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Moment Functional and Orthogonal Trigonometric Polynomials of Semi-Integer Degree

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In this paper we consider algebraic properties of orthogonal trigonometric polynomials of semi-integer degree. We investigate the theory of orthogonality with respect to a general linear functional, which maps the space of trigonometric polynomials to the real numbers. Under certain conditions imposed on the linear functional, we prove the existence of a sequence of orthogonal trigonometric polynomials of semi-integer degree, which satisfies three-term recurrence relations when it is treated in a suitable matrix settings.

Key Words. trigonometric polynomial; semi-integer degree; orthogonality; moment functional; recurrence relation.

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1 Introduction

The first results on orthogonal trigonometric polynomials of semi-integer degree were given in 1959 by Abram Haimovich Turetzkii (see [7]). They are connected with quadrature rules with an even maximal trigonometric degree of exactness in the case of an odd number of nodes. A trigonometric polynomial of semi-integer degree $n+\frac{1}{2}$ is a trigonometric function of the following form

$$\sum_{\nu=0}^{n}\left[c_\nu \cos\Bigl(\nu+\frac{1}{2}\Bigr)x + d_\nu \sin\Bigl(\nu+\frac{1}{2}\Bigr)x\right], \tag{1}$$

where $c_\nu, d_\nu \in \mathbb{R}$, $|c_n|+|d_n| \neq 0$. The coefficients c_n and d_n are called the leading coefficients.

Let us denote by $\mathcal{T}_n$, $n \in \mathbb{N}_0$, the linear space of all trigonometric polynomials of degree less than or equal to n, i.e., the linear span of the following set $\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\}$, by $\mathcal{T}_n^{1/2}$, $n \in \mathbb{N}_0$, the linear space of all trigonometric polynomials of semi-integer degree less than or equal to $n + \frac{1}{2}$, i.e., the linear span of $\{\cos(k + \frac{1}{2})x, \sin(k + \frac{1}{2})x : k = 0, 1, \ldots, n\}$, and by $\mathcal{T}$ and $\mathcal{T}^{1/2}$ the set of all trigonometric polynomials and the set of trigonometric polynomials of semi-integer degree, respectively.

For an integrable and nonnegative weight function $w(x)$ on the interval $[0, 2\pi)$, vanishing there only on a set of a measure zero, and a given set x_ν, $\nu = 0, 1, \ldots, 2n$, of distinct points in $[0, 2\pi)$, Turetzkii in [7] considered an interpolatory quadrature rule of the form

$$\int_0^{2\pi} t(x)w(x)\,dx = \sum_{\nu=0}^{2n} w_\nu t(x_\nu), \quad t \in \mathcal{T}_n. \tag{2}$$

Such a quadrature rule can be obtained from the trigonometric interpolation polynomial (cf. [2], [4]). A simple generalization dealing with a translation of the interval $[0, 2\pi)$ was given in [5]. Thus, the mentioned problem can be considered on any interval whose length is equal to 2π, i.e., on any interval of the form $[L, L + 2\pi)$, $L \in \mathbb{R}$.

Definition 1. *A quadrature rule of the form*

$$\int_L^{L+2\pi} f(x)w(x)\,dx = \sum_{\nu=0}^{n} w_\nu f(x_\nu) + R_n(f),$$

where $L \in \mathbb{R}$, $L \le x_0 < x_1 < \cdots < x_n < L + 2\pi$, has a trigonometric degree of exactness equal to d if $R_n(f) = 0$ for all $f \in \mathcal{T}_d$ and there exists $g \in \mathcal{T}_{d+1}$ such that $R_n(g) \neq 0$.

Turetzkii tried to increase the trigonometric degree of exactness of a quadrature rule (2) in such a way that he did not specify in advance the nodes x_ν, $\nu = 0, 1, \ldots, 2n$. His approach was a simulation of the development of Gaussian quadrature rules for algebraic polynomials. He proved that the trigonometric degree of exactness of the quadrature rule (2) is $2n$ if and only if the nodes x_ν $(\in [0, 2\pi))$, $\nu = 0, 1, \ldots, 2n$, are zeros of a trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ which is orthogonal on $[0, 2\pi)$ with respect to the weight function $w(x)$ to every trigonometric polynomial of a semi-integer degree less than or equal to $n - \frac{1}{2}$. It is said that such a quadrature rule is of Gaussian type, because it has the maximal trigonometric degree of exactness.

The trigonometric polynomial of semi-integer degree $A_{n+\frac{1}{2}}$, which is orthogonal on $[0, 2\pi)$ with respect to a weight function $w(x)$ to every trigonometric polynomial of a semi-integer degree less than or equal to $n - \frac{1}{2}$, with given leading coefficients c_n and d_n, is uniquely determined (see [7, §3]) and it has in $[0, 2\pi)$ exactly $2n + 1$ distinct simple zeros (see [7, Theorem 3]).

Orthogonal trigonometric polynomials of semi-integer degree were studied in detail in [5], where two choices of the leading coefficients for such orthogonal systems were considered: $c_n = 1$, $d_n = 0$ and $c_n = 0$, $d_n = 1$. It was proved that such orthogonal systems satisfied some five–term recurrence relations. Also, a numerical method for constructing Gaussian type quadratures based on the five–term recurrence relations was presented. For some special weight functions the explicit formulas for the five-term recursion coefficients were obtained in [6].

A concept of orthogonality in the space $\mathcal{T}^{1/2}$ can be considered more generally. Namely, it is known that orthogonal algebraic polynomials can be defined with respect to a moment functional (see [1], [4]). In this paper we consider orthogonal trigonometric polynomials of semi-integer degree with respect to a linear functional defined on the vector space $\mathcal{T}$. The paper is organized as follows. The second section gives a general concept of trigonometric polynomials of semi-integer degree which are orthogonal with respect to a given linear functional. It is also an introduction to a very suitable matrix notation for this purpose. The third section establishes the existence of three-term recurrence relations. Also, the corresponding Christofell-Darboux formulas are proved.

2 Orthogonality with respect to a moment functional

Definition 2. *Let m_0 be a real number, $\{m_n^C\}$, $\{m_n^S\}$, $n \in \mathbb{N}$, two sequences of real numbers, and let $\mathcal{L}$ be a linear functional defined on the vector space $\mathcal{T}$ by*

$$\mathcal{L}[1] = m_0, \quad \mathcal{L}[\cos nx] = m_n^C, \quad \mathcal{L}[\sin nx] = m_n^S, \quad n \in \mathbb{N}.$$

Then $\mathcal{L}$ is called the moment functional determined by m_0 and by the sequences $\{m_n^C\}$, $\{m_n^S\}$.

For a 2×2 type matrix $[t_{ij}]$, whose entries are trigonometric polynomials, for the brevity we denote by $\mathcal{L}[[t_{ij}]]$ the following 2×2 type matrix $[\mathcal{L}[t_{ij}]]$.

