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BOUNDS FOR THE (p,ν)–EXTENDED BETA FUNCTION AND CERTAIN CONSEQUENCES

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Dedicated to the 96th birthday anniversary of Professor Walter Gautschi

(*Communicated by L. Mihoković*)

Abstract. Our main goal in this article is to present both upper and lower bounds for the (p,ν)-extended Gauss' hypergeometric function and the related confluent hypergeometric (or Kummer's) function, the modified Bessel function of the second kind, with the extended Gautschi–Pinelis inequality (upper bounds) and with the aid of the classical Bernoulli inequality (lower bounds) and also inferring associated functional bounds for the (p,ν)-extended Beta function.

1. Introduction and preliminaries

Very recently, Parmar *et al.* introduced the so–called (p,ν)-extended Beta function [8, p. 93, Eq. (13)]

$$\mathrm{B}_{p,\nu}(x,y) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}\,(1-t)^{y-\frac{3}{2}}\,K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\mathrm{d}t, \tag{1.1}$$

where $\Re(p)>0$; $\min\{\Re(x),\Re(y)\}>0$ and $\sqrt{p}$ takes its principal value; in the case $p=0$ we consider $\min\{\Re(x),\Re(y)\}>\frac{1}{2}$. Here $K_\mu(z)$ stands for the modified Bessel function of the second kind (in other words Macdonald function) of the order μ [6, p. 251, Eq. 10.27.4]

$$K_\mu(x) = \frac{\pi}{2}\,\frac{I_{-\mu}(x)-I_\mu(x)}{\sin(\pi\mu)}, \qquad \mu\in\mathbb{C}\setminus\mathbb{Z},$$

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else $\lim_{\mu\to n} K_\mu(z) = K_n(z)$ is used for any $n \in \mathbb{Z}$. Accordingly, the (p,ν)-extended Gauss' and (p,ν)-extended Kummer hypergeometric functions are [8, p. 98, Eqs. (40–41)]

$$F_{p,\nu}(a,b;c;z) = \sum_{n\geqslant 0} (a)_n \frac{\mathrm{B}_{p,\nu}(b+n,\,c-b)}{\mathrm{B}(b,\,c-b)} \frac{z^n}{n!}, \qquad p \geqslant 0; \Re(c) > \Re(b) > 0\,; |z| < 1, \tag{1.2}$$

$$\Phi_{p,\nu}(b;c;z) = \sum_{n\geqslant 0} \frac{\mathrm{B}_{p,\nu}(b+n,\,c-b)}{\mathrm{B}(b,\,c-b)} \frac{z^n}{n!}, \qquad p \geqslant 0; \Re(c) > \Re(b) > 0, \tag{1.3}$$

respectively. For $\nu = 0$ equations (1.1), (1.2) and (1.3), reduce to p-extended Beta function introduced by Chaudhry *et al.* [1, p. 20, Eq. (1.7)]

$$\mathrm{B}_p(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t\,, \qquad \Re(p) \geqslant 0\,; \min\{\Re(x),\Re(y)\} > 0, \tag{1.4}$$

and subsequently, the p-extended Gauss' hypergeometric and the p-Kummer (or confluent) hypergeometric functions [2, pp. 591–2, Eqs. (2.2–2.3)]

$$F_p(a,b;c;z) = \sum_{n\geqslant 0}^{\infty} (a)_n \frac{\mathrm{B}_p(b+n,\,c-b)}{\mathrm{B}(b,\,c-b)} \frac{z^n}{n!}, \qquad p \geqslant 0; \Re(c) > \Re(b) > 0\,; |z| < 1$$

$$\Phi_p(b;c;z) = \sum_{n\geqslant 0} \frac{\mathrm{B}_p(b+n,\,c-b)}{\mathrm{B}(b,\,c-b)} \frac{z^n}{n!}, \qquad p \geqslant 0\,; \Re(c) > \Re(b) > 0,$$

respectively. Also, we invite the reader to visit the related publications [5,9,7] and also to consult [11, p. 350, Eq. (1.13)].

The main purpose of this short note are to obtain integral representations and allied bounding inequalities for the functions $\mathrm{B}_{p,\nu}(x,\,y)$, $F_{p,\nu}(a,b;c;z)$ and $\Phi_{p,\nu}(a,b;c;z)$ in the widest possible range of the parameters involved.

