Suppose all zeros of the polynomial $Q$ are bigger than $\zeta$, i.e., $\zeta < \lambda_v, v = 1, \ldots, n$, then we have

$$Q(x) = C \prod_{v=1}^n (x - \lambda_v) = C \prod_{v=1}^n (x - \zeta - (\lambda_v - \zeta)) = C \prod_{v=1}^n (y - \lambda_v^*) = Q^*(y),$$

where we put $\lambda_v^* = \lambda_v - \zeta$, $v = 1, \ldots, n$, and $y = x - \zeta$. We can express elementary symmetric functions of the polynomial $Q^*$ using elementary symmetric functions of the polynomial $Q$; we have

$$\sigma_{n,k}^* = \sum_{(i_1, \ldots, i_k)} (\lambda_{i_1}^* - \zeta) \cdots (\lambda_{i_k}^* - \zeta),$$

i.e.,

$$\sigma_{n,k}^* = \sum_{j=0}^k (-1)^{k-j} \binom{n-j}{k-j} \zeta^{k-j} \sigma_{n,j}.$$  (3.8)

We can apply Theorem 2.3 to the polynomial $Q^*$, since $\lambda_v^* > 0, v = 1, \ldots, n$. Therefore,

$$p_k^n(\sigma_n^*) > 0, \quad k \in \mathbb{N}_0,$$

and

$$p_k^n(\sigma_n^*) = \delta_{k,0} + (-1)^{k-1} \sum_{v=\max\{0, k-n\}}^{k-1} (-1)^v (\sigma_{n,n}^*)^{k-n-1} \sigma_{n,n-k}^* p_v^n(\sigma_n^*),$$

where $\sigma_n^* = (\sigma_{n,0}^*, \ldots, \sigma_{n,n}^*)$ and $\sigma_n = (\sigma_{n,0}, \ldots, \sigma_{n,n})$. Using (3.8), we get

$$p_k^n(\sigma_n^*) = p_k^n(\sigma_n^*) > 0$$

and the recurrence relation for the polynomials $p_k^n(\sigma_n)$ is given by

$$p_k^n(\sigma_n) = \delta_{k,0} + (-1)^{k-1} \sum_{v=\max\{0, k-n\}}^{k-1} (-1)^v \left( \sum_{j=0}^n (-1)^{n-j} \zeta^{n-j} \sigma_{n,j} \right)^{k-v-1}.$$
\[
\times \left( \sum_{j=0}^{n-k+\nu} (-1)^{n-k+\nu-j} \binom{n-j}{n-k+\nu-j} \sigma_{n,j} \right) p_v^{n,\zeta} (\sigma_n). \tag{3.9}
\]

We have proved the following result:

**Theorem 3.1.** If the zeros of the polynomial $Q$ are bigger than $\zeta$, then
\[
p_{k}^{n,\zeta} (\sigma_n) > 0, \quad k \in \mathbb{N}_0,
\]
where the polynomials $p_{k}^{n,\zeta} (\sigma_n), k \in \mathbb{N}_0$, are generated using the recurrence (3.9).

In the case $\zeta = 0$, we have $p_{k}^{n,0} (\sigma_n) = p_{k}^{n} (\sigma_n)$.

4. Determinant representation of $p_{k}^{n}$

It is not surprising that our polynomials $p_{k}^{n} (\sigma_n)$ can be represented in a determinant form. Namely, we have the following result:

**Theorem 4.1.** The polynomial $p_{k}^{n} (\sigma_n), k \in \mathbb{N}$, admits the following determinant representation:
\[
p_{k}^{n} (\sigma_n) = \begin{vmatrix}
\beta_0 & -1 & & & & & & & O \\
\beta_1 & \beta_0 & -1 & & & & & & \\
\beta_2 & \beta_1 & \beta_0 & -1 & & & & & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 & -1 & & & \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
& & & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 & -1 \\
& & & & O & & & & \\
& & & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 & \\
\end{vmatrix}, \tag{4.1}
\]

where $\beta_v = (-1)^v \sigma_{n,n}^{v} \sigma_{n,n-v-1}, v = 0, 1, \ldots, n - 1$.

