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Weighted L^2-Analogues of Bernstein's Inequality and Classical Orthogonal Polynomials

ALLAL GUESSAB

Department of Applied Mathematics, University of Pau, I.P.R.A., Avenue de l'Université, 64000 Pau, France

AND

GRADIMIR V. MILOVANOVIĆ*

Department of Mathematics, University of Niš, Faculty of Electronic Engeneering, P.O. Box 73, 18000 Niš, Yugoslavia

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Let $\mathscr{P}_n$ be the class of algebraic polynomials of degree at most n. Some weighted L^2-analogues of the Bernstein's inequality for polynomials $P \in \mathscr{P}_n$ are investigated and a connection with the classical orthogonal polynomials is given. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $\mathscr{P}_n$ be the class of algebraic polynomials of degree at most n. A standard form of Bernstein's theorem for algebraic polynomials is the following:

THEOREM A. *Let $P \in \mathscr{P}_n$ and $|P(t)| \leqslant 1$ $(-1 \leqslant t \leqslant 1)$, then*

$$|P'(t)| \leqslant \frac{n}{\sqrt{1-t^2}}, \qquad -1 < t < 1. \tag{1.1}$$

The equality is attained at the points $t = t_\nu = \cos((2\nu - 1)\pi/2n)$, $1 \leqslant \nu \leqslant n$, if and only if $P(t) = \gamma T_n(t)$, where T_n is the Chebyshev polynomial of the first kind of degree n and $|\gamma| = 1$.

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Bernstein [2] proved this result at the same time when he gave the corresponding theorem for trigonometric polynomials. However, inequality (1.1) in the present form first appeared in print in the paper of Fekete [7] who attributes the proof to Fejér [6]. Bernstein [3] attributes the proof to E. Landau.

Using the uniform norm on $[-1, 1]$, $\|f\|_\infty = \max_{-1 \leqslant t \leqslant 1} |f(t)|$, this result can be stated in the form

$$\|\sqrt{1-t^2}\, P'(t)\|_\infty \leqslant n\, \|P\|_\infty. \tag{1.2}$$

In this paper we consider weighted L^2-analogues of this inequality using the weights $t \mapsto w(t)$ of the classical orthogonal polynomials, i.e., the Jacobi weight $w(t) = (1-t)^\alpha (1+t)^\beta$ $(\alpha, \beta > -1)$ on $(-1, 1)$, the generalized Laguerre weight $w(t) = t^s e^{-t}$ $(s > -1)$ on $(0, +\infty)$, and finally the Hermite weight $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$. These weights satisfy the differential equation of the first order

$$\frac{d}{dt}(A(t)\, w(t)) = B(t)\, w(t),$$

where the function $t \mapsto A(t)$ is given by

$$A(t) = \begin{cases} 1-t^2, & \text{in the Jacobi case,} \\ t, & \text{in the generalized Laguerre case,} \\ 1, & \text{in the Hermite case,} \end{cases} \tag{1.3}$$

and $t \mapsto B(t)$ is a polynomial of the first degree.

2. Main Result

Let w be the weight of the classical orthogonal polynomials ($w \in CW$) and A be given by (1.3).

Using the norm $\|f\|_w^2 = (f, f)$, where

$$(f, g) = \int_a^b w(t)\, f(t)\, g(t)\, dt \qquad (f, g \in L^2[a, b]), \tag{2.1}$$

we consider the following problem connected with the Bernstein's inequality (1.2): *Determine the best constant* $C_{n,m}(w)$ $(1 \leqslant m \leqslant n)$ *such that the inequality*

$$\|A^{m/2} P^{(m)}\|_w \leqslant C_{n,m}(w)\, \|P\|_w \tag{2.2}$$

holds for all $P \in \mathscr{P}_n$.

At first, we note if $w \in CW$ then the corresponding classical orthogonal polynomial $t \mapsto Q_n(t)$ is a particular solution of the differential equation of the second order

$$\frac{d}{dt}\left(A(t)\, w(t)\, \frac{dy}{dt}\right) + \lambda_n w(t)\, y = 0, \tag{2.3}$$

where

$$\lambda_n = -n(\tfrac{1}{2}(n-1)\, A''(0) + B'(0)). \tag{2.4}$$

The kth derivative of Q_n is also the classical orthogonal polynomial, with respect to the weight $t \mapsto w_k(t) = A(t)^k\, w(t)$ and satisfies the differential equation

$$\frac{d}{dt}\left(A(t)\, w_k(t)\, \frac{dy}{dt}\right) + \lambda_{n,k}\, w_k(t)\, y = 0, \tag{2.5}$$

where

$$\lambda_{n,k} = -(n-k)(\tfrac{1}{2}(n+k-1)\, A''(0) + B'(0)). \tag{2.6}$$

We note that (2.6) reduces to (2.4) for $k = 0$, i.e., $\lambda_{n,0} = \lambda_n$.

The classical orthogonal polynomials can be specificated as the Jacobi polynomials $P_n^{(\alpha,\beta)}$ ($\alpha, \beta > -1$), the generalized Laguerre polynomials L_n^s ($s > -1$) and finally as the Hermite polynomials H_n.

