A NOTE ON A FURTHER EXTENSION OF GAUSS’S
SECOND SUMMATION THEOREM WITH AN
APPLICATION TO THE EXTENSION OF TWO
WELL-KNOWN COMBINATORIAL IDENTITIES

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ABSTRACT. Recently, Masjed Jamei and Koepf established the extension of
several classical summation theorems (including Gauss’s second summation
theorem). Our aim in this paper is to establish a further extension of Gauss’s
second summation formulas due to Masjed Jamei and Koepf in the most gen-
eral form. The result is then applied to obtain extensions of (i) Knuth’s old
sum (or the Reed Dawson identity) and (ii) Riordan’s identity in the most
geneneral form. A few interesting results are obtained as special cases of our
main findings.

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1. Introduction and preliminaries
The following well-known combinatorial sums known as Knuth’s old sum [10], or
alternatively as the Reed Dawson identities [13], are
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{-k} \binom{2k}{k} = 2^{-2n} \binom{2n}{n} = \left( \frac{1}{2} \right)_n
\]  

(1.1)

and

\[
\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} 2^{-k} \binom{2k}{k} = 0,
\]

(1.2)

where \((\alpha)_n\) denotes the Pochhammer symbol (or the rising factorial) for any complex number \(\alpha \neq 0\) defined by

\[(\alpha)_n = \begin{cases} 
\alpha(\alpha+1) \cdots (\alpha+n-1), & n \in \mathbb{N}, \\
1, & n = 0.
\end{cases}
\]

It is of interest to mention that Reed Dawson presented the above identities in a private communication to Riordan who recorded them in his well-known book [13, p. 71].

Several different proofs of the above sums have been given in the literature; see the survey paper by Prodinger [10] and references therein. In 1974, Andrews [1, p. 478] established the above sums by employing the Gauss second summation theorem [9, (15.4.28)] given by

\[
2F1 \left[ \begin{array}{c} a, b \\ \frac{1}{2}(a+b+1), \frac{1}{2} \end{array} \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \right) \Gamma \left( \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \right)}.
\]

(1.3)

In 2004, Choi et al. [2] utilized the following terminating hypergeometric identities recorded, for example in [11, pp. 126–127]

\[
2F1 \left[ \begin{array}{c} -2n, \alpha \\ 2a \frac{1}{2} \end{array} \right] = \frac{\left( \frac{1}{2} \right)_n}{(\alpha + \frac{1}{2})_n},
\]

(1.4)

and

\[
2F1 \left[ \begin{array}{c} -2n, \alpha \\ 2a \frac{1}{2} \end{array} \right] = 0.
\]

(1.5)

Also, the following well-known combinatorial identities established by Riordan [13] are seen to be closely related to (1.1) and (1.2) viz.

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n+1}{k+1} 2^{-k} \binom{2k}{k} = \left( \frac{3}{2} \right)_n
\]

(1.6)

and

\[
\sum_{k=0}^{2n+1} (-1)^k \binom{2n+2}{k+1} 2^{-k} \binom{2k}{k} = \left( \frac{3}{2} \right)_n.
\]

(1.7)

Riordan [13] established (1.6) and (1.7) by the method of inverse relations.

On the other hand, in the theory of hypergeometric and generalized hypergeometric series, classical summation theorems for the series \(2F1, 3F2, 4F3, 5F4\) and others play a key role. During 1992–2011, considerable progress was made in generalizing the above-mentioned classical summation theorems. For this, we refer to the research papers by Lavoie et al. [4, 5, 6], Kim et al. [3], Rakha and Rathie [12] and Milovanović et al. [8].

More recently, in 2018, various summation theorems have been extended and generalized by Masjed-Jamei and Koepf [7]. In our present investigation, however,
we are only interested in the following generalization of Gauss's second summation theorem [7, Eq. (21)] given by

\[
\begin{align*}
3F_2 \left[ \begin{array}{c}
\frac{1}{2} a, b, \frac{1}{2} \n
\end{array} \right. & \left. \frac{1}{2} (a + b + 1), m \right] = \\
\frac{\Gamma(m) \Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} \right) \Gamma(1 + a - m) \Gamma(1 + b - m)}{2^{1-m} \Gamma(a) \Gamma(b) \Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} - m \right)} \\
& \times \left\{ \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} - m \right)}{\Gamma \left( 1 + \frac{1}{2} (a - m) \right) \Gamma \left( 1 + \frac{1}{2} (b - m) \right)} - \left( \begin{array}{c}
\frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} \n
\end{array} \right) \right. \\
& \left. \frac{1}{2} \right] \right\}^{(m-2)} \right)
\end{align*}
\]

