Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 429 (2008) 2401–2414
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Positive definite solutions of some matrix equations ☆

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Received 15 April 2007; accepted 5 February 2008
Available online 25 March 2008
Submitted by L. Cvetkovic

Dedicated to Professor Richard Varga on the occasion of his 80th birthday

Abstract

In this paper we investigate some existence questions of positive semi-definite solutions for certain classes of matrix equations known as the generalized Lyapunov equations. We present sufficient and necessary conditions for certain equations and only sufficient for others.

AMS classification: 15A24; 15A48; 42A82; 47A62

Keywords: Matrix equation; Lyapunov matrix equation; Positive definite matrix; Positive definite function; Fourier transform; Bochner's theorem

1. Introduction

Recently, Bathia and Drisi [3] studied questions related to the positive semi-definiteness of solutions of the following matrix equations:

$$AX + XA = B,$$
$$A^2X + 2tAXA + XA^2 = B,$$
$$A^3X + t(A^2XA + AXA^2) + XA^3 = B,$$

☆ The work was supported by the Serbian Ministry of Science (Project #144004G) and the Swiss National Science Foundation (SCOPES Joint Research Project No. IB7320-111079).
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0024-3795/$ - see front matter
doi:10.1016/j.laa.2008.02.001

$$A^4X + tA^3XA + 6A^2XA^2 + tAXA^3 + XA^4 = B,$$
$$A^4X + 4A^3XA + tA^2XA^2 + 4AXA^3 + XA^4 = B, \tag{1.1}$$

where A is a given positive definite matrix and matrix B is positive semi-definite. First equation is known to be the Lyapunov equation and has a great deal with the analysis of the stability of motion.

Second equation has been studied by Kwong [10] and he succeeded to give an answer about the existence of the positive semi-definite solutions. In [3] necessary and sufficient conditions are given for the parameter t in order that Eq. (1.1) have positive semi-definite solutions, provided that B is positive semi-definite. For numerous other references see [2,4–10]. There is also a strong connection between the question of positive semi-definite solutions of these equations and various inequalities involving unitarily equivalent matrix norms (see [2,6–9]).

We briefly recall that a matrix A is positive definite, provided it is symmetric and for every vector $\mathbf{x} \neq 0$ we have $(A\mathbf{x}, \mathbf{x}) > 0$. A matrix A is positive semi-definite, provided it is symmetric and for every $\mathbf{x}$ we have $(A\mathbf{x}, \mathbf{x}) \geqslant 0$.

In this paper we investigate the existence question of positive semi-definite solutions of a general form of Eq. (1.1). Introducing characteristic polynomials for these equations, in Section 4 we present some sufficient conditions for the existence of these solutions. In the last section we present results concerning some specific equations.

2. Characteristic polynomial

We denote by $2\mathbb{N}$ and $2\mathbb{N} - 1$ sets of even and odd natural numbers. First, we prove a simple lemma to give a motivation for results of this paper.

Lemma 2.1. *Suppose we are given matrix equation*

$$\sum_{\nu=0}^{m} a_\nu A^{m-\nu} X A^\nu = B, \tag{2.1}$$

where B is positive semi-definite, A is positive definite, and $a_\nu = a_{m-\nu} \in \mathbb{R}$, $\nu = 0, 1, \ldots, m$, $a_0 = a_m > 0$. If the function $t \mapsto \varphi_m(t)$, defined by

$$\frac{1}{\varphi_m(t)} = \begin{cases} \sum\limits_{\nu=0}^{m/2-1} a_\nu \cosh\left(\frac{m}{2} - \nu\right) t + \frac{1}{2} a_{m/2}, & m \in 2\mathbb{N}, \\ \sum\limits_{\nu=0}^{(m-1)/2} a_\nu \cosh\left(\frac{m}{2} - \nu\right) t, & m \in 2\mathbb{N} - 1 \end{cases} \tag{2.2}$$

is positive semi-definite, then Eq. (2.1) has a positive semi-definite solution. If equation has positive semi-definite solution for any positive definite matrix A then function φ_m is positive semi-definite.