For each $k \in \mathbb{N}_0$ let denote by $\mathbf{x}^k$ the column vector

$$\mathbf{x}^k = \Big[\cos\Big(k + \frac{1}{2}\Big)x \quad \sin\Big(k + \frac{1}{2}\Big)x\Big]^T.$$

For $k, j \in \mathbb{N}_0$, let define matrices $\mathbf{m}_{k,j}$ by

$$\mathbf{m}_{k,j} = \mathcal{L}[\mathbf{x}^k(\mathbf{x}^j)^T]. \tag{3}$$

By definition, $\mathbf{m}_{k,j}$ is a matrix of type 2×2 and its elements are linear combinations of the moments m_0, $\{m_n^C\}$, $\{m_n^S\}$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}_0$, the matrices $\mathbf{m}_{k,j}$, $k, j = 0, 1, \ldots, n$, are used to define the so-called *moment matrix*

$$M_n = [\mathbf{m}_{k,j}]_{k,j=0}^n. \tag{4}$$

We denote its determinant by Δ_n, i.e.,

$$\Delta_n = \det M_n. \tag{5}$$

Lemma 1. *The moment matrix M_n, $n \in \mathbb{N}_0$, is a symmetric matrix.*

Proof. It is easy to see that all of the matrices $\mathbf{m}_{k,k}$, $k = 0, 1, \ldots, n$, are symmetric. Since

$$\mathcal{L}[\mathbf{x}^j(\mathbf{x}^k)^T] = \mathcal{L}\left[\left(\mathbf{x}^k(\mathbf{x}^j)^T\right)^T\right],$$

it follows that $\mathbf{m}_{j,k} = \mathbf{m}_{k,j}^T$, $k, j = 0, 1, \ldots, n$, i.e., the moment matrix M_n is symmetric. $\square$

If $\mathcal{L}$ is a moment functional and $A_{n+\frac{1}{2}}(x)$ is a trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ such that $\mathcal{L}[A_{n+\frac{1}{2}}t] = 0$ for every $t \in \mathcal{T}_{n-1}^{1/2}$, then $A_{n+\frac{1}{2}}$ is an orthogonal trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ with respect to the moment functional $\mathcal{L}$. One can start with the basis $\left\{\cos(k+\frac{1}{2})x, \sin(k+\frac{1}{2})x : k = 0, 1, \ldots, n\right\}$ of $\mathcal{T}_n^{1/2}$ and use the Gramm-Schmidt orthogonalization method to generate a new basis, whose elements are mutually orthogonal with respect to $\mathcal{L}$. It is obvious that in any basis of $\mathcal{T}_n^{1/2}$, for all $k = 0, 1, \ldots, n$ we have two linearly independent trigonometric polynomials of the same semi-integer degree $k + \frac{1}{2}$.

The orthogonal trigonometric polynomials of semi-integer degree with respect to a suitable weight function w on $[0, 2\pi)$, considered in [7], [5] and [6], are orthogonal trigonometric polynomials of semi-integer degree with respect to the linear functional $\mathcal{L}_w$, defined by

$$\mathcal{L}_w[t] := \int_0^{2\pi} t(x)\, w(x)\, dx, \quad t \in \mathcal{T}. \tag{6}$$

It is required that the orthogonal trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ is orthogonal to every element of $\mathcal{T}_{n-1}^{1/2}$. As a matter of fact, the orthogonality is considered only in terms of trigonometric polynomials of different semi-integer degrees, i.e., trigonometric polynomials of the same semi-integer degree have to be orthogonal to all trigonometric polynomials of lower semi-integer degrees, but they may not be orthogonal among themselves. So, we follow this idea when we define orthogonal trigonometric polynomials of semi-integer degree with respect to a moment functional. Let us denote by

$$\mathbf{A}_k(x) = \left[A_{k+\frac{1}{2}}^{(1)}(x) \quad A_{k+\frac{1}{2}}^{(2)}(x)\right]^T, \quad k \in \mathbb{N}_0,$$

the vector whose elements are two linearly independent trigonometric polynomials of semi-integer degree $k + \frac{1}{2}$. We use the following notation

$$\mathcal{S}\{\mathbf{A}_0(x), \ldots, \mathbf{A}_n(x)\} = \left\{A_{\frac{1}{2}}^{(1)}(x), A_{\frac{1}{2}}^{(2)}(x), \ldots, A_{n+\frac{1}{2}}^{(1)}(x), A_{n+\frac{1}{2}}^{(2)}(x)\right\}, \ n \in \mathbb{N}_0,$$

for the set consisting of components of the vectors $\mathbf{A}_k(x)$, $k = 0, 1, \ldots, n$.

We may also call $\mathbf{A}_k(x)$ a trigonometric polynomial of semi-integer degree $k + \frac{1}{2}$. By $\mathbf{0}$ we denote the zero vector $[0 \quad 0]^T$, as well as the 2×2 type zero matrix, which will be clear from the context, and finally, by I and $\widehat{I}$ we denote the identity matrix of type 2×2 and the matrix

$$\widehat{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively.

Definition 3. *Let $\mathcal{L}$ be a moment functional. A sequence of trigonometric polynomials of semi-integer degree $\{\mathbf{A}_n(x)\}_{n=0}^{+\infty}$ is said to be orthogonal with respect to $\mathcal{L}$ if the following conditions are satisfied:*

$$\mathcal{L}[\mathbf{x}^k \mathbf{A}_n^T] = 0, \quad k < n; \quad \mathcal{L}[\mathbf{x}^n \mathbf{A}_n^T] = K_n, \tag{7}$$

where K_n, $n \in \mathbb{N}_0$, is an invertible 2×2 type matrix.

Notice that in Definition 3 is indirectly assumed that $\mathcal{L}$ permits the existence of such an orthogonal sequence $\{\mathbf{A}_n(x)\}_{n=0}^{+\infty}$.

Lemma 2. *Let $\mathcal{L}$ be a moment functional and $\{\mathbf{A}_k\}_{k=0}^{+\infty}$ be a sequence of orthogonal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$. Then the set $\mathcal{S}\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$ forms a basis for $\mathcal{T}_n^{1/2}$, $n \in \mathbb{N}_0$.*

Proof. Since $\dim(\mathcal{T}_n^{1/2}) = 2n + 2$ and the set $\mathcal{S}\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$ has $2n + 2$ elements, only we need to prove is a linear independence. Let consider the sum $\mathbf{a}_0^T \mathbf{A}_0 + \mathbf{a}_1^T \mathbf{A}_1 + \cdots + \mathbf{a}_n^T \mathbf{A}_n$, where $\mathbf{a}_k = [a_k^1 \;\; a_k^2]^T$, $a_k^j \in \mathbb{R}$, $k = 0, 1, \ldots, n$, $j = 1, 2$. Multiplying the previous sum from the right hand side by the $(\mathbf{x}^k)^T$ and applying $\mathcal{L}$, due to orthogonality, it follows from $\mathbf{a}_0^T \mathbf{A}_0 + \mathbf{a}_1^T \mathbf{A}_1 + \cdots + \mathbf{a}_n^T \mathbf{A}_n = \mathbf{0}$ that $\mathbf{a}_k^T K_k^T = \mathbf{0}$. Since K_k is invertible, it follows that $\mathbf{a}_k = \mathbf{0}$. Therefore, $\mathcal{S}\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$ is a linearly independent system, which forms a basis for $\mathcal{T}_n^{1/2}$. □

Orthogonal trigonometric polynomials of semi-integer degree $\mathbf{A}_n$, $n \in \mathbb{N}_0$, can be written as

$$\mathbf{A}_n = C_{n,n} \mathbf{x}^n + C_{n,n-1} \mathbf{x}^{n-1} + \cdots + C_{n,0} \mathbf{x}^0, \tag{8}$$

where $C_{n,k}$, $k = 0, 1, \ldots, n$, are 2×2 type real matrices. The matrix $C_{n,n}$ is called the *leading coefficient* of $\mathbf{A}_n$.