2. Bounding inequalities for the (p,ν)-extended hypergeometric functions

In this section, our main goal is to derive an upper bound for the (p,ν)-extended Beta function $\mathrm{B}_{p,\nu}(x,y)$ presented in (1.1). With the aid of this upper bound we obtain consequent bounds for (p,ν)-extended Gaussian hypergeometric $F_{p,\nu}$ and (p,ν)-extended Kummer's confluent hypergeometric $\Phi_{p,\nu}$ *via* its series representations (1.2) and (1.3).

2.1. Upper bound for (p,ν)-extended Beta function

Firstly, we establish the upper bound for the (p,ν)-extended Beta function $\mathrm{B}_{p,\nu}$ which shall imply *a fortiori* the bound for the hypergeometric function $F_{p,\nu}$. For this task we need the following result [3, p. 17, Eq. (5.2)]

$$\left|K_{\nu+\frac{1}{2}}(z)\right| < \frac{\sqrt{\pi}\,\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \frac{\Gamma\big(2\nu+1,\Re(z)\big)}{\big(\Re(z)\big)^{2\nu+1}}, \qquad \nu,\Re(z) > 0\,, \tag{2.1}$$

where the upper incomplete gamma function

$$\Gamma(a,x) = \int_x^\infty t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}t\,, \qquad \Re(a), \Re(x) > 0\,,$$

is employed. Consequently, since $\Gamma(a,x) < \Gamma(a)$, there holds [3, p. 17]

$$\left|K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\right| < \frac{1}{2}\left(\frac{2|p|t(1-t)}{\Re^2(p)}\right)^{\nu+\frac{1}{2}}\Gamma\left(\nu+\tfrac{1}{2}\right), \quad \Re(p) > 0, t \in (0,1)\,. \tag{2.2}$$

The immediate implication of (2.2) follows by means of (1.1).

LEMMA 1. *For all* $\Re(p) > 0$, $\nu > 0$, $\min\{\Re(x), \Re(y)\} > 0$ *and* $t \in (0,1)$, *we have*

$$\left|\mathrm{B}_{p,\nu}(x,y)\right| \leqslant \frac{2^\nu |p|^{\nu+1}\,\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(\Re(p))^{2\nu+1}}\,\mathrm{B}(x+\nu, y+\nu)\,.$$

This upper bound plays an important role in the whole section applied either for sum or integral representations (indirectly) of the families of (p,ν)-extended special functions.

3. Refined bounds upon $K_\mu(z)$ and $B_{p,\nu}(x,y)$ and related consequences

The estimate $\Gamma(a,x) < \Gamma(a); x > 0$ applied to (2.1) in obtaining the bound (2.2) is pretty rough. Hence, to refine the upper–bound by virtue of (2.1) we use the findings on the bounding inequalities for the real parameter upper incomplete Gamma function by Pinelis [10], who's results precised certain appropriate classical results by Gautschi [4]. Namely, Pinelis reported on bilateral bounds on the incomplete gamma function $\Gamma(a,x)$ for real a and $x > 0$. These bounds $V_a(x)$, say, are exact in the sense that $V_a(x) \sim \Gamma(a,x)$ when $x \downarrow 0$, and $V_a(x) \sim \Gamma(a,x)$ for $x \to \infty$. In recalling Pinelis' results, regarding only the upper bounds for the upper incomplete Gamma, we retain the notations introduced by him.

THEOREM A. [10, p. 1262, Theorem 1.1] *Let*

$$b_a := \begin{cases} \Gamma(a+1)^{\frac{1}{a-1}} & a \in (-1,\infty)\setminus\{1\}, \\ \mathrm{e}^{1-\gamma} & a = 1, \end{cases} \tag{3.1}$$

where γ *denotes the Euler–Mascheroni constant, and*

$$G_a(x) := \begin{cases} \mathrm{e}^{-x}x^{-2} & a = -1, \\ \mathrm{e}^{-x}\,\dfrac{(x+b_a)^a - x^a}{a\,b_a} & a \in (-1,\infty)\setminus\{0\}, \\ \mathrm{e}^{-x}\log\left(1+\dfrac{1}{x}\right) & a = 0. \end{cases} \tag{3.2}$$