Proof. Since A is a positive definite matrix, its eigenvectors create a basis. Hence, we can use the system of eigenvectors as a basis in which the matrix A has a diagonal form, with eigenvalues on its diagonal. Denote the eigenvalues by λ_ν, $\nu = 1, \ldots, n$. Then Eq. (2.1), in the previous basis with $X = (x_{i,j})$ and $B = (b_{i,j})$, can be represented in the form

$$\sum_{\nu=0}^{m} a_\nu \lambda_i^{m-\nu} \lambda_j^\nu x_{i,j} = b_{i,j}, \quad i, j = 1, \ldots, n,$$

i.e.,

$$x_{i,j} = \frac{b_{i,j}}{\sum_{\nu=0}^{m} a_\nu \lambda_i^{m-\nu} \lambda_j^{\nu}}, \quad i, j = 1, \ldots, n.$$

If we denote with $C = (c_{i,j})$ a matrix with entries

$$c_{i,j} = \frac{1}{\sum_{\nu=0}^{m} a_\nu \lambda_i^{m-\nu} \lambda_j^{\nu}}, \quad i, j = 1, \ldots, n, \tag{2.3}$$

we can recognize that the matrix X is a direct or Schur product of the matrices C and B, so that if C and B are positive semi-definite, X is also positive semi-definite.

Since the eigenvalues λ_ν, $\nu = 1, \ldots, n$, are positive, we can represent them in the form $\lambda_i = e^{x_i}$, where $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. Applying this to the matrix C, we get for its elements and m even

$$\begin{aligned} c_{i,j} &= \frac{e^{-m/2(x_i+x_j)}}{\sum_{\nu=0}^{m} a_\nu e^{(m/2-\nu)x_i} e^{(\nu-m/2)x_j}} \\ &= \frac{e^{-m/2(x_i+x_j)}}{\sum_{\nu=0}^{m} a_\nu e^{(m/2-\nu)(x_i-x_j)}} \\ &= \frac{1}{2} \frac{e^{-m/2(x_i+x_j)}}{\sum_{\nu=0}^{m/2-1} a_\nu \cosh\left(\frac{m}{2}-\nu\right)(x_i-x_j) + \frac{1}{2}a_{m/2}}. \end{aligned}$$

Similarly, for m odd, for elements $c_{i,j}$ of C we get the following expression

$$\begin{aligned} c_{i,j} &= \frac{e^{-m/2(x_i+x_j)}}{\sum_{\nu=0}^{m} a_\nu e^{(m/2-\nu)x_i} e^{(\nu-m/2)x_j}} \\ &= \frac{1}{2} \frac{e^{-m/2(x_i+x_j)}}{\sum_{\nu=0}^{(m-1)/2} a_\nu \cosh\left(\frac{m}{2}-\nu\right)(x_i-x_j)}. \end{aligned}$$

Two matrices X and Y are said to be congruent if there exists a non-singular matrix Z such that $X = Z^* Y Z$. It is known that congruency preserves definiteness. In both cases, for even and odd m, our matrix C is congruent with a matrix with elements

$$\begin{cases} \dfrac{1}{\sum_{\nu=0}^{m/2-1} a_\nu \cosh\left(\frac{m}{2}-\nu\right)(x_i-x_j) + \frac{1}{2}a_{m/2}}, & m \in 2\mathbb{N}, \\ \dfrac{1}{\sum_{\nu=0}^{(m-1)/2} a_\nu \cosh\left(\frac{m}{2}-\nu\right)(x_i-x_j)}, & m \in 2\mathbb{N}-1, \end{cases}$$

where in both cases the congruency matrix Z is a diagonal matrix with entries $(1/\sqrt{2})e^{-m/2x_i}$, $i = 1, \ldots, n$.

Now we introduce the function $t \mapsto \varphi_m(t)$ by (2.2). Then φ_m is going to be positive semi-definite if and only if our matrix in (2.3) is positive semi-definite. □

According to the Bochner's theorem (see [13, p. 17,12, p. 290]) the function φ_m is positive semi-definite if and only if its Fourier transform is nonnegative on the real line. Hence, we can answer the existence question of positive semi-definite solutions of Eq. (2.1), provided we are able to answer the question whether the function φ_m is positive semi-definite, conditioned matrix B is positive semi-definite.

Next we want to show that we can express denominator of the functions φ_m as polynomials in $\cosh(t/2)$. We have the following auxiliary result:

Lemma 2.2. *For $n \in \mathbb{N}$, we have*

$$\cosh nt = \sum_{j=0}^{[n/2]} (-1)^j A_{n,j} \cosh^{n-2j} t, \tag{2.4}$$

where

$$A_{n,j} = \sum_{\nu=j}^{[n/2]} \binom{n}{2\nu} \binom{\nu}{j}. \tag{2.5}$$

Proof. Using Moivre formula, we have

$$\cos nt = \mathrm{Re}(\cos t + \mathrm{i} \sin t)^n = \sum_{\nu=0}^{[n/2]} \binom{n}{2\nu} (-1)^\nu \cos^{n-2\nu} t \sin^{2\nu} t.$$