Lemma 3. *Let $\mathcal{L}$ be a moment functional and $\mathbf{A}_n$, $n \in \mathbb{N}_0$, be an orthogonal trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ with respect to $\mathcal{L}$. Then the leading coefficient $C_{n,n}$ is an invertible matrix.*

Proof. According to Lemma 2, there exists a matrix $C'_{n,n}$ such that

$$\mathbf{x}^n = C'_{n,n} \mathbf{A}_n + \mathbf{B}_{n-1},$$

where $\mathbf{B}_{n-1}$ is a vector whose components belong to $\mathcal{T}_{n-1}^{1/2}$. Comparing coefficients of $\mathbf{x}^n$, we obtain $C'_{n,n} C_{n,n} = I$, which implies that $C_{n,n}$ is invertible. □

If $\{\mathbf{A}_n\}$ is a sequence of orthogonal trigonometric polynomials of semi-integer degree and $C_{n,n}$ denotes the leading coefficient of $\mathbf{A}_n$, then $\widetilde{\mathbf{A}}_n = C_{n,n}^{-1}\mathbf{A}_n$ yields the corresponding *monic* sequence $\{\widetilde{\mathbf{A}}_n\}$ of orthogonal trigonometric polynomials of semi-integer degree. Namely, we have

$$\mathcal{L}[\mathbf{x}^k\widetilde{\mathbf{A}}_n^T] = \mathcal{L}[\mathbf{x}^k\mathbf{A}_n^T(C_{n,n}^{-1})^T] = 0, \quad k = 0, 1, \ldots, n-1,$$

and

$$\mathcal{L}[\mathbf{x}^n\widetilde{\mathbf{A}}_n^T] = \mathcal{L}[\mathbf{x}^n\mathbf{A}_n^T(C_{n,n}^{-1})^T] = K_n(C_{n,n}^{-1})^T,$$

and $K_n(C_{n,n}^{-1})^T$ is invertible matrix by Lemma 3.

For an orthogonal system of trigonometric polynomials of semi-integer degree $\{\mathbf{A}_n\}$ with respect to a moment functional $\mathcal{L}$, let us denote by $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, the following matrix

$$\boldsymbol{\mu}_n = \mathcal{L}[\mathbf{A}_n\mathbf{A}_n^T]. \tag{9}$$

It is obvious that the matrix $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, given by (9) is symmetric.

Lemma 4. *Let $\mathcal{L}$ be a moment functional and $\mathbf{A}_n$, $n \in \mathbb{N}_0$, be an orthogonal trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ with respect to $\mathcal{L}$. Then the matrix $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, given by* (9) *is invertible.*

Proof. Since $\boldsymbol{\mu}_n = \mathcal{L}[\mathbf{A}_n\mathbf{A}_n^T] = C_{n,n}\mathcal{L}[\mathbf{x}^n\mathbf{A}_n^T] = C_{n,n}K_n$, it is invertible according to Lemma 3. $\square$

Theorem 1. *Let $\mathcal{L}$ be a moment functional and $\mathbf{A}_n$, $n \in \mathbb{N}_0$, an orthogonal trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ with respect to $\mathcal{L}$. Then $\mathbf{A}_n$ is uniquely determined by the matrix K_n.*

Proof. Suppose contrary that there exist $\mathbf{A}_n$ and $\mathbf{A}'_n$, both satisfying the orthogonality conditions (7) with the same K_n. Let $C_{n,n}$ and $C'_{n,n}$ denote the leading coefficients of $\mathbf{A}_n$ and $\mathbf{A}'_n$, respectively. Since the system $\mathcal{S}\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$ forms a basis of $\mathcal{T}_n^{1/2}$, the elements of $\mathbf{A}'_n$ can be written in terms of that basis. So, there exist 2×2 type matrices C_k, $k = 0, 1, \ldots, n$, such that

$$\mathbf{A}'_n = C_n\mathbf{A}_n + C_{n-1}\mathbf{A}_{n-1} + \cdots + C_0\mathbf{A}_0.$$

Multiplying the both hand sides of the above equation from the right by $\mathbf{A}_k^T$, $k = 0, 1, \ldots, n-1$, and applying the moment functional $\mathcal{L}$, we get that $C_k\mathcal{L}[\mathbf{A}_k\mathbf{A}_k^T] = 0$, $k = 0, 1, \ldots, n-1$, by orthogonality. According to Lemma 4, it follows that $C_k = 0$ for all $k = 0, 1, \ldots, n-1$, i.e., $\mathbf{A}'_n = C_n\mathbf{A}_n$. Comparing the leading coefficients leads to $C'_{n,n} = C_nC_{n,n}$, i.e., $C_n = C'_{n,n}C_{n,n}^{-1}$ and $\mathbf{A}_n = C_{n,n}C'^{-1}_{n,n}\mathbf{A}'_n$. By using (7) we obtain

$$K_n = \mathcal{L}[\mathbf{x}^n\mathbf{A}_n^T] = \mathcal{L}[\mathbf{x}^n\mathbf{A}'^T_n](C_{n,n}C'^{-1}_{n,n})^T = K_n(C_{n,n}C'^{-1}_{n,n})^T,$$

which implies that $C_{n,n}C'^{-1}_{n,n} = I$. Thus, $C_{n,n} = C'_{n,n}$ and $\mathbf{A}_n = \mathbf{A}'_n$. $\square$

Theorem 2. *Let $\mathcal{L}$ be a moment functional. A system of orthogonal trigonometric polynomials of semi-integer degree with respect to the moment functional $\mathcal{L}$ exists if and only if $\Delta_n \neq 0$, $n \in \mathbb{N}_0$.*

Proof. Using the expanded form (8) of $\mathbf{A}_n$ and the matrices $\mathbf{m}_{k,j}$ defined by (3), we get

$$\begin{aligned} \mathcal{L}[\mathbf{x}^k \mathbf{A}_n^T] &= \mathcal{L}[\mathbf{x}^k (C_{n,n}\mathbf{x}^n + C_{n,n-1}\mathbf{x}^{n-1} + \cdots + C_{n,0}\mathbf{x}^0)^T] \\ &= \mathbf{m}_{k,n} C_{n,n}^T + \mathbf{m}_{k,n-1} C_{n,n-1}^T + \cdots + \mathbf{m}_{k,0} C_{n,0}^T. \end{aligned}$$

It is easy to see that the orthogonality conditions (7) are equivalent to the following system of linear equations

$$M_n \begin{bmatrix} C_{n,0}^T \\ \vdots \\ C_{n,n-1}^T \\ C_{n,n}^T \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ K_n \end{bmatrix}, \tag{10}$$

where M_n is the moment matrix, defined by (4).

Let us suppose that a system of orthogonal trigonometric polynomials of semi-integer degree with respect to the moment functional $\mathcal{L}$ exists. For each K_n it is unique by Theorem 1. Hence, the system of equations (10) has a unique solution, which implies that $\Delta_n \neq 0$.

Let us now suppose that $\Delta_n \neq 0$. Then for each invertible matrix K_n the system of equations (10) has a unique solution $(C_{n,0}, \ldots, C_{n,n})$. Let denote $\mathbf{A}_n = \sum_{k=0}^{n} C_{n,k}\mathbf{x}^k$. The system (10) is equivalent to the following

$$\mathcal{L}[\mathbf{x}^k \mathbf{A}_n^T] = 0, \quad k = 0, 1, \ldots, n-1; \quad \mathcal{L}[\mathbf{x}^n \mathbf{A}_n^T] = K_n,$$

i.e., orthogonal trigonometric polynomials of semi-integer degree with respect to the moment functional $\mathcal{L}$ exist. □

Definition 4. *A moment functional $\mathcal{L}$ is said to be regular if $\Delta_n \neq 0$ for all $n \in \mathbb{N}_0$.*

Definition 5. *A moment functional $\mathcal{L}$ is said to be positive definite if for all $t \in \mathfrak{T}^{1/2}$, $t \neq 0$, the following inequality $\mathcal{L}[t^2] > 0$ holds.*

Theorem 3. *If a moment functional $\mathcal{L}$ is positive definite, then $\Delta_n > 0$ for all $n \in \mathbb{N}_0$.*