Then for any $a \geqslant -1$ *and* $x > 0$ *we have*

$$\Gamma(a,x) < G_a(x), \qquad a \in [-1,1) \cup (2,3) \cup (3,\infty),$$
$$\Gamma(2,x) = G_2(x) = \mathrm{e}^{-x}(1+x),$$

whilst for $a \in (1,2)$ *the inequality is reversed.*

REMARK 1. The cases $a = 1,3$ read

$$\Gamma(1,x) = G_1(x) = \mathrm{e}^{-x},$$
$$\Gamma(3,x) = \mathrm{e}^{-x}(2+2x+x^2) < \mathrm{e}^{-x}(2+\sqrt{6}\,x+x^2) = G_3(x),$$

but these estimates are not to attractive, since in (2.1) imply the values $K_{\nu+\frac12}(x)$, $\nu = 0,1$ for which we do not need upper bounds, being

$$K_{\frac12}(x) = \mathrm{e}^{-x}\sqrt{\frac{\pi}{2x}}; \quad K_{\frac32}(x) = \mathrm{e}^{-x}\sqrt{\frac{\pi}{2x}}\left(1+\frac{1}{x}\right).$$

The modified Bessel function of the second kind $K_\mu(x)$ is even function of its order $\mu \in \mathbb{R}$, moreover it is positive for real argument $x \in \mathbb{R}$. Therefore, we restrict our investigation to real variable case in deriving refinements of (2.1).

PROPOSITION 1. *Let* $x > 0$. *Then we have*

$$K_{\nu+\frac12}(x) < \begin{cases} \dfrac{\sqrt{\pi}\,G_{2\nu+1}(x)}{\Gamma(\nu+1)\,(2x)^{\nu+\frac12}}, & \nu \in \left(\frac12,1\right) \cup (1,\infty), \\ \mathrm{e}^{-x}\left(1+\frac{1}{x}\right), & \nu = \frac12, \end{cases} \tag{3.3}$$

where $G_{2\nu+1}(x)$ *is given in* (3.1) *and* (3.2) *respectively.*

Proof. Consider the estimate (2.1). Then treat the incomplete Gamma function with Pinelis results in Theorem A, pointing out that since the parameter $\nu > 0$ by assumption in (2.1), Pinelis' results we apply in the parameter range $a = 2\nu+1 > 1$. The implication (3.3) is straightforward. □

PROPOSITION 2. *For all* $p > 0$ *and* $t \in (0,1)$ *we have*

$$K_{\nu+\frac12}\left(\frac{p}{t(1-t)}\right) < \begin{cases} \dfrac{c_{p,\nu}}{\big(t(1-t)\big)^{\nu+\frac12}} \exp\left\{-\dfrac{p}{t(1-t)}\right\}, & \nu \in \left(\frac12,1\right) \cup (1,\infty), \\ \left(1+\dfrac{t}{p}(1-t)\right) \exp\left\{-\dfrac{p}{t(1-t)}\right\}, & \nu = \frac12, \end{cases} \tag{3.4}$$

where

$$c_{p,\nu} = \frac{\sqrt{\pi}\,p^{\nu+\frac12}\left\{\left(1+\dfrac{b_{2\nu+1}}{4p}\right)^{2\nu+1} - 1\right\}}{2^{\nu+\frac12}\,\Gamma(\nu+1)\,(2\nu+1)\,b_{2\nu+1}}.$$

Proof. Consider the suitably transformed appropriate Pinelis' bound (3.2)

$$G_{2\nu+1}\Big(\frac{p}{t(1-t)}\Big) = \frac{p^{2\nu+1}\big(t(1-t)\big)^{-2\nu-1}}{(2\nu+1)\,b_{2\nu+1}}\,\exp\Big\{-\frac{p}{t(1-t)}\Big\}\,h_{p,\nu}(t)\,,$$

where

$$h_{p,\nu}(t) = \Big(1+\frac{b_{2\nu+1}}{p}\,t(1-t)\Big)^{2\nu+1} - 1\,.$$

Being

$$h'_{p,\nu}(t) = \frac{(2\nu+1)b_{2\nu+1}}{p}\,\Big(1+\frac{b_{2\nu+1}}{p}\,t(1-t)\Big)^{2\nu}(1-2t)\,,$$

the only stationary point of $h_{p,\nu}(t)$ is $t_1 = \frac{1}{2}$, as the other two solutions of $h'_{p,\nu}(t) = 0$ are outside of the unit interval. Now $h_{p,\nu}(t)\uparrow$ for $t\in(0,\frac{1}{2})$ and $h_{p,\nu}(t)\downarrow$ if $t\in(\frac{1}{2},1)$, it is

$$\sup_{0<t<1} h_{p,\nu}(t) = h_{p,\nu}(\tfrac{1}{2}) = \Big(1+\frac{b_{2\nu+1}}{4p}\Big)^{2\nu+1} - 1\,.$$