**Proof.** Introducing $\beta_v = (-1)^v \sigma_{n,n}^{v} \sigma_{n,n-v-1}, v = 0, 1, \ldots, n - 1$, and $\beta_{-1} = -1$, the recurrence relation (2.3) becomes
\[
\sum_{v=\max(0,i-n)}^{i} \beta_{i-v-1} p_v^{n} (\sigma_n) = -\delta_{i,0}, \quad i \in \mathbb{N}_0.
\]
For \( i = 0, 1, \ldots, k, k \geq n \), it gives the following system of linear equations:

\[
\begin{align*}
-1 & = -1, \\
0 & = 0, \\
0 & = 0, \\
\vdots & : \\
0 & = 0, \\
0 & = 0, \\
\vdots & : \\
0 & = 0.
\end{align*}
\]

Since the determinant of this system is equal to \((\beta_{-1})^k = (-1)^k \neq 0\), we can solve it for \(p^n_k(\sigma_n)\) using Cramer’s rule, which leads to the following determinant representation:

\[
p^n_k(\sigma_n) = \frac{1}{(-1)^k} \left| \begin{array}{cccc}
\beta_{-1} & 1 & & \\
\beta_0 & \beta_{-1} & & \\
\beta_1 & \beta_0 & \beta_{-1} & \\
\vdots & \ddots & \ddots & \\
\beta_{n-1} & \cdots & \beta_1 & \beta_0 & \beta_{-1} \\
\beta_{n-1} & \cdots & \beta_1 & \beta_0 & 0 \\
\beta_{n-1} & \cdots & \beta_1 & 0 & 0
\end{array} \right|.
\]

Expanding this determinant with respect to the last column, we get (4.1). \(\square\)

5. Special cases

5.1. Case of a single zero of multiplicity \(n\)

Suppose that polynomial \(Q\) has a single zero \(\lambda_1\) of multiplicity \(n\), i.e., let \(Q(x) = (x - \lambda_1)^n\). In this special case, the elementary symmetric functions \(\sigma_{n,k}\) have the following values:

\[
\sigma_{n,k} = \binom{n}{k} \lambda_1^k, \quad k \in \mathbb{N}_0,
\]

which is verified easily recalling the definition of the elementary symmetric functions and recalling the number of combinations of \(n\) elements of \(k\)th class.

**Theorem 5.1.** If the polynomial \(Q\) has a single zero \(\lambda_1\) of multiplicity \(n\), then

\[
p^n_k(\sigma_n) = \binom{k + n - 1}{n - 1} \lambda_1^{k(n-1)}.
\]

**Proof.** Obviously, for \(k = 0\), there is nothing to prove since our statement becomes

\[
p^n_0(\sigma_n) = \binom{0 + n - 1}{n - 1} \lambda_1^{0(n-1)} = 1.
\]
Thus, we assume $k > 0$. If we replace values for $p_{k}^{n}(\sigma_{n})$ given in the statement into (2.3), we get

$$\binom{k + n - 1}{n - 1} = (-1)^{k-1} \sum_{\nu = \max\{0, k-n\}}^{k-1} (-1)^{\nu} \binom{n}{n-k+\nu} \binom{\nu + n - 1}{n - 1},$$

which can be reduced to

$$\sum_{\nu = \max\{0, k-n\}}^{k} (-1)^{\nu} \binom{n}{k-\nu} \binom{\nu + n - 1}{n - 1} = 0. \quad (5.1)$$