Proof. Let us assume that $\mathcal{L}$ is positive definite. Let $\mathbf{v}$ be an eigenvector of the moment matrix M_n, corresponding to an eigenvalue λ. For a trigonometric polynomial of semi-integer degree $n+\frac{1}{2}$ defined by $t(x) = \sum_{k=0}^{n} \mathbf{v}_k^T \mathbf{x}^k$, it follows that $\mathbf{v}^T M_n \mathbf{v} = \mathcal{L}[t^2] > 0$. On the other hand, $\mathbf{v}^T M_n \mathbf{v} = \lambda \|\mathbf{v}\|^2$, which implies that $\lambda > 0$. Therefore, all eigenvalues are positive and then $\Delta_n = \det M_n > 0$. □

According to Theorems 2 and 3 we have the following result:

Corollary 1. *For a positive definite moment functional* $\mathcal{L}$*, there exists a system of orthogonal trigonometric polynomials of semi-integer degree with respect to* $\mathcal{L}$*, i.e., every positive definite moment functional is regular.*

Definition 6. *Let* $\mathcal{L}$ *be a positive definite moment functional. A system of trigonometric polynomials of semi-integer degree* $\{\mathbf{A}_n^*(x)\}_{n=0}^{+\infty}$ *is said to be orthonormal with respect to* $\mathcal{L}$ *if the following conditions are satisfied*

$$\mathcal{L}[\mathbf{A}_m^*(\mathbf{A}_n^*)^T] = \delta_{m,n} I, \quad m, n \in \mathbb{N}_0, \tag{11}$$

where $\delta_{m,n}$ *is Kronecker delta function.*

Lemma 5. *Let* $\mathcal{L}$ *be a regular moment functional and let* $\{\mathbf{A}_n\}$ *be a system of orthogonal trigonometric polynomials of semi-integer degree with respect to* $\mathcal{L}$*. Then* $\mathcal{L}$ *is a positive definite moment functional if and only if all of the matrices* $\boldsymbol{\mu}_n$*,* $n \in \mathbb{N}_0$*, given by* (9)*, are positive definite.*

Proof. If $\mathcal{L}$ is a positive definite moment functional, then for any nonzero vector $\mathbf{a}$ with real entries, $t(x) = \mathbf{a}^T \mathbf{A}_n(x)$ is a nonzero trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ by Lemma 2. Therefore, $\mathbf{a}^T \boldsymbol{\mu}_n \mathbf{a} = \mathcal{L}(t^2) > 0$, which means that $\boldsymbol{\mu}_n$ is a positive definite matrix.

Let us now suppose that all of the matrices $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, given by (9), are positive definite. According to Lemma 2, every nonzero trigonometric polynomial of semi-integer degree $n + \frac{1}{2}$ can be represented in the following form $t(x) = \sum_{k=0}^{n} t_k(x)$, where $t_k(x) = \mathbf{a}_k^T \mathbf{A}_k(x)$, $k = 0, 1, \ldots, n$, and $\mathbf{a}_n$ differs from the zero vector. Because of orthogonality we get

$$\mathcal{L}[t^2] = \sum_{k=0}^{n} \mathcal{L}[t_k^2] = \sum_{k=0}^{n} \mathbf{a}_k^T \boldsymbol{\mu}_k \mathbf{a}_k,$$

which is positive since all of the matrices $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, are positive definite. □

Theorem 4. *If* $\mathcal{L}$ *is a positive definite moment functional, then there exists a system of orthonormal trigonometric polynomials of semi-integer degree* $\{\mathbf{A}_n^*(x)\}$ *with respect to* $\mathcal{L}$*.*

Proof. Let $\{\mathbf{A}_n\}$ be a system of orthogonal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$, and let $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, be the matrix defined by (9). According to Lemma 5, the matrix $\boldsymbol{\mu}_n$ is positive definite. Let $\boldsymbol{\nu}_n$ be the positive definite square root of $\boldsymbol{\mu}_n$, i.e., the unique positive definite matrix such that $\boldsymbol{\mu}_n = \boldsymbol{\nu}_n \boldsymbol{\nu}_n$ (see [8]). Since $\boldsymbol{\mu}_n$ is a symmetric matrix, the matrix $\boldsymbol{\nu}_n$ is also symmetric. Let define $\mathbf{A}_n^*(x) = \boldsymbol{\nu}_n^{-1} \mathbf{A}_n(x)$. Then we have

$$\mathcal{L}[\mathbf{A}_n^*(\mathbf{A}_n^*)^T] = \boldsymbol{\nu}_n^{-1} \mathcal{L}[\mathbf{A}_n \mathbf{A}_n^T] \boldsymbol{\nu}_n^{-1} = \boldsymbol{\nu}_n^{-1} \boldsymbol{\mu}_n \boldsymbol{\nu}_n^{-1} = I,$$

which proves the assertion. □

Remark 1. *It is easy to see that a system of orthonormal trigonometric polynomials of semi-integer degree with respect to a positive definite moment functional* $\mathcal{L}$ *is not unique. As a matter of fact, if* $\{\mathbf{A}_n^*\}$ *is an orthonormal system, then for any orthogonal* 2×2 *type matrix* O_n, $\{O_n\mathbf{A}_n^*\}$ *is also an orthonormal system with respect to the same moment functional* $\mathcal{L}$. *Moreover, if* $\mathbf{A}_n^*$ *and* $\widehat{\mathbf{A}}_n^*$ *are two vectors of orthonormal trigonometric polynomials of semi-integer degree with respect to a positive definite moment functional* $\mathcal{L}$, *then* $\mathbf{A}_n^*$ *and* $\widehat{\mathbf{A}}_n^*$ *differ by multiplication by an orthogonal* 2×2 *type matrix.*

3 Three-term recurrence relations

It is well known that orthogonal algebraic polynomials satisfy the three-term recurrence relation (see [1], [3], [4]). Such a recurrence relation is one of the most important piece of information for the constructive and computational use of orthogonal polynomials. Knowledge of the recursion coefficients allows the zeros of orthogonal polynomials to be computed as eigenvalues of a symmetric tridiagonal matrix, and with them the Gaussian quadrature rule, and also allows an efficient evaluation of expansions in orthogonal polynomials.

For the orthogonal trigonometric polynomials of semi-integer degree with respect to a regular moment functional $\mathcal{L}$, there exists three-term recurrence relations in a vector-matrix form. Actually, two kinds of recurrence relations exist, the first one with cosine function, and the second one with sine function.

3.1 Three-term recurrence relation with cosine function

Theorem 5. *Let* $\mathcal{L}$ *be a regular moment functional and* $\{\mathbf{A}_n\}$ *be a system of orthogonal trigonometric polynomials of semi-integer degree with respect to* $\mathcal{L}$. *Then,*

$$2\cos x\mathbf{A}_n = \boldsymbol{\gamma}_n^C\mathbf{A}_{n+1} + \boldsymbol{\alpha}_n^C\mathbf{A}_n + \boldsymbol{\beta}_n^C\mathbf{A}_{n-1}, \quad n = 0, 1, \ldots; \quad \mathbf{A}_{-1} = \mathbf{0}, \tag{12}$$

where $\boldsymbol{\beta}_0^C$ *is arbitrary* 2×2 *type matrix and* $\boldsymbol{\alpha}_n^C$, $\boldsymbol{\beta}_n^C$ *and* $\boldsymbol{\gamma}_n^C$ *are* 2×2 *type matrices given by*

$$\begin{aligned}
\boldsymbol{\gamma}_n^C &= \mathcal{L}[2\cos x\mathbf{A}_n\mathbf{A}_{n+1}^T]\boldsymbol{\mu}_{n+1}^{-1}, \quad n \in \mathbb{N}_0, \\
\boldsymbol{\alpha}_n^C &= \mathcal{L}[2\cos x\mathbf{A}_n\mathbf{A}_n^T]\boldsymbol{\mu}_n^{-1}, \quad n \in \mathbb{N}_0, \\
\boldsymbol{\beta}_n^C &= \boldsymbol{\mu}_n(\boldsymbol{\gamma}_{n-1}^C)^T\boldsymbol{\mu}_{n-1}^{-1}, \quad n \in \mathbb{N}.
\end{aligned} \tag{13}$$