for \( m \in \mathbb{N} \). Here, and in what follows, we denote by \( \genfrac{[}{]}{0pt}{}{(m)}{p \ F_q} (z) \) the hypergeometric series truncated after \( m + 1 \) terms, namely

\[
\genfrac{[}{]}{0pt}{}{(m)}{p \ F_q} \left[ a_1, \ldots, a_p; b_1, \ldots, b_q; z \right] = \sum_{n=0}^{m} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},
\]

so that

\[
\genfrac{[}{]}{0pt}{}{(-1)}{p \ F_q} (z) = 0, \quad \genfrac{[}{]}{0pt}{}{(0)}{p \ F_q} (z) = 1, \quad \genfrac{[}{]}{0pt}{}{(1)}{p \ F_q} (z) = 1 + \frac{a_1 \cdots a_p}{b_1 \cdots b_q} z
\]

and so on. Clearly, for \( m = 1 \), (1.8) reduces to Gauss's second summation theorem (1.3).

The paper is organized as follows. In Section 2, we establish the extension of the result (1.8). In Section 3, we get an interesting combinatorial identity in the most general form with the help of the result given in Section 2. Some corollaries are given in Section 4 and concluding remarks in 5. Otherwise, several interesting combinatorial identities, including Knuth's old sum (or the Reed Dawson identities) and Riordan's identity, have been obtained as special cases of our main results. For this, we require the following results obtained by Masjed-Jamei and Koepf [7]

\[
p \ F_q \left[ a_1, \ldots, a_{p-1}, 1; b_1, \ldots, b_{q-1}, m; z \right] = \Gamma(b_1) \cdots \Gamma(b_{q-1}) \Gamma(a_1 - m + 1) \cdots \Gamma(a_{p-1} - m + 1) \frac{(m-1)!}{\Gamma(a_1 - b_1 - m + 1) \cdots \Gamma(b_{q-1} - a_{p-1} - m + 1)} \frac{z^{m-1}}{m-1}
\]

and by Rakha and Rathie [12]

\[
2F_1 \left[ \begin{array}{c}
\frac{1}{2} a, b \n
\end{array} \right. & \left. \frac{1}{2} (a + b + i + 1); \frac{1}{2} \right] = \\
\frac{\Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} i + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a - \frac{1}{2} b - \frac{1}{2} i + \frac{1}{2} \right)}{2^{i-3} \Gamma(b) \Gamma \left( \frac{1}{2} a - \frac{1}{2} b + \frac{1}{2} i + \frac{1}{2} \right)} \times \\
& \sum_{r=0}^{i} \frac{(-1)^r (i \choose r)}{r} \frac{\Gamma \left( \frac{1}{2} b + \frac{1}{2} r \right)}{\Gamma \left( \frac{1}{2} a + \frac{1}{2} r - \frac{1}{2} i + \frac{1}{2} \right)}
\]

for \( i = 0, 1, 2, \ldots \).

2. Further extension of summation formula (1.8)

In this section, we establish the extension of the summation formula (1.8) asserted in the following theorem.
THEOREM 2.1. For \( i \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \), the following result holds true:

\[
3F_2\left[ \begin{array}{c}
\frac{a}{2}, b, 1 \\
\frac{1}{2}(a+b+i+1), \frac{1}{2}
\end{array} \right] = \frac{2^{m-1}\Gamma(m) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}\right) \Gamma(1+a-m) \Gamma(1+b-m)}{\Gamma(a)\Gamma(b) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{3}{2} - m\right)} \\
\times \left\{ A_{i,m}(a,b) - 2F_1\left[ \begin{array}{c}
1+a-m, 1+b-m \\
\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{3}{2} - m, \frac{1}{2}
\end{array} \right] \right\},
\tag{2.1}
\]

where

\[
A_{i,m}(a,b) = \frac{2^{b-m} \Gamma\left(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}i + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{3}{2} - m\right)}{\Gamma(1+b-m) \Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}\right)} \\
\times \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}b - \frac{1}{2}m + \frac{1}{2}r\right)}{\Gamma(1 + \frac{1}{2}a - \frac{1}{2}m + \frac{1}{2}r - \frac{1}{2}i)}.
\]