Using an expansion of the geometric series, it is easy to verify the following expansion:

$$\frac{(1 + x)^n}{(n - 1)!} d^{n-1} \frac{x^{n-1}}{1 + x} = \sum_{\ell = 0}^{+\infty} x^{\ell} \sum_{\nu = \max\{0, \ell - n\}}^{\ell} (-1)^{\nu} \binom{n}{\ell - \nu} \binom{\nu + n - 1}{n - 1}. \quad (5.2)$$

However, using the Bézout’s theorem, we get

$$\frac{d^{n-1}}{dx^{n-1}} \frac{x^{n-1}}{1 + x} = \frac{d^{n-1}}{dx^{n-1}} \frac{r_{n-2} + (-1)^{n-1}}{1 + x} = \frac{d^{n-1}}{dx^{n-1}} \frac{(-1)^{n-1}}{1 + x} = \frac{(n - 1)!}{(1 + x)^n},$$

since $r_{n-2}$ is a polynomial of degree $n - 2$. Using this fact, (5.2) is transformed into

$$1 = \sum_{\ell = 0}^{+\infty} x^{\ell} \sum_{\nu = \max\{0, \ell - n\}}^{\ell} (-1)^{\nu} \binom{n}{\ell - \nu} \binom{\nu + n - 1}{n - 1},$$

which means that all coefficients with $x^{\ell}, \ell \in \mathbb{N}$, on the right-hand side must be zero, i.e., (5.1) holds for $k \in \mathbb{N}$. \hfill \Box

### 5.2. Case $n = 2$

In the case $n = 2$, we already stated Theorem 2.4 in Section 2. We give now a proof of this theorem.

**Proof of Theorem 2.4.** In the case $n = 2$, the representation (4.1) reduces to a determinant of a tridiagonal matrix. Namely,

$$p_{k}^{2}(\sigma_{2}) = \begin{vmatrix} \beta_{0} & -1 & \text{O} \\ \beta_{1} & \beta_{0} & -1 \\ \text{O} & \beta_{1} & \beta_{0} \end{vmatrix},$$

where $\beta_{0} = \sigma_{2,1}$ and $\beta_{1} = -\sigma_{2, 2}\sigma_{2,0}$.

Using the well-known relation for determinants of the tridiagonal matrices (see [9]), we have the following recurrence $p_{k}^{2}(\sigma_{2}) = \beta_{0} p_{k-1}^{2}(\sigma_{2}) + \beta_{1} p_{k-2}^{2}(\sigma_{2})$, i.e.,

$$p_{k}^{2}(\sigma_{2}) = \sigma_{2,1} p_{k-1}^{2}(\sigma_{2}) - \sigma_{2,2} \sigma_{2,0} p_{k-2}^{2}(\sigma_{2}), \quad k > 2. \quad (5.3)$$
The rest of the proof goes inductively. Namely, we suppose that

\[ p_{k-1}^2(\sigma_2) = \sum_{v=0}^{[(k-1)/2]} a_{2,v}^k \sigma_{2,1}^{k-1-2v} (\sigma_{2,0} \sigma_{2,2})^v \]  
and

\[ p_{k-2}^2(\sigma_2) = \sum_{v=0}^{[(k-2)/2]} a_{2,v}^k \sigma_{2,1}^{k-2-2v} (\sigma_{2,0} \sigma_{2,2})^v. \]

Then, using (5.3), we find

\[ p_k^2(\sigma_2) = \sum_{v=0}^{[(k-1)/2]} a_{2,v}^k \sigma_{2,1}^{k-2v} (\sigma_{2,0} \sigma_{2,2})^v - \sum_{v=0}^{[(k-2)/2]} a_{2,v}^k \sigma_{2,1}^{k-2-2v} (\sigma_{2,0} \sigma_{2,2})^{v+1} \]

\[ = a_{2,0}^k \sigma_{2,1}^k + \sum_{v=1}^{[(k-1)/2]} (a_{2,v}^k - a_{2,v-1}^k) \sigma_{2,1}^{k-2v} (\sigma_{2,2} \sigma_{2,0})^v \]

\[ - \frac{1}{2} (1 + (-1)^k) a_{2,[k-2]/2}^k \sigma_{2,1}^{k-2[k/2]} (\sigma_{2,2} \sigma_{2,0})^{[k/2]}, \]

and using this relation, we get the relations (2.6). It is easy to check that \( p_1^2 \) and \( p_2^2 \) have representations as it is stated in this theorem.