Proof. Since the components of $2\cos x\mathbf{A}_n$ are trigonometric polynomials of semi-integer degree $n + 1 + \frac{1}{2}$, they can be represented as a linear combination of orthogonal trigonometric polynomials of semi-integer degree at most $n + 1 + \frac{1}{2}$ by Lemma 2. Therefore, in a vector notation, there exist 2×2 type matrices C_k, $k = 0, 1, \ldots, n + 1$, such that

$$2\cos x\mathbf{A}_n = C_{n+1}\mathbf{A}_{n+1} + C_n\mathbf{A}_n + \cdots + C_0\mathbf{A}_0.$$

Multiplying the both hand sides of the previous equation by $\mathbf{A}_k^T$, $k = 0, 1, \ldots, n-2$, from the right and applying the moment functional $\mathcal{L}$, due to orthogonality we obtain $C_k \boldsymbol{\mu}_k = \mathcal{L}[2\cos x \mathbf{A}_n \mathbf{A}_k^T] = 0$, which implies that $C_k = 0$, since $\boldsymbol{\mu}_k$ is an invertible matrix. Therefore, the three-term recurrence relation (12) holds.

Let us now multiply the both hand sides of the equation (12) by $\mathbf{A}_n^T$ from the right and apply the moment functional $\mathcal{L}$. Due to orthogonality we obtain

$$\mathcal{L}[2\cos x \mathbf{A}_n \mathbf{A}_n^T] = \boldsymbol{\alpha}_n^C \mathcal{L}[\mathbf{A}_n \mathbf{A}_n^T] = \boldsymbol{\alpha}_n^C \boldsymbol{\mu}_n,$$

which yields the expression for $\boldsymbol{\alpha}_n^C$. In the similar way, multiplying the equation (12) by $\mathbf{A}_{n+1}^T$ from the right and applying the moment functional $\mathcal{L}$, because of orthogonality, we obtain the expression for $\boldsymbol{\gamma}_n^C$.

To finish the proof, we write the recurrence relation (12) with $n+1$ instead of n, transpose the written equation, multiply by $\mathbf{A}_n$ from the left and apply $\mathcal{L}$. Then, we obtain

$$\mathcal{L}[2\cos x \mathbf{A}_n \mathbf{A}_{n+1}^T] = \boldsymbol{\mu}_n \boldsymbol{\beta}_{n+1}^C,$$

i.e., $\boldsymbol{\gamma}_n^C \boldsymbol{\mu}_{n+1} = \boldsymbol{\mu}_n \boldsymbol{\beta}_{n+1}^T$. Changing n by $n-1$, it is easy to get what is stated. □

Remark 2. *Although the matrix coefficient $\boldsymbol{\beta}_0^C$ in (12) can be chosen arbitrarily, since it multiplies $\mathbf{A}_{-1} = \mathbf{0}$, it is convenient for later purposes to define $\boldsymbol{\beta}_0^C = \boldsymbol{\mu}_0$.*

Lemma 6. *All of the matrices $\boldsymbol{\gamma}_n^C$, $n \in \mathbb{N}_0$, and $\boldsymbol{\beta}_n^C$, $n \in \mathbb{N}$, in (12) are invertible.*

Proof. Writing $\mathbf{A}_n$ and $\mathbf{A}_{n+1}$ in the recurrence relation (12) in the expanded forms (8) and comparing the highest coefficients at both hand sides it follows that $C_{n,n} = \boldsymbol{\gamma}_n C_{n+1,n+1}$. Since the matrices $C_{n,n}$ and $C_{n+1,n+1}$ are invertible by Lemma 3, the matrix $\boldsymbol{\gamma}_n$, is also invertible for all $n \in \mathbb{N}_0$. The assertion for the matrix $\boldsymbol{\beta}_n$, $n \in \mathbb{N}$, follows from the last equation in (13) and Lemma 4. □

We proved in Theorem 4 the existence of an orthonormal sequence $\{\mathbf{A}_n^*\}$ of trigonometric polynomials of semi-integer degree for a positive definite moment functional $\mathcal{L}$. For such a case, the recurrence relation can be considered, too. The steps in proof are the same as in Theorem 5 with $\boldsymbol{\mu}_n = I$, $n \in \mathbb{N}_0$.

Theorem 6. *Let $\mathcal{L}$ be a positive definite moment functional and $\{\mathbf{A}_n^*\}$ be a system of orthonormal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$. Then,*

$$2\cos x \mathbf{A}_n^* = \boldsymbol{\beta}_{n+1}^{*C} \mathbf{A}_{n+1}^* + \boldsymbol{\alpha}_n^{*C} \mathbf{A}_n^* + (\boldsymbol{\beta}_n^{*C})^T \mathbf{A}_{n-1}^*, \; n = 0, 1, \ldots; \; \mathbf{A}_{-1} = \mathbf{0}, \quad (14)$$

where $\boldsymbol{\beta}_0^{*C}$ is arbitrary 2×2 type matrix and $\boldsymbol{\alpha}_n^{*C}$ and $\boldsymbol{\beta}_n^{*C}$ are 2×2 type matrices given by

$$\boldsymbol{\alpha}_n^{*C} = \mathcal{L}[2\cos x \mathbf{A}_n^* (\mathbf{A}_n^*)^T], \quad \boldsymbol{\beta}_n^{*C} = \mathcal{L}[2\cos x \mathbf{A}_{n-1}^* (\mathbf{A}_n^*)^T], \quad n \in \mathbb{N}_0. \quad (15)$$

Remark 3. *It is easy to see that each $\boldsymbol{\alpha}_n^{*C}$, $n \in \mathbb{N}_0$, is symmetric and all of the matrices $\boldsymbol{\beta}_n^{*C}$, $n \in \mathbb{N}$, are invertible.*

3.2 Three-term recurrence relation with sine function

Let $\{\mathbf{A}_n\}$ be a system of orthogonal trigonometric polynomials of semi-integer degree with respect to a regular moment functional $\mathcal{L}$.

Since the components of $2\sin x\mathbf{A}_n$ are trigonometric polynomials of semi-integer degree $n+1+\frac{1}{2}$, they can also be represented as a linear combination of orthogonal trigonometric polynomials of semi-integer degree at most $n+1+\frac{1}{2}$ by Lemma 2. A consequence of this fact is that one can consider the following representation

$$2\sin x\mathbf{A}_n = S_{n+1}\mathbf{A}_{n+1} + S_n\mathbf{A}_n + \cdots + S_0\mathbf{A}_0,$$

for some matrices S_k, $k = 0, 1, \ldots, n+1$. By using the above equation, in analogous way as in the proof of Theorem 5, the following result can be proved.

Theorem 7. *Let $\mathcal{L}$ be a regular moment functional and $\{\mathbf{A}_n\}$ be a system of orthogonal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$. Then,*

$$2\sin x\mathbf{A}_n = \boldsymbol{\gamma}_n^S\mathbf{A}_{n+1} + \boldsymbol{\alpha}_n^S\mathbf{A}_n + \boldsymbol{\beta}_n^S\mathbf{A}_{n-1}, \quad n = 0, 1, \ldots; \quad \mathbf{A}_{-1} = \mathbf{0}, \tag{16}$$

where $\boldsymbol{\alpha}_n^S$, $\boldsymbol{\beta}_n^S$ and $\boldsymbol{\gamma}_n^S$ are 2×2 type matrices given by

$$\begin{aligned} \boldsymbol{\gamma}_n^S &= \mathcal{L}[2\sin x\mathbf{A}_n\mathbf{A}_{n+1}^T]\boldsymbol{\mu}_{n+1}^{-1}, \quad n \in \mathbb{N}_0, \\ \boldsymbol{\alpha}_n^S &= \mathcal{L}[2\sin x\mathbf{A}_n\mathbf{A}_n^T]\boldsymbol{\mu}_n^{-1}, \quad n \in \mathbb{N}_0, \\ \boldsymbol{\beta}_n^S &= \boldsymbol{\mu}_n(\boldsymbol{\gamma}_{n-1}^S)^T\boldsymbol{\mu}_{n-1}^{-1}, \quad n \in \mathbb{N}. \end{aligned} \tag{17}$$

The matrix coefficient $\boldsymbol{\beta}_0^S$ can be chosen arbitrarily, but we define it as $\boldsymbol{\beta}_0^C = \boldsymbol{\mu}_0$.