PROOF. The proof of the result (2.1) asserted by the theorem is quite straightforward. For this, if we set \( z = 1/2 \), \( p = 3 \), \( q = 2 \), \( a_1 = a \), \( a_2 = b \) and \( b_1 = (a+b+i+1)/2 \) in (1.9), then after a little simplification, we have

\[
3F_2\left[ \begin{array}{c}
\frac{a}{2}, b, 1 \\
\frac{1}{2}(a+b+i+1), \frac{1}{2}
\end{array} \right] = \frac{2^{m-1}\Gamma(m) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2}\right) \Gamma(1+a-m) \Gamma(1+b-m)}{\Gamma(a)\Gamma(b) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{3}{2} - m\right)} \\
\times \left\{ 2F_1\left[ \begin{array}{c}
1+a-m, 1+b-m \\
\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{3}{2} - m, \frac{1}{2}
\end{array} \right] - 2F_1\left[ \begin{array}{c}
1+a-m, 1+b-m \\
\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{3}{2} - m, \frac{1}{2}
\end{array} \right] \right\}. \tag{2.2}
\]

We now observe that the first \( 2F_1 \) appearing on the right-hand side of (2.2) can be evaluated with the help of the result (1.10) and we easily arrive at the desired result (2.1). This completes the proof of (2.1) asserted in the theorem. \( \square \)

COROLLARY 2.2. In (2.1), if we set \( i = 0 \), we at once get the known result (1.8) due to Masjed-Jamei and Koepf.

We conclude this section by remarking that by giving values to \( i \) in (2.1), we can obtain several results in compact form. Also, an interesting application of this result will be given in the next section.

3. A new combinatorial identity

In this section, we shall establish a new combinatorial identity asserted in the following theorem.
Theorem 3.1. For \( i \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \), the following result holds true:

\[
\sum_{k=0}^{n} (-1)^k \binom{n+i}{k+i} 2^{-k} \frac{\binom{2k}{k}}{(n-k+m-1)} = \frac{2^{n+m-1} \sqrt{\pi} \Gamma(m+1)}{\Gamma(n+m+1) \Gamma(n+i+m) \Gamma(-n-m+\frac{3}{2})} \\
\times \left\{ \frac{2^{-n-m-i} \sqrt{\pi} \Gamma(-n-m+\frac{3}{2})}{\Gamma(i+\frac{1}{2}) \Gamma(1-n-m-i)} \sum_{r=0}^{i} (-1)^r \left( \frac{1}{r} \Gamma \left( \frac{1}{2} - \frac{1}{2}n - \frac{1}{2}m - \frac{1}{2}i + \frac{1}{2}r \right) \right) \right\} \\
- \binom{m-2}{2} \left\{ 1 - n - m, 1 - n - m - i, -n - m + \frac{3}{2}, 1 \right\} \right\}.
\]

(3.1)

Proof. The derivation of the identity (3.1) follows from an application of (1.8). Denoting the left-hand side of (3.1) by \( S \), we have

\[
S = \sum_{k=0}^{n} (-1)^k \binom{n+i}{k+i} 2^{-k} \frac{\binom{2k}{k}}{(n-k+m-1)}.
\]

Now, reversing the order of summation by replacing \( k \) by \( n - k \), we have

\[
S = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+i}{n-k+i} 2^{-n+k} \frac{\binom{2n-2k}{n-k}}{(n-k+m-1)}.
\]

Making use of the identities

\[
(n-k)! = \frac{(-1)^k n!}{(-n)_k}, \quad (2n-2k)! = \frac{2^{2n-2k} \left( \frac{1}{2} \right)_n n!}{(-n)_k \left( \frac{1}{2} - n \right)_k}.
\]

we have, after some simplification,

\[
S = \frac{(-1)^n 2^n \left( \frac{1}{2} \right)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(-n-i)_k(1)_k 2^{-k}}{\left( \frac{1}{2} - n \right)_k (m)_k k!}
= \frac{(-1)^n 2^n \left( \frac{1}{2} \right)_n}{n!} \binom{m-2}{2} \left\{ -n, -n - i, 1 \right\} \left\{ \frac{1}{2} - n, m, \frac{1}{2} \right\}.
\]

We now observe that the \( \binom{m-2}{2} \) appearing in this last result can be evaluated with the help of (2.1) and after some calculation, we easily arrive at the right-hand side of (3.1). This completes the proof of (3.1).

\[ \square \]

4. Corollaries

In this section, we mention some special cases of Theorem 3.1.