To prove the last relation, we can check directly

\[ (-1)^v \binom{k-v}{v} = (-1)^v \binom{k-1-v}{v} - (-1)^{v-1} \binom{k-1-v}{v-1}, \]

which, after dividing by \((-1)^v\), becomes the basic binomial identity (see [9, p. 53]). \( \square \)

5.3. Cases \( k = 3, 4, 5, 6 \)

In this subsection we present special cases for \( k = 3, 4, 5, 6 \) and \( n \) arbitrary. The case \( k = 2 \) is already known in the literature (see [10, p. 73]) and we present it for completeness, hence, for \( k = 2 \) we have

\[ p_2^n(\sigma_n) = \sigma_{n,n-1}^2 - \sigma_{n,n}^2 \sigma_{n,n-2} > 0, \quad n \in \mathbb{N}. \]

For bigger values of \( k \) we can use some computer algebra, for example \textit{Mathematica}, \textit{Maple}, to construct the polynomials \( p_k^n(\sigma_n) \). We present our results in the following statement.

\textbf{Theorem 5.2.} We have

\[ p_3^n(\sigma_n) = \sigma_{n,n-1}^3 - 2\sigma_{n,n-2} \sigma_{n,n-1} \sigma_{n,n} + \sigma_{n,n-3} \sigma_{n,n}^2, \]

\[ p_4^n(\sigma_n) = \sigma_{n,n-1}^4 - 3\sigma_{n,n-2} \sigma_{n,n-1}^2 \sigma_{n,n} + \sigma_{n,n-2}^2 \sigma_{n,n}^2 + 2\sigma_{n,n-3} \sigma_{n,n-1} \sigma_{n,n}^2 - \sigma_{n,n-4} \sigma_{n,n}^3, \]

\[ p_5^n(\sigma_n) = \sigma_{n,n-1}^5 - 4\sigma_{n,n-2} \sigma_{n,n-1}^3 \sigma_{n,n} + 3\sigma_{n,n-2}^2 \sigma_{n,n}^2 + 3\sigma_{n,n-3} \sigma_{n,n-1} \sigma_{n,n}^2 - 2\sigma_{n,n-3} \sigma_{n,n-2} \sigma_{n,n}^3 - 2\sigma_{n,n-4} \sigma_{n,n-1} \sigma_{n,n}^3 + \sigma_{n,n-5} \sigma_{n,n}^5, \]
\[
p^n_6(\sigma_n) = \sigma^8_{n,n-1} - 5\sigma_{n-2}^4\sigma_{n,n-1}\sigma_{n,n} + 6\sigma_{n-2}^2\sigma_{n,n-1}^2\sigma_{n,n} + 4\sigma_{n-3}^3\sigma_{n,n-1}\sigma_{n,n}^2
\]
\[
- \sigma^3_{n,n-2}\sigma^3_{n,n} - 6\sigma_{n-3}^3\sigma_{n,n-2}\sigma_{n,n-1}\sigma_{n,n}^3 - 3\sigma_{n-4}^4\sigma_{n,n-1}\sigma_{n,n}^3
\]
\[
+ \sigma_{n,n-2}^4\sigma_{n,n}^3 + 2\sigma_{n-4}^4\sigma_{n,n-2}\sigma_{n,n}^4 + 2\sigma_{n-5}^4\sigma_{n,n-1}\sigma_{n,n}^4 - \sigma_{n,n-6}\sigma_{n,n}^5,
\]
for any \(n \in \mathbb{N}\).