Analogously as in Lemma 6, we can prove that all of the matrices $\boldsymbol{\gamma}_n^S$, $n \in \mathbb{N}_0$, and $\boldsymbol{\beta}_n^S$, $n \in \mathbb{N}$, are invertible.

With $\boldsymbol{\mu}_n = I$, $n \in \mathbb{N}_0$, the following result can be easily proved.

Theorem 8. *Let $\{\mathbf{A}_n^*\}$ be a system of orthonormal trigonometric polynomials of semi-integer degree with respect to a positive definite moment functional $\mathcal{L}$. Then we have the following three-term recurrence relation with sine function,*

$$2\sin x\mathbf{A}_n^* = \boldsymbol{\beta}_{n+1}^{*S}\mathbf{A}_{n+1}^* + \boldsymbol{\alpha}_n^{*S}\mathbf{A}_n^* + (\boldsymbol{\beta}_n^{*S})^T\mathbf{A}_{n-1}^*, \quad n = 0, 1, \ldots; \quad \mathbf{A}_{-1} = \mathbf{0}, \tag{18}$$

where $\boldsymbol{\alpha}_n^{*S}$ and $\boldsymbol{\beta}_n^{*S}$ are 2×2 type matrices given by

$$\boldsymbol{\alpha}_n^{*S} = \mathcal{L}[2\sin x\mathbf{A}_n^*(\mathbf{A}_n^*)^T], \quad \boldsymbol{\beta}_n^{*S} = \mathcal{L}[2\sin x\mathbf{A}_{n-1}^*(\mathbf{A}_n^*)^T], \quad n \in \mathbb{N}_0. \tag{19}$$

The matrix coefficient $\boldsymbol{\beta}_0^{*S}$ can be chosen arbitrarily.

Also, each $\boldsymbol{\alpha}_n^{*S}$, $n \in \mathbb{N}_0$, is symmetric and all of the matrices $\boldsymbol{\beta}_n^{*S}$, $n \in \mathbb{N}$, are invertible.

3.3 Monic orthogonal trigonometric polynomials of semi-integer degree

In the sequel, by $\{\mathbf{A}_n(x)\}$ we will denote the sequence of the monic orthogonal trigonometric polynomials of semi-integer degree with respect to a regular moment functional $\mathcal{L}$. Thus, $\mathbf{A}_n(x)$ is a vector of two trigonometric polynomials of semi-integer degree, such that the first one is with the leading cosine function, and the second one with the leading sine function. We use the following quite natural notation

$$\mathbf{A}_n(x) = \begin{bmatrix} A^C_{n+\frac{1}{2}}(x) \\ A^S_{n+\frac{1}{2}}(x) \end{bmatrix},$$

where $A^C_{n+\frac{1}{2}}(x)$ and $A^S_{n+\frac{1}{2}}(x)$ have the following expanded forms

$$A^C_{n+\frac{1}{2}}(x) = \cos\Big(n+\frac{1}{2}\Big)x + \sum_{\nu=0}^{n-1}\Big[c_\nu^{(n)}\cos\Big(\nu+\frac{1}{2}\Big)x + d_\nu^{(n)}\sin\Big(\nu+\frac{1}{2}\Big)x\Big], \quad (20)$$

$$A^S_{n+\frac{1}{2}}(x) = \sin\Big(n+\frac{1}{2}\Big)x + \sum_{\nu=0}^{n-1}\Big[f_\nu^{(n)}\cos\Big(\nu+\frac{1}{2}\Big)x + g_\nu^{(n)}\sin\Big(\nu+\frac{1}{2}\Big)x\Big], \quad (21)$$

for some real coefficients $c_\nu^{(n)}$, $d_\nu^{(n)}$, $f_\nu^{(n)}$ and $g_\nu^{(n)}$, $\nu = 0, 1, \ldots, n-1$.

For a monic system of orthogonal trigonometric polynomials of semi-integer degree $\{\mathbf{A}_n\}$ with respect to a regular moment functional $\mathcal{L}$, the matrix $\boldsymbol{\gamma}_n^C$ in (12) is the identity matrix I (see proof of Lemma 6), hence, recurrence relation (12) has the following form

$$2\cos x\mathbf{A}_n = \mathbf{A}_{n+1} + \boldsymbol{\alpha}_n^C\mathbf{A}_n + \boldsymbol{\beta}_n^C\mathbf{A}_{n-1}, \quad n = 0, 1, \ldots; \quad \mathbf{A}_{-1} = \mathbf{0}. \quad (22)$$

Here, we have $\boldsymbol{\beta}_n^C = \boldsymbol{\mu}_n\boldsymbol{\mu}_{n-1}^{-1}$, $n \in \mathbb{N}$, $\boldsymbol{\beta}_0^C = \boldsymbol{\mu}_0$.

When the monic orthogonal trigonometric polynomials of semi-integer degree are in question, the situation with the recurrence relation with sine function is something different from the case with cosine function. The reason for this lies in the following simple equality

$$2\sin x \begin{bmatrix} \cos(k+\frac{1}{2})x \\ \sin(k+\frac{1}{2})x \end{bmatrix} = \begin{bmatrix} \sin(k+1+\frac{1}{2})x - \sin(k-\frac{1}{2})x \\ -\cos(k+1+\frac{1}{2})x + \cos(k-\frac{1}{2})x \end{bmatrix}.$$

In order to obtain the recurrence relation with sine function for the monic orthogonal trigonometric polynomials of semi-integer degree we only need to see the following equalities:

$$\widehat{I}\begin{bmatrix} \sin(k+\frac{1}{2})x \\ -\cos(k+\frac{1}{2})x \end{bmatrix} = \begin{bmatrix} \cos(k+\frac{1}{2})x \\ \sin(k+\frac{1}{2})x \end{bmatrix}, \quad \widehat{I}^2 = -I.$$

For the monic system of orthogonal trigonometric polynomials of semi-integer degree $\{\mathbf{A}_n\}$ with respect to a regular moment functional $\mathcal{L}$, the recurrence relation (16) has the following form

$$2\sin x\mathbf{A}_n = -\widehat{I}\mathbf{A}_{n+1} + \boldsymbol{\alpha}_n^S\mathbf{A}_n + \boldsymbol{\beta}_n^S\mathbf{A}_{n-1}, \quad n = 0, 1, \ldots; \quad \mathbf{A}_{-1} = \mathbf{0}. \qquad (23)$$

Since $(-\widehat{I})^T = \widehat{I}$, from the last equation in (17) and (23) we get here $\boldsymbol{\beta}_n^S = \boldsymbol{\mu}_n\widehat{I}\boldsymbol{\mu}_{n-1}^{-1}$, $n \in \mathbb{N}$.