Accordingly, for all $p>0$ and $t\in(0,1)$

$$G_{2\nu+1}\Big(\frac{p}{t(1-t)}\Big) \leqslant \frac{p^{2\nu+1}\Big\{\Big(1+\frac{b_{2\nu+1}}{4p}\Big)^{2\nu+1}-1\Big\}}{(2\nu+1)\,b_{2\nu+1}\,\big(t(1-t)\big)^{2\nu+1}}\,\exp\Big\{-\frac{p}{t(1-t)}\Big\}\,.$$

So, (3.4) immediately follows. $\square$

The related result about the uniform upper bound including the B_p function by Chaudhry *et al.* (1.4) for the real parameter Beta function $\mathrm{B}_{p,\nu}(x,y)$ reads as follows.

PROPOSITION 3. *For all* $p>0$, $\nu\in(\frac{1}{2},1)\cup(1,\infty)$ *and for all* $\min\{x,y\}>\nu+1$ *we have*

$$\mathrm{B}_{p,\nu}(x,y) \leqslant \frac{p^{\nu+1}\Big\{\Big(1+\frac{b_{2\nu+1}}{4p}\Big)^{2\nu+1}-1\Big\}}{2^{\nu}\,(2\nu+1)\,\Gamma(\nu+1)\,b_{2\nu+1}}\,\mathrm{B}_p(x-\nu-1,y-\nu-1)\,. \tag{3.5}$$

Moreover, the upper bound in terms of the Eulerian Beta *function reads*

$$\mathrm{B}_{p,\nu}(x,y) \leqslant \frac{p^{\nu+1}\,\mathrm{e}^{-4p}\Big\{\Big(1+\frac{b_{2\nu+1}}{4p}\Big)^{2\nu+1}-1\Big\}}{2^{\nu}\,(2\nu+1)\,\Gamma(\nu+1)\,b_{2\nu+1}}\,\mathrm{B}(x-\nu-1,y-\nu-1)\,. \tag{3.6}$$

Proof. Starting with the integral definition (1.1), by virtue of the estimate (3.4) we conclude

$$\begin{aligned}\mathrm{B}_{p,\nu}(x,y) &< \sqrt{\frac{2p}{\pi}}\,c_{p,\nu}\int_0^1 t^{x-\nu-2}(1-t)^{y-\nu-2}\exp\Big\{-\frac{p}{t(1-t)}\Big\}\,\mathrm{d}t\\ &= \sqrt{\frac{2p}{\pi}}\,c_{p,\nu}\,\mathrm{B}_p(x-\nu-1,y-\nu-1),\end{aligned}$$

which completes the proof of the statement (3.5). In continuation we remark that for $p > 0$ and for all $t \in (0,1)$ we have the estimate

$$\exp\Big\{-\frac{p}{t(1-t)}\Big\} \leqslant \mathrm{e}^{-4p}.$$

This completes the proof of (3.6). □

The results in bounding the real parameter (p,ν)-extended Gaussian hypergeometric function $F_{p,\nu}$ and the (p,ν)-extended Kummer confluent hypergeometric function $\Phi_{p,\nu}$ follow by the findings presented in Proposition 2, that is by the relations (3.4).

THEOREM 1. *For all $p \geqslant 0, a > 0, \nu \in (\frac12,1)\cup(1,\infty)$, $\min\{b,\frac12 c, c-b\} > \nu+1$ and for all $|z| < 1$, we have*

$$\big|F_{p,\nu}(a,b;c;z)\big| \leqslant k_{p,\nu}\,\frac{\mathrm{B}(b-\nu-1,c-b-\nu-1)}{\mathrm{B}(b,\,c-b)}\,{}_2F_1(a,b-\nu-1;c-2\nu-2;|z|),$$

where

$$k_{p,\nu} = \frac{p^{\nu+1}\,\mathrm{e}^{-4p}\,\Big\{\Big(1+\dfrac{b_{2\nu+1}}{4p}\Big)^{2\nu+1}-1\Big\}}{2^{\nu}\,(2\nu+1)\,\Gamma(\nu+1)\,b_{2\nu+1}}.$$