**Proof.** Direct calculation. \(\square\)

6. Application to the theory of orthogonal polynomials

Here we want to present an inspiration for our result. Suppose that we have a linear functional \(\mathcal{L}\) acting on the space of all algebraic polynomials \(\mathcal{P}\). The moments of the functional \(\mathcal{L}\) are given by \(m_k = \mathcal{L}(x^k), k \in \mathbb{N}_0\). Suppose that the sequence of moments satisfies the following recurrence relation:

\[
\sum_{\nu=0}^{k} \frac{m_{\nu}}{\nu!} \sum_{\ell=-j}^{j} a_{\ell} \frac{((\ell + \alpha)h)^{k-\nu}}{(k-\nu)!} = \delta_{k,0}, \quad k \in \mathbb{N}_0,
\]

(6.1)

where \(h, \alpha\) and \(a_{\ell}, \ell = -j, \ldots, j\), are arbitrary real numbers.

Let a polynomial \(Q\) of degree \(n = 2j\) be defined by

\[
Q(x) = \sum_{\ell=-j}^{j} a_{\ell}x^{\ell+j}
\]

and such that all its zeros be positive and different from 1. Hence, we can request that this polynomial \(Q\) is normalized such that \(Q(1) = 1\).

We are interested in a representation of the linear functional \(\mathcal{L}\). It is known (see [2]) that every linear functional \(\mathcal{L}\) acting on the space of all polynomials can be represented with a function of the bounded variation \(\phi\) using Stieltjes–Lebesgue integral

\[
\mathcal{L}(p) = \int_{\mathbb{R}} p(x) d\phi(x), \quad p \in \mathcal{P}.
\]

Under a condition that \(\mathcal{L}\) is positive definite, the function \(\phi\) is nondecreasing. For the special case when zeros of \(Q\) are bigger than 1, we can state that the respective linear functional \(\mathcal{L}\) is positive definite. Moreover, we can reconstruct the measure which represents it. Thus, we have the following result:

**Theorem 6.1.** If all zeros \(\lambda_\nu, \nu = 1, \ldots, n\), counting multiplicities, of the polynomial \(Q\) are bigger than 1, then the linear functional \(\mathcal{L}\) admits the representation

\[
\mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i p((i + j - \alpha)h),
\]
where
\[ Q_i = \frac{1}{i!} \frac{d^i}{dx^i} \left( \frac{1}{Q(x)} \right) \bigg|_{x=0}, \quad i \in \mathbb{N}_0. \]

The functional \( \mathcal{L} \) is positive definite.

**Proof.** Denote by \( f \) the generating function for the sequence of moments \( m_k, k \in \mathbb{N}_0 \), i.e.,
\[ f(u) = \sum_{k=0}^{+\infty} \frac{m_k}{k!} u^k. \]

The recurrence relation (6.1) for the moments has as a consequence the following relation:
\[ f(u) Q(\exp(hu)) = \exp((j - \alpha)hu). \]

As it is well known, we can expand the rational function \( 1/Q(x) \) into the partial fraction decomposition
\[ \frac{1}{Q(x)} = \sum_{v=1}^{M} \sum_{\ell=1}^{M_v} \frac{Q_{v,\ell}}{(x - \mu_{v})^{\ell}}, \quad (6.2) \]
where we assume that \( Q \) has \( M \) distinct zeros \( \mu_{v} \) with the multiplicities \( M_v, v = 1, \ldots, M \).