3.4 Orthonormal trigonometric polynomials of semi-integer degree

If $\mathcal{L}$ is a positive definite moment functional, then by $\{\mathbf{A}_n^*(x)\}$ we will denote the sequence of the orthonormal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$, given by $\mathbf{A}_n^*(x) = \boldsymbol{\nu}_n^{-1}\mathbf{A}_n(x)$, where the matrix $\boldsymbol{\nu}_n$ is the positive square root of the matrix $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$. As it was said, $\{\mathbf{A}_n(x)\}$ is a sequence of the monic trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$. Then, the recursion coefficients $\boldsymbol{\beta}_n^{*C}$ and $\boldsymbol{\beta}_n^{*S}$ are given as follows

$$\begin{aligned} \boldsymbol{\beta}_n^{*C} &= \mathcal{L}[2\cos x\mathbf{A}_{n-1}^*(\mathbf{A}_n^*)^T] = \boldsymbol{\nu}_{n-1}^{-1}\mathcal{L}[2\cos x\mathbf{A}_{n-1}\mathbf{A}_n^T]\boldsymbol{\nu}_n^{-1} \\ &= \boldsymbol{\nu}_{n-1}^{-1}\mathcal{L}[\mathbf{A}_n\mathbf{A}_n^T]\boldsymbol{\nu}_n^{-1} = \boldsymbol{\nu}_{n-1}^{-1}\boldsymbol{\mu}_n\boldsymbol{\nu}_n^{-1} = \boldsymbol{\nu}_{n-1}^{-1}\boldsymbol{\nu}_n; \end{aligned} \qquad (24)$$

$$\begin{aligned} \boldsymbol{\beta}_n^{*S} &= \mathcal{L}[2\sin x\mathbf{A}_{n-1}^*(\mathbf{A}_n^*)^T] = \boldsymbol{\nu}_{n-1}^{-1}\mathcal{L}[2\sin x\mathbf{A}_{n-1}\mathbf{A}_n^T]\boldsymbol{\nu}_n^{-1} \\ &= \boldsymbol{\nu}_{n-1}^{-1}\mathcal{L}[-\widehat{I}\mathbf{A}_n\mathbf{A}_n^T]\boldsymbol{\nu}_n^{-1} = -\boldsymbol{\nu}_{n-1}^{-1}\widehat{I}\boldsymbol{\mu}_n\boldsymbol{\nu}_n^{-1} = -\boldsymbol{\nu}_{n-1}^{-1}\widehat{I}\boldsymbol{\nu}_n. \end{aligned} \qquad (25)$$

Some simple properties of the recursion coefficients matrices $\boldsymbol{\alpha}_n^{*C}$, $\boldsymbol{\alpha}_n^{*S}$, $n \in \mathbb{N}_0$, and $\boldsymbol{\beta}_n^{*C}$, $\boldsymbol{\beta}_n^{*S}$, $n \in \mathbb{N}$, of the recurrence relations (14) and (18) are given in Subsections 3.1 and 3.2. The following result gives some connections between these coefficients.

Theorem 9. *Let $\{\mathbf{A}_n^*(x)\}$ be the sequence of the orthonormal trigonometric polynomials of semi-integer degree with respect to a positive definite moment functional $\mathcal{L}$, satisfying the three-term recurrence relations* (14) *and* (18)*. Then the recursion coefficients matrices satisfy the following commutativity conditions:*

$$\begin{aligned} &\boldsymbol{\beta}_k^{*C}\boldsymbol{\beta}_{k+1}^{*S} = \boldsymbol{\beta}_k^{*S}\boldsymbol{\beta}_{k+1}^{*C} \\ &\boldsymbol{\beta}_{k+1}^{*C}\boldsymbol{\alpha}_{k+1}^{*S} + \boldsymbol{\alpha}_k^{*C}\boldsymbol{\beta}_{k+1}^{*S} = \boldsymbol{\alpha}_k^{*S}\boldsymbol{\beta}_{k+1}^{*C} + \boldsymbol{\beta}_{k+1}^{*S}\boldsymbol{\alpha}_{k+1}^{*C} \\ &(\boldsymbol{\beta}_k^{*C})^T\boldsymbol{\beta}_k^{*S} + \boldsymbol{\alpha}_k^{*C}\boldsymbol{\alpha}_k^{*S} + \boldsymbol{\beta}_{k+1}^{*C}(\boldsymbol{\beta}_{k+1}^{*S})^T \\ &\qquad\qquad = (\boldsymbol{\beta}_k^{*S})^T\boldsymbol{\beta}_k^{*C} + \boldsymbol{\alpha}_k^{*S}\boldsymbol{\alpha}_k^{*C} + \boldsymbol{\beta}_{k+1}^{*S}(\boldsymbol{\beta}_{k+1}^{*C})^T, \end{aligned} \qquad (26)$$

for $k \geq 0$, where $\boldsymbol{\beta}_0^{*C} = \boldsymbol{\beta}_0^{*S} = \mathbf{0}$.

Proof. Using the recurrence relations (14) and (18), there are two different ways of calculating the matrices

$$\mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k+2}^*)^T],\ \mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k+1}^*)^T],\ \mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k}^*)^T],$$

which lead to the desired commutativity equalities. Thus, using recurrence relations and the fact that $\{\mathbf{A}_n^*(x)\}$ is orthonormal with respect to $\mathcal{L}$, we have

$$\begin{aligned}
\mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k+2}^*)^T] &= \mathcal{L}[2\cos x \mathbf{A}_k^* 2\sin x (\mathbf{A}_{k+2}^*)^T] \\
&= \mathcal{L}\Big[\big(\boldsymbol{\beta}_{k+1}^{*C}\mathbf{A}_{k+1}^* + \boldsymbol{\alpha}_k^{*C}\mathbf{A}_k^* + (\boldsymbol{\beta}_k^{*C})^T\mathbf{A}_{k-1}^*\big) \times \\
&\quad \times\big((\mathbf{A}_{k+3}^*)^T(\boldsymbol{\beta}_{k+3}^{*S})^T + (\mathbf{A}_{k+2}^*)^T(\boldsymbol{\alpha}_{k+2}^{*S})^T + (\mathbf{A}_{k+1}^*)^T\boldsymbol{\beta}_{k+2}^{*S}\big)\Big] \\
&= \mathcal{L}[\boldsymbol{\beta}_{k+1}^{*C}\mathbf{A}_{k+1}^*(\mathbf{A}_{k+1}^*)^T\boldsymbol{\beta}_{k+2}^{*S}] = \boldsymbol{\beta}_{k+1}^{*C}\boldsymbol{\beta}_{k+2}^{*S},
\end{aligned}$$

and, analogously,

$$\mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k+2}^*)^T] = \mathcal{L}[2\sin x \mathbf{A}_k^* 2\cos x (\mathbf{A}_{k+2}^*)^T] = \boldsymbol{\beta}_{k+1}^{*S}\boldsymbol{\beta}_{k+2}^{*C},$$

which leads to the first equation in (26).

Further, from

$$\begin{aligned}
\mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k+1}^*)^T] &= \mathcal{L}[2\cos x \mathbf{A}_k^* 2\sin x (\mathbf{A}_{k+1}^*)^T] \\
&= \mathcal{L}\Big[\big(\boldsymbol{\beta}_{k+1}^{*C}\mathbf{A}_{k+1}^* + \boldsymbol{\alpha}_k^{*C}\mathbf{A}_k^* + (\boldsymbol{\beta}_k^{*C})^T\mathbf{A}_{k-1}^*\big) \times \\
&\quad \times\big((\mathbf{A}_{k+2}^*)^T(\boldsymbol{\beta}_{k+2}^{*S})^T + (\mathbf{A}_{k+1}^*)^T(\boldsymbol{\alpha}_{k+1}^{*S})^T + (\mathbf{A}_{k}^*)^T\boldsymbol{\beta}_{k+1}^{*S}\big)\Big] \\
&= \boldsymbol{\beta}_{k+1}^{*C}\boldsymbol{\alpha}_{k+1}^{*S} + \boldsymbol{\alpha}_k^{*C}\boldsymbol{\beta}_{k+1}^{*S},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k+1}^*)^T] &= \mathcal{L}[2\sin x \mathbf{A}_k^* 2\cos x (\mathbf{A}_{k+1}^*)^T] \\
&= \boldsymbol{\beta}_{k+1}^{*S}\boldsymbol{\alpha}_{k+1}^{*C} + \boldsymbol{\alpha}_k^{*S}\boldsymbol{\beta}_{k+1}^{*C},
\end{aligned}$$

we obtain the second equation in (26). Notice, that we here use the fact that matrices $\boldsymbol{\alpha}_k^{*C}$ and $\boldsymbol{\alpha}_k^{*S}$ are symmetric.