Accordingly, in the same parameters' range

$$\big|\Phi_{p,\nu}(b;c;z)\big| \leqslant k_{p,\nu}\,\frac{\mathrm{B}(b-\nu-1,c-b-\nu-1)}{\mathrm{B}(b,\,c-b)}\,\Phi(b-\nu-1;c-2\nu-2;|z|). \quad (3.7)$$

Proof. Consider the modulus $|F_{p,\nu}(a,b;c;z)|$ for $p \geqslant 0$; $\Re(c) > \Re(b) > 0$, when $|z| < 1$. Then we have the estimate

$$|F_{p,\nu}(a,b;c;z)| \leqslant \sum_{n\geqslant 0}(a)_n\,\frac{\mathrm{B}_{p,\nu}(b+n,\,c-b)}{\mathrm{B}(b,\,c-b)}\,\frac{|z|^n}{n!}.$$

By (3.6) we conclude

$$\begin{aligned}|F_{p,\nu}(a,b;c;z)| &\leqslant \frac{k_{p,\nu}\,\mathrm{B}(b-\nu-1,c-b-\nu-1)}{\mathrm{B}(b,c-b)}\sum_{n\geqslant 0}(a)_n\,\frac{(b-\nu-1)_n}{(c-2\nu-2)_n}\,\frac{|z|^n}{n!}\\ &= \frac{k_{p,\nu}\,\mathrm{B}(b-\nu-1,c-b-\nu-1)}{\mathrm{B}(b,c-b)}\,{}_2F_1(a,b-\nu-1;c-2\nu-2;|z|),\end{aligned}$$

being all summands in the right–hand–side series positive.

By similar argumentation we deduce the bound (3.7) inferred for the Kummer function. □

4. Lower bounds established *via* Bernoulli's inequality

To expand the parameter space of ν to negative values, consider the integral representation of the modified Bessel function of the second kind [6, p. 252, Eq. 10.32.8]

$$K_{\nu+\frac{1}{2}}(z) = \frac{\sqrt{\pi}\left(\frac{z}{2}\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \int_1^\infty \mathrm{e}^{-zt}\,(t^2-1)^\nu\,\mathrm{d}t\,, \qquad \Re(\nu) > -1,\, |\arg(z)| < \tfrac{\pi}{2}. \tag{4.1}$$

We get the following *lower* bound in terms of the upper incomplete Gamma function. We notice that since (2.1) here we also consider the same conditions, that is $\nu > 0$ with the positive argument $x = \Re(z) > 0$.

PROPOSITION 4. *For all* $\nu \in (-1,0)\cup(1,\infty)$ *and for all* $x > 0$ *we have*

$$K_{\nu+\frac{1}{2}}(x) > \frac{\sqrt{\pi}\,\nu(4\nu-2-x^2)}{(2x)^{\nu+\frac{1}{2}}\,\Gamma(\nu+1)}\,\Gamma(2\nu-1,x) + \frac{\sqrt{\pi}(2\nu+x)x^{\nu-\frac{3}{2}}}{2^{\nu+\frac{1}{2}}\,\Gamma(\nu+1)}\,\mathrm{e}^{-x}. \tag{4.2}$$

Proof. Firstly, we transform the integral representation formula (4.1) getting

$$K_{\nu+\frac{1}{2}}(x) = \frac{\sqrt{\pi}\left(\frac{x}{2}\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \int_1^\infty \mathrm{e}^{-xt}\,t^{2\nu}(1-t^{-2})^\nu\,\mathrm{d}t\,.$$

Estimating $(1-t^{-2})^\nu > 1-\nu t^{-2}$ in the integrand by means of the Bernoulli inequality for $\nu \in (-1,0)$ and $t \geqslant 1$, we conclude

$$\begin{aligned} K_{\nu+\frac{1}{2}}(x) &> \frac{\sqrt{\pi}\,x^{\nu+\frac{1}{2}}}{2^{\nu+\frac{1}{2}}\,\Gamma(\nu+1)} \int_1^\infty \mathrm{e}^{-xt}\,t^{2\nu}(1-\nu t^{-2})\,\mathrm{d}t \\ &= \frac{\sqrt{\pi}}{(2x)^{\nu+\frac{1}{2}}\,\Gamma(\nu+1)}\left[\Gamma(2\nu+1,x) - \nu x^2\,\Gamma(2\nu-1,x)\right]. \end{aligned} \tag{4.3}$$