Using this partial fraction decomposition, the expansion of the geometric series and the expansion of the function \( \exp(x) \), we have for \( u \) sufficiently close to zero
\[ f(u) = \sum_{v=1}^{M} \sum_{\ell=1}^{M_v} \frac{Q_{v,\ell} \exp((j - \alpha)hu)}{(\exp(hu) - \mu_{v})^{\ell}} = \sum_{k=0}^{+\infty} \frac{u^k}{k!} \sum_{i=0}^{+\infty} \frac{(i + j - \alpha)h)^k}{\mu_{v}^{i}}. \]

Now, from this equation we can identify the moments as
\[ m_k = \sum_{i=0}^{+\infty} (i + j - \alpha)h)^k \sum_{v=1}^{M} \sum_{\ell=1}^{M_v} (-1)^\ell \binom{\ell + i - 1}{i} \frac{Q_{v,\ell}}{\mu_{v}^{\ell+i}}, \quad k \in \mathbb{N}_0. \]

On the other side, from (6.2), we have
\[ Q_i = \frac{1}{i!} \frac{d^i}{dx^i} \left( \frac{1}{Q(x)} \right) \bigg|_{x=0} = \sum_{v=1}^{M} \sum_{\ell=1}^{M_v} (-1)^\ell \binom{\ell + i - 1}{i} \frac{Q_{v,\ell}}{\mu_{v}^{\ell+i}}, \]
so that we can interpret our linear functional in the following form:
\[ \mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i p((i + j - \alpha)h). \]
Since we know that $\sigma_{n,\alpha}^i Q_i = p_i(\sigma_n) Q_0$, according to Lemma 2.2, and since

$$Q_0 = \prod_{v=1}^{n} \frac{-\lambda_v}{1 - \lambda_v} > 0,$$

using Theorem 2.3, we conclude that $Q_i > 0$ and our functional $L$ is positive definite. □

Of course, since our result cannot depend on the order of enumeration of the coefficients $a_\ell$, $\ell = -j, \ldots, j$, of the polynomial $Q$, if all zeros of $Q$ are positive and smaller than 1, then we can change the order of enumeration of the coefficients $a_\ell$, $\ell = -j, \ldots, j$, and have all zeros of such $Q$ bigger than 1. However, we need to change the sign of $h$ and $\alpha$ also. Now, our representation Theorem 6.1 is applicable, so that we have the following corollary.

**Corollary 6.2.** If all zeros $\lambda_v$, $v = 1, \ldots, n$, counting multiplicities, of the polynomial $Q$ are positive and smaller than 1, then the linear functional $L$ admits the representation

$$L(p) = \sum_{i=0}^{+\infty} Q_i^* p(-(i + j + \alpha)h),$$

where

$$Q_i^* = \left. \frac{d^i}{i! dx^i} \left( \frac{1}{x^n Q(1/x)} \right) \right|_{x=0}, \quad i \in \mathbb{N}_0.$$ 

The functional $L$ is positive definite.

Using the same arguments, we can state the similar results provided a sequence of moments satisfies the following recurrence relation:

$$\sum_{v=0}^{k} \frac{m_v}{v!} \sum_{\ell=-j}^{j-1} a_\ell \frac{((\ell + \alpha + 1/2)h)^{k-v}}{(k-v)!} = \delta_{k,0}, \quad k \in \mathbb{N}_0. \quad (6.3)$$

Then, we define polynomial $Q$ in the following way:

$$Q(x) = \sum_{\ell=-j}^{j-1} a_\ell x^{\ell+j}.$$ 

Obviously, the polynomial $Q$ has degree $n = 2j - 1$. Using the same arguments as before, we can state the following result:

**Theorem 6.3.** If all zeros $\lambda_v$, $v = 1, \ldots, n$, counting multiplicities, of the polynomial $Q$ are bigger than 1, then the linear functional $L$ admits the representation

$$L(p) = \sum_{i=0}^{+\infty} Q_i p((i + j - \alpha - 1/2)h),$$
where
\[ Q_i = \frac{1}{i!} \frac{d^i}{dx^i} \left( \frac{1}{Q(x)} \right) \Bigg|_{x=0}, \quad i \in \mathbb{N}_0. \]

The functional \( L \) is positive definite.