Finally, from

$$\begin{aligned}
\mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k}^*)^T] &= \mathcal{L}[2\cos x \mathbf{A}_k^* 2\sin x (\mathbf{A}_{k}^*)^T] \\
&= \mathcal{L}\Big[\big(\boldsymbol{\beta}_{k+1}^{*C}\mathbf{A}_{k+1}^* + \boldsymbol{\alpha}_k^{*C}\mathbf{A}_k^* + (\boldsymbol{\beta}_k^{*C})^T\mathbf{A}_{k-1}^*\big) \times \\
&\quad \times\big((\mathbf{A}_{k+1}^*)^T(\boldsymbol{\beta}_{k+1}^{*S})^T + (\mathbf{A}_{k}^*)^T(\boldsymbol{\alpha}_{k}^{*S})^T + (\mathbf{A}_{k-1}^*)^T\boldsymbol{\beta}_{k}^{*S}\big)\Big] \\
&= \boldsymbol{\beta}_{k+1}^{*C}(\boldsymbol{\beta}_{k+1}^{*S})^T + \boldsymbol{\alpha}_k^{*C}\boldsymbol{\alpha}_k^{*S} + (\boldsymbol{\beta}_k^{*C})^T\boldsymbol{\beta}_k^{*S},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}[4\cos x \sin x \mathbf{A}_k^*(\mathbf{A}_{k}^*)^T] &= \mathcal{L}[2\sin x \mathbf{A}_k^* 2\cos x (\mathbf{A}_{k}^*)^T] \\
&= \boldsymbol{\beta}_{k+1}^{*S}(\boldsymbol{\beta}_{k+1}^{*C})^T + \boldsymbol{\alpha}_k^{*S}\boldsymbol{\alpha}_k^{*C} + (\boldsymbol{\beta}_k^{*S})^T\boldsymbol{\beta}_k^{*C},
\end{aligned}$$

we get the third equation in (26). □

3.5 Christoffel-Darboux formulas

As a direct corollary of the three-term recurrence relation for algebraic orthogonal polynomials the Christoffel-Darboux formula can be proved (see [1], [3], [4]). According to the fact that we proved three-term recurrence relations for orthogonal trigonometric polynomials of semi-integer degree, a similar formula can be expected in this trigonometric case. Actually, since we have two recurrence relation, we have two Christoffel-Darboux formulas.

Theorem 10 (Christoffel-Darboux formulas)**.** *Let $\{\mathbf{A}_n^*\}$ be a sequence of orthonormal trigonometric polynomials of semi-integer degree with respect to a positive definite linear functional. Then, for all $x, y \in \mathbb{R}$, and for all nonnegative integers n, the following formula*

$$\sum_{k=0}^{n} (\mathbf{A}_k^*(x))^T \mathbf{A}_k^*(y) = \frac{(\boldsymbol{\beta}_{n+1}^{*C} \mathbf{A}_{n+1}^*(x))^T \mathbf{A}_n^*(y) - (\mathbf{A}_n^*(x))^T (\boldsymbol{\beta}_{n+1}^{*C} \mathbf{A}_{n+1}^*(y))}{2(\cos x - \cos y)},$$

$$\sum_{k=0}^{n} (\mathbf{A}_k^*(x))^T \mathbf{A}_k^*(y) = \frac{(\boldsymbol{\beta}_{n+1}^{*S} \mathbf{A}_{n+1}^*(x))^T \mathbf{A}_n^*(y) - (\mathbf{A}_n^*(x))^T (\boldsymbol{\beta}_{n+1}^{*S} \mathbf{A}_{n+1}^*(y))}{2(\sin x - \sin y)}$$

hold.

Proof. Put $\sigma_{-1} = 0$ and

$$\sigma_k = \left(\boldsymbol{\beta}_{k+1}^{*C} \mathbf{A}_{k+1}^*(x)\right)^T \mathbf{A}_k^*(y) - (\mathbf{A}_k^*(x))^T \left(\boldsymbol{\beta}_{k+1}^{*C} \mathbf{A}_{k+1}^*(y)\right), \quad k = 0, 1, \ldots, n.$$

By using the three-term recurrence relation (14), we get

$$\begin{aligned}
\sigma_k &= \Big(2\cos x \mathbf{A}_k^*(x) - \boldsymbol{\alpha}_k^{*C} \mathbf{A}_k^*(x) - (\boldsymbol{\beta}_k^{*C})^T \mathbf{A}_{k-1}^*(x)\Big)^T \mathbf{A}_k^*(y) \\
&\quad - \left(\mathbf{A}_k^*(x)\right)^T \Big(2\cos y \mathbf{A}_k^*(y) - \boldsymbol{\alpha}_k^{*C} \mathbf{A}_k^*(y) - (\boldsymbol{\beta}_k^{*C})^T \mathbf{A}_{k-1}^*(y)\Big) \\
&= 2(\cos x - \cos y)(\mathbf{A}_k^*(x))^T \mathbf{A}_k^*(y) - (\mathbf{A}_k^*(x))^T \left((\boldsymbol{\alpha}_k^{*C})^T - \boldsymbol{\alpha}_k^{*C}\right) \mathbf{A}_k^*(y) \\
&\quad - \Big(\left(\mathbf{A}_{k-1}^*(x)\right)^T \boldsymbol{\beta}_k^{*C} \mathbf{A}_k^*(y) - (\mathbf{A}_k^*(x))^T (\boldsymbol{\beta}_k^{*C})^T \mathbf{A}_{k-1}^*(y)\Big).
\end{aligned}$$

Since the all of the matrices $\boldsymbol{\alpha}_k^{*C}$ are symmetric (see Remark 3), the second term on the right hand side of the previous expression of σ_k is equal to zero. The third term of the same expression can be written as follows

$$\begin{aligned}
&\left(\mathbf{A}_k^*(x)\right)^T (\boldsymbol{\beta}_k^{*C})^T \mathbf{A}_{k-1}^*(y) - \left(\mathbf{A}_{k-1}^*(x)\right)^T \boldsymbol{\beta}_k^{*C} \mathbf{A}_k^*(y) \\
&\qquad = \Big(\boldsymbol{\beta}_k^{*C} \mathbf{A}_k^*(x)\Big)^T \mathbf{A}_{k-1}^*(y) - \left(\mathbf{A}_{k-1}^*(x)\right)^T \boldsymbol{\beta}_k^{*C} \mathbf{A}_k^*(y) = \sigma_{k-1}.
\end{aligned}$$

Therefore, we have

$$\sigma_k = 2(\cos x - \cos y)(\mathbf{A}_k^*(x))^T \mathbf{A}_k^*(y) + \sigma_{k-1},$$

i.e.,

$$(\cos x - \cos y)\big(\mathbf{A}_k^*(x)\big)^T \mathbf{A}_k^*(y) = \sigma_k - \sigma_{k-1}.$$

Summing the previous equality for all $k = 0, 1, \ldots, n$, we get the first formula.

In the same way, by using the three-term recurrence relation (18), the second formula can be proved. □

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