THEOREM 2.1. *For all $P \in \mathscr{P}_n$ the inequality* (2.2) *holds with the best constant*

$$C_{n,m}(w) = \sqrt{\lambda_{n,0}\lambda_{n,1} \cdots \lambda_{n,m-1}}, \tag{2.7}$$

where $\lambda_{n,k}$ is given by (2.6). *The equality is attained in* (2.2) *if and only if P is a constant multiple of the classical polynomial Q_n orthogonal with respect to the weight function $w \in CW$.*

Proof. Suppose that $P \in \mathscr{P}_n$. Then we can put $P(t) = \sum_{\nu=0}^{n} a_\nu Q_\nu(t)$ and consider the linear functional

$$L_\nu[P] = \frac{d}{dt}\left(A(t)\, w(t)\, \frac{dP(t)}{dt}\right) + \lambda_\nu w(t)\, P(t).$$

Since $L_\nu[Q_\nu] \equiv 0$, we get

$$L_n[P] = \sum_{\nu=0}^{n} (\lambda_n - \lambda_\nu)\, a_\nu w(t)\, Q_\nu(t).$$

Using the inner product (2.1) we obtain

$$(w^{-1}L_n[P], P) = \sum_{\nu=0}^{n} (\lambda_n - \lambda_\nu)\, a_\nu^2\, \|Q_\nu\|_w^2. \tag{2.8}$$

Integration by parts, we find that

$$(w^{-1}L_n[P], P) = -\|\sqrt{A}P'\|_w^2 + \lambda_n\, \|P\|_w^2.$$

Since $\lambda_\nu \leqslant \lambda_n$ for $\nu \leqslant n$, from the last equality and (2.8) we conclude that the inequality

$$\|\sqrt{A}\, P'\|_w \leqslant \sqrt{\lambda_n}\, \|P\|_w$$

holds. Thus, we proved (2.7) for $m = 1$. The equality case follows from the fact that

$$(w^{-1}L_n[P], P) = \sum_{\nu=0}^{n} (\lambda_n - \lambda_\nu)\, a_\nu^2\, \|Q_\nu\|_w^2 = 0$$

if and only if $a_\nu = 0$ for $\nu = 0, 1, ..., n-1$ and a_n is an arbitrary real constant. Therefore, $P(t) = a_n Q_n(t)$.

If we use the differential equation (2.5), instead of (2.3), we get the inequality

$$\|A^{k/2}P^{(k)}\|_w \leqslant \sqrt{\lambda_{n,k-1}}\, \|A^{(k-1)/2}P^{(k-1)}\|_w \qquad (P \in \mathscr{P}_n),$$

with equality if and only if $P(t) = a_n Q_n(t)$.

Finally, iterating this inequality for $k = 1, ..., m$, we finish the proof. ∎

Remark 2.1. In [1] Agarwal and Milovanović obtained the inequality

$$(2\lambda_n + B'(0))\, \|\sqrt{A}\, P'\|_w^2 \leqslant \lambda_n^2\, \|P\|_w^2 + \|AP''\|_w^2 \qquad (P \in \mathscr{P}_n),$$

which generalizes a previous result of Varma [14] for the Hermite polynomials.

3. Special Cases

In this section we consider inequality (2.2) for the Jacobi weight, the generalized Laguerre weight, and for the Hermite weight. Then, we have

$$\lambda_{n,k} = \begin{cases} (n-k)(n+k+\alpha+\beta+1), & \text{in the Jacobi case,} \\ n-k, & \text{in the generalized Laguerre case,} \\ 2(n-k), & \text{in the Hermite case.} \end{cases}$$

Corollary 3.1. *Let $w(t) = (1-t)^\alpha (1+t)^\beta$ ($\alpha, \beta > -1$). Then, for every $P \in \mathscr{P}_n$, the inequality*

$$\|(1-t^2)^{m/2} P^{(m)}\|_w \leqslant \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}}\,\|P\|_w,$$

holds, with equality if and only if $P(t) = cP_n^{(\alpha,\beta)}(t)$.

Remark 3.1. Daugavet and Rafal'son [4] and Konjagin [9] considered the extremal problems of the form

$$\|P^{(m)}\|_{p,\nu} \leqslant A_{n,m}(r, \mu; p, \nu)\,\|P\|_{r,\mu} \qquad (P \in \mathscr{P}_n),$$

where

$$\|f\|_{r,\nu} = \left(\int_{-1}^{1} |f(t)(1-t^2)^\mu|^r\,dt\right)^{1/r}, \qquad 0 \leqslant r < +\infty,$$

$$= \operatorname*{ess\,sup}_{-1\leqslant t\leqslant 1} |f(t)|\,(1-t^2)^\mu, \qquad r = +\infty.$$

The case when $p = r \geqslant 1$, $\mu = \nu = 0$, and $m = 1$, was considered by Hille, Szegő, and Tamarkin [8]. The exact constant $A_{n,m}(r, \mu; p, \nu)$ is known in a few cases, for example, $A_{n,1}(+\infty, 0; 1, 0) = 2n$ and

$$A_{n,m}(2, \mu; 2, \mu + m/2) = \sqrt{\frac{n!\,\Gamma(n+4\mu+m+1)}{(n-m)!\,\Gamma(n+4\mu+1)}}.$$

The last case, in fact, is our result with the Gegenbauer weight ($\alpha = \beta = 2\mu$).

In the generalized Laguerre case, Theorem 2.1 reduces to:

Corollary 3.2. *Let $w(t) = t^s e^{-t}$ ($s > -1$) on $(0, +\infty)$. Then for every $P \in \mathscr{P}_n$ we have*

$$\|t^{m/2} P^{(m)}\|_w \leqslant \sqrt{\frac{n!}{(n-m)!}}\,\|P\|_w,$$

with equality if and only if $P(t) = cL_n^s(t)$.

The Hermite case with the weight $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$ is the simplest. Then the best constant is $C_{n,m}(w) = 2^{m/2}\sqrt{n!/(n-m)!}$. This result can be found in the Ph.D. Thesis of Shampine [12] (see also, Dörfler [5] and Milovanović [10]). The case $m = 1$ was investigated by Schmidt [11] and Turán [13].

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