For the case when all zeros are positive and smaller than 1, we have the following corollary:

**Corollary 6.4.** If all zeros \( \lambda_\nu, \nu = 1, \ldots, n \), counting multiplicities, of the polynomial \( Q \) are positive and smaller than 1, then the linear functional \( L \) admits the representation

\[ L(p) = \sum_{i=0}^{+\infty} Q_i^* p(-(i + j + \alpha - 1/2)h), \]

where
\[ Q_i^* = \frac{1}{i!} \frac{d^i}{dx^i} \left( \frac{1}{x^n Q(1/x)} \right) \Bigg|_{x=0}, \quad i \in \mathbb{N}_0. \]

The functional \( L \) is positive definite.

As an illustrative example, we consider the case when the polynomial \( Q \) has only one zero \( \lambda_1 \) with multiplicity \( M_1 \) (in this case, of course, \( n = M_1 \)). Since the polynomial \( Q \) is normalized \((Q(1) = 1)\), we have

\[ Q(x) = \left( \frac{x - \lambda_1}{1 - \lambda_1} \right)^{M_1}. \]

Now, from this equation we can read

\[ a_\ell = \frac{(-1)\ell+j}{(1-\lambda_1)^{M_1}} \binom{M_1}{\ell+j} \lambda_1^{\ell+j}, \quad \ell = -j, \ldots, [M_1/2], \]

where \( j = [(M_1 + 1)/2] \). So that moments satisfy the following recurrence relation

\[ \sum_{\nu=0}^{k} \frac{[M_1/2]}{\nu!} \sum_{\ell=-j}^{k} a_\ell \frac{((\ell + \alpha + j - M_1/2)h)^{k-\nu}}{(k-\nu)!} = \delta_{k,0}, \quad k \in \mathbb{N}_0. \]

Assuming \( \lambda_1 > 1 \), we know, according to Theorems 6.1 and 6.3, that the sequence of moments can be represented as a sequence of moments of the linear functional \( L \) of the following form:

\[ L(p) = \sum_{i=0}^{+\infty} Q_i p((i - \alpha + M_1/2)h), \quad Q_i = \frac{M_1^i}{i!} \frac{(\lambda_1 - 1)^{M_1}}{\lambda_1^{M_1+i}}. \]

It is easy to see that the functional \( L \) is positive definite as we expect according to the previous theorems.
It can be checked easily that this linear functional coincides with the Meixner linear functional of the first kind (see [2]). Actually, the Meixner polynomials of the first kind (see [2, p. 161]) are orthogonal with respect to the linear functional

\[ \mathcal{L}_M(p) = \sum_{i=0}^{+\infty} p(i) \frac{c^i(\beta)_i}{i!}, \quad p \in \mathcal{P}, \; c \in (0, 1), \; \beta > 0. \]

Our linear functional \( \mathcal{L} \) coincides with this one if we choose \( c = 1/\lambda_1, \; \beta = M_1, \; \alpha = 0, \; h = 1 \) and if we apply the shift for \( M_1/2 \).

The case \( \lambda_1 \in (0, 1) \), applying the Corollaries 6.2 and 6.4, leads again to the Meixner polynomials of the first kind which are orthogonal with respect to the linear functional

\[ \mathcal{L}_M(p) = \sum_{i=0}^{+\infty} p(-(i + \beta)) \frac{c^{-i}(\beta)_i}{i!}, \quad p \in \mathcal{P}, \; c > 1, \; \beta > 0, \]

where we need to choose \( c = 1/\lambda_1, \; \beta = M_1, \; h = 1 \) and again we need to shift for \( M_1/2 \).

This gives directly the following result:

**Theorem 6.5.** Assume that a sequence of moments \( m_k, \; k \in \mathbb{N}_0 \), satisfies the recurrence relation (6.1) or (6.3), with zeros of \( Q \), being all bigger than 1 or positive and smaller than 1. Then, there exists a sequence of polynomials orthogonal with respect to the corresponding linear functional \( \mathcal{L} \).

**References**