Corollary 4.1. In (3.1), if we take \( i = 0 \), we obtain the following combinatorial identity which is also of a general nature:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{-k} \frac{\binom{2k}{k}}{(n-k+m-1)} = \frac{2^{n+m-1} \Gamma \left( \frac{1}{2} \right) \Gamma(m) \Gamma(n+1)}{\Gamma(-n-m+\frac{3}{2}) \Gamma^2(n+m)} \\
\times \left\{ \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( -n-m+\frac{3}{2} \right)}{\Gamma^2 \left( 1 - \frac{1}{2}n - \frac{1}{2}m \right)} \right\}^{(m-2)} \binom{m-2}{2} \left\{ -n - m, 1 - n - m, \frac{1}{2} \right\}.
\]

For \( m = 1, 2 \) and \( 3 \), we respectively obtain the following interesting combinatorial identities for even or odd \( n \).
(i) For \( m = 1 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2^k)}{n-k+1} = \binom{\frac{1}{2}}{\nu} \quad \text{for } n = 2\nu \text{ (even)},
\]
\[
\binom{1}{\nu} \quad \text{for } n = 2\nu + 1 \text{ (odd)},
\]
which is Knuth’s old sum or the Reed Dawson identities.

(ii) For \( m = 2 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2^k)}{(n-k+1)} = \frac{2^{2\nu} \binom{\frac{3}{2}}{\nu} \binom{\frac{5}{2}}{\nu}}{\nu^2 \binom{\frac{3}{2}}{\nu} \binom{\frac{5}{2}}{\nu}}, \quad \text{for } n = 2\nu \text{ (even)},
\]
\[
\frac{(1)_{\nu} \binom{\frac{3}{2}}{\nu}}{4(2)_{\nu}(2)_{\nu}} \quad \text{for } n = 2\nu + 1 \text{ (odd)}.
\]

(iii) For \( m = 3 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2^k)}{(n-k+2)} = \frac{1}{2} \left[ \binom{\frac{1}{2}}{\nu}(1)_{\nu} + \frac{2^{2\nu} \binom{\frac{3}{2}}{\nu} \binom{\frac{5}{2}}{\nu}}{\binom{\frac{3}{2}}{\nu} (2)_{\nu}(2)_{\nu}} \right], \quad \text{for } n = 2\nu \text{ (even)}
\]
\[
- \frac{2^{2\nu+1} \binom{\frac{3}{2}}{\nu} \binom{\frac{5}{2}}{\nu}}{3 (1)_{\nu} \binom{\frac{3}{2}}{\nu}}, \quad \text{for } n = 2\nu + 1 \text{ (odd)}.
\]

Similarly, other results can be obtained.

**Corollary 4.2.** In (3.1), if we take \( i = 1 \), we obtain the following combinatorial identity which is also of a general nature:
\[
\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{(2^k)}{(n-k+1)} = \frac{2^{n+m-1} \sqrt{\pi} \Gamma(m) \Gamma(n+2)}{\Gamma(-n-m+\frac{3}{2}) \Gamma(n+m+1)} \left\{ 2\sqrt{\pi} \Gamma(-n-m+\frac{3}{2}) \right\}
\]
\[
\times \left[ \frac{1}{\Gamma^2 \left( -\frac{1}{2}n - \frac{1}{2}m + \frac{1}{2} \right)} - \frac{1}{\Gamma \left( -\frac{1}{2}n - \frac{1}{2}m \right) \Gamma \left( 1 - \frac{1}{2}n - \frac{1}{2}m \right)} \right]^{(m-2)}_{2F_1} \left[ \frac{1 - n - m}{-n - m + \frac{3}{2}}, \frac{1}{2} \right].
\]

For \( m = 1, 2 \) and 3, we respectively obtain the following interesting combinatorial identities for even or odd \( n \).
(i) For \( m = 1 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} 2^{-k} \binom{2k}{k} = \frac{\left(\frac{3}{2}\right)_n}{(1)_n}, \quad \text{for } n = 2\nu \text{ (even)} \quad \text{and } n = 2\nu + 1 \text{ (odd)},
\]
which are the Riordan identities.

(ii) For \( m = 2 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} 2^{-k} \binom{2k}{k} \frac{2^{2\nu-1} \frac{3}{2}_n}{(3)_{2\nu}}, \quad \text{for } n = 2\nu \text{ (even)},
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} 2^{-k} \binom{2k}{k} \frac{2^{2\nu-1} \frac{3}{2}_n}{(4)_{2\nu}}, \quad \text{for } n = 2\nu + 1 \text{ (odd)}.
\]

(iii) For \( m = 3 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} 2^{-k} \binom{2k}{k} \frac{2^{2\nu-1} \frac{5}{3}_n}{(5)_{2\nu}}, \quad \text{for } n = 2\nu \text{ (even)}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} 2^{-k} \binom{2k}{k} \frac{2^{2\nu-1} \frac{5}{3}_n}{(6)_{2\nu}}, \quad \text{for } n = 2\nu + 1 \text{ (odd)}.
\]

Similarly, other results can be obtained.