For all $\nu \in (-1,0)\cup(1,\infty)$ and $x > 0$ the obtained expression is positive, therefore the lower bound (4.2) is not redundant. Now, by the use of the recurrence formula [6, p. 178, Eq. 8.8.2]

$$\Gamma(a+1,x) = a\,\Gamma(a,x) + x^a\,\mathrm{e}^{-x}$$

twice, we deduce the two–step recurrence relation

$$\begin{aligned} \Gamma(2\nu+1,x) - \nu x^2\,\Gamma(2\nu-1,x) &= \nu(4\nu-2-x^2)\,\Gamma(2\nu-1,x) \\ &\quad + (2\nu+x)x^{2\nu-1}\,\mathrm{e}^{-x}, \end{aligned}$$

which transforms (4.3) into the asserted inequality. □

The immediate consequence of Proposition 4 is the following result:

PROPOSITION 5. *For all* $p > 0, \nu \in (-1,0) \cup (1,\infty)$, $x,y > 0$ *and* $\min\{x,y\} > \nu+1 > 0$ *we have*

$$\begin{aligned} B_{p,\nu}(x,y) > \frac{p^\nu}{2^{\nu+1}\Gamma(\nu+1)} \int_0^1 & u^{x-y-2}(1-u)^{y-\nu-2} \\ & \times \Bigg\{ \nu\Big(4\nu-2-\frac{p^2}{u^2(1-u)^2}\Big)\Gamma\Big(2\nu-1,\frac{p}{u(1-u)}\Big) \\ & + \Big(2\nu+\frac{p}{u(1-u)}\Big)\Big(\frac{p}{u(1-u)}\Big)^{2\nu-1}\exp\Big\{\frac{p}{u(1-u)}\Big\}\Bigg\}\,\mathrm{d}u. \end{aligned} \tag{4.4}$$

REMARK 2. Firstly, we quote that for $\nu \in (0,1)$ the Bernoulli inequality used above is reversed. However, Propositions 1 and 2 (partially, but successfully) cover this case. Next, bearing in mind that the right–hand–side integral in (4.4) hard to obtain in a closed form; moreover, Pinelis' Theorem A does not cover lower bound for $\Gamma(2\nu-1,x)$ for negative ν, we leave the further estimations in (4.4) and of the related (p,ν)-extended hypergeometric functions for another address.

Finally, let us point out that the Bernoulli inequality approach gives poor results in estimating the modified Bessel function of the second kind $K_{\nu+\frac{1}{2}}(x)$, which is clearly observable from the Figure 1. Namely, in Figure 1 we display the quotient of the left and right hand side in the inequality (4.2), *viz.*

$$Q_\nu(x) = \frac{2^{\nu+\frac{1}{2}}\Gamma(\nu+1)x^{\nu+\frac{1}{2}}K_{\nu+\frac{1}{2}}(x)}{\sqrt{\pi}\,\nu(4\nu-2-x^2)\Gamma(2\nu-1,x)+\sqrt{\pi}(2\nu+x)x^{2\nu-1}\mathrm{e}^{-x}},$$

for several suitable values of the parameter ν. The cases when $Q_\nu(k) > 1$ ($\nu = -0.55$, $\nu = -0.3$ and $\nu = 1.5$) are shown in red color, while those when $Q_\nu(k) < 1$ ($\nu = 0.1$ and $\nu = 0.45$) in blue.

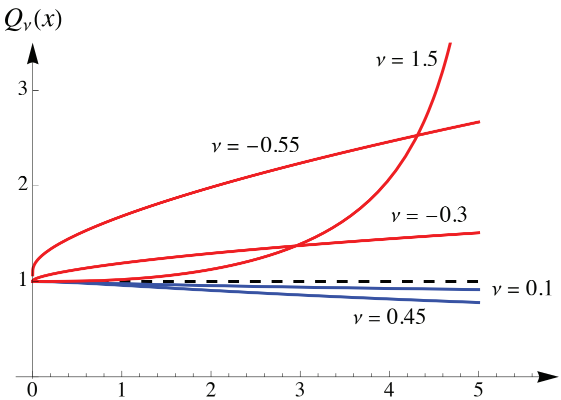

Figure 1: *Quotient* $Q_\nu(x)$ *for* $x \in (0,5)$ *for certain values of* ν

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