**Corollary 4.3.** In (3.1), if we take \( i = 2 \), we obtain the following combinatorial identity which is also of a general nature:
\[
\sum_{k=0}^{n} (-1)^k \binom{n+2}{k+2} 2^{-k} \binom{2k}{k} \frac{(2^m-1)_{n-k}}{(n-k+1)_{n-k+1}}
\]
\[
= \frac{2^{n+m-1} \sqrt{\pi} \Gamma(m+1) \Gamma(n+3)}{\Gamma(-n-m+\frac{3}{2}) \Gamma(n+m+1)} \sum_{k=0}^{n} \binom{m-1}{n-k} \frac{2^{n-m} \Gamma(-n-m+\frac{3}{2})}{3 \Gamma(-n-m)}
\]
\[
\times \left[ \frac{\Gamma(-\frac{m}{2} n - \frac{m}{2} m - \frac{1}{2})}{\Gamma(-\frac{m}{2} n - \frac{m}{2} m + \frac{1}{2})} - 2 \frac{\Gamma(-\frac{m}{2} n - \frac{m}{2} m)}{\Gamma(-\frac{m}{2} n - \frac{m}{2} m + \frac{1}{2})} + \frac{\Gamma(-\frac{m}{2} n - \frac{m}{2} m + \frac{1}{2})}{\Gamma(-\frac{m}{2} n - \frac{m}{2} m + 1)} \right]
\]
\[
\times \binom{m-2}{n-k} \binom{1 - n - m, -n - m - 1 - \frac{1}{2}}{-n - m + \frac{3}{2}, \frac{1}{2}}.
\]

For \( m = 1, 2 \) and \( 3 \), we respectively obtain the following interesting combinatorial identities for even or odd \( n \).

(i) For \( m = 1 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n+2}{k+2} 2^{-k} \binom{2k}{k} = \frac{2^{\frac{3}{2}_n}}{3(1)_n} \binom{2\nu + \frac{3}{2}}{2\nu}, \quad \text{for } n = 2\nu \text{ (even)}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{n+2}{k+2} 2^{-k} \binom{2k}{k} = \frac{2^{\frac{3}{2}_n}}{3(1)_n} (2\nu + 3), \quad \text{for } n = 2\nu + 1 \text{ (odd)}.
\]
(ii) For $m = 2$,

$$
\sum_{k=0}^{n} (-1)^k \binom{n+2}{k+2} 2^{-k} \binom{2k}{n-k+1} = \begin{cases} \\
\frac{2\nu+2}{3} \left[ \frac{\left(\frac{4}{2}\right)_\nu}{(2)_\nu} + 2^{2\nu-1} \frac{\left(\frac{2}{2}\right)_{2\nu}}{(4)_{2\nu}} \right], & \text{for } n = 2\nu \text{ (even)} \\
\frac{2\nu+2}{4} \left(\frac{5}{2}\right)_\nu - \frac{3}{2\nu+4} \frac{2^{2\nu-1} \left(\frac{5}{2}\right)_{2\nu}}{(3)_{2\nu}}, & \text{for } n = 2\nu + 1 \text{ (odd)}. 
\end{cases}
$$

(iii) For $m = 3$,

$$
\sum_{k=0}^{n} (-1)^k \binom{n+2}{k+2} 2^{-k} \binom{2k}{n-k+2} = \begin{cases} \\
\frac{2\nu+3}{6} \left(\frac{3}{2}\right)_\nu + 2^{2\nu-2} \frac{\left(\frac{5}{2}\right)_{2\nu}}{(5)_{2\nu}} \left[ 1 - \frac{(2\nu+2)(2\nu+4)}{2(2\nu+\frac{7}{2})} \right], & \text{for } n = 2\nu \text{ (even)}, \\
\frac{2\nu+3}{6} \left(\frac{3}{2}\right)_\nu + 2^{2\nu-2} \frac{\left(\frac{7}{2}\right)_{2\nu}}{(6)_{2\nu}} \left[ 1 - \frac{(2\nu+3)(2\nu+5)}{2(2\nu+\frac{7}{2})} \right], & \text{for } n = 2\nu + 1 \text{ (odd)}. 
\end{cases}
$$

Similarly, other results can be obtained.

5. Concluding remarks

In this note, we have provided a further extension of Gauss’s second summation formula due to Masjed-Jamei and Koepf in the most general form. The result is then applied to obtain the extension of (i) Knuth’s old sum (or Reed Dawson identities) and (ii) Riordan’s identity in the most general form. Several known, as well as new, combinatorial identities have also given as special cases of our main findings. All the main results and the special cases have been verified numerically using MATHEMATICA 12.1.1.

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References

A note on further extension of Gauss’s second summation theorem