Changing $t := \mathrm{i}t$, and using $\cos \mathrm{i}t = \cosh t$, $\sin \mathrm{i}t = \mathrm{i} \sinh t$, together with the identity $\sinh^2 t = \cosh^2 t - 1$, we get

$$\begin{aligned}
\cosh nt &= \sum_{\nu=0}^{[n/2]} \binom{n}{2\nu} \cosh^{n-2\nu} t \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^{\nu-j} \cosh^{2j} t \\
&= \sum_{j=0}^{[n/2]} (-1)^j \cosh^{n-2j} t \sum_{\nu=j}^{[n/2]} \binom{n}{2\nu} \binom{\nu}{j} \\
&= \sum_{j=0}^{[n/2]} (-1)^j A_{n,j} \cosh^{n-2j} t,
\end{aligned}$$

where the coefficients $A_{n,j}$ are given by (2.5). □

Using the previous lemmas we can represent the denominator in the function φ_m as a polynomial in $\cosh t$.

Lemma 2.3. *Let m be an even number. Then*

$$\varphi_m(t) = \frac{1}{\sum_{\ell=0}^{m/2} \cosh^\ell t \sum_{j=\ell}^{m/2} \frac{(-1)^j}{2^j} \binom{j}{\ell} \sum_{\nu=j}^{m/2} \frac{a_{m/2-\nu}}{1+\delta_{\nu,0}} (-1)^\nu A_{2\nu,\nu-j}}. \tag{2.6}$$

In the case m is an odd integer, we have

$$\begin{aligned}
\varphi_m(t) &= \frac{1}{\sum_{j=0}^{(m-1)/2} (-1)^j \cosh^{2j+1} \frac{t}{2} \sum_{\nu=j}^{(m-1)/2} a_{\frac{m-1}{2}-\nu} (-1)^\nu A_{2\nu+1,\nu-j}} \\
&= \frac{1}{\cosh \frac{t}{2} \sum_{\ell=0}^{(m-1)/2} \cosh^\ell t \sum_{j=\ell}^{(m-1)/2} \frac{(-1)^j}{2^j} \binom{j}{\ell} \sum_{\nu=j}^{(m-1)/2} a_{\frac{m-1}{2}-\nu} (-1)^\nu A_{2\nu+1,\nu-j}}.
\end{aligned}$$

Proof. The proof can be given using equality (2.4). According to (2.2) and (2.4), for m even we get

$$\begin{aligned}\frac{1}{\varphi_m(t)} &= \sum_{\nu=0}^{m/2} \frac{a_{\frac{m}{2}-\nu}}{1+\delta_{\nu,0}} \cosh \nu t \\ &= \sum_{\nu=0}^{m/2} \frac{a_{\frac{m}{2}-\nu}}{1+\delta_{\nu,0}} \sum_{j=0}^{\nu} (-1)^{\nu-j} A_{2\nu,\nu-j} \cosh^{2j} \frac{t}{2} \\ &= \sum_{j=0}^{m/2} \frac{(-1)^j}{2^j} \sum_{\ell=0}^{j} \binom{j}{\ell} \cosh^{\ell} t \sum_{\nu=j}^{m/2} \frac{a_{\frac{m}{2}-\nu}}{1+\delta_{\nu,0}} (-1)^{\nu} A_{2\nu,\nu-j} \\ &= \sum_{\ell=0}^{m/2} \cosh^{\ell} t \sum_{j=\ell}^{m/2} \frac{(-1)^j}{2^j} \binom{j}{\ell} \sum_{\nu=j}^{m/2} \frac{a_{\frac{m}{2}-\nu}}{1+\delta_{\nu,0}} (-1)^{\nu} A_{2\nu,\nu-j}.\end{aligned}$$

Similarly, for odd m, we obtain

$$\begin{aligned}\frac{1}{\varphi_m(t)} &= \sum_{\nu=0}^{(m-1)/2} a_{\frac{m-1}{2}-\nu} \cosh(2\nu+1)\frac{t}{2} \\ &= \sum_{\nu=0}^{(m-1)/2} a_{\frac{m-1}{2}-\nu} \sum_{j=0}^{\nu} (-1)^{\nu-j} \cosh^{2j+1} \frac{t}{2} A_{2\nu+1,nu-j} \\ &= \cosh \frac{x}{2} \sum_{j=0}^{(m-1)/2} (-1)^j \cosh^{2j} \frac{t}{2} \sum_{\nu=j}^{(m-1)/2} a_{\frac{m-1}{2}-\nu} (-1)^{\nu} A_{2\nu+1,\nu-j} \\ &= \cosh \frac{t}{2} \sum_{\ell=0}^{(m-1)/2} \cosh^{\ell} t \sum_{j=\ell}^{(m-1)/2} \frac{(-1)^j}{2^j} \binom{j}{\ell} \sum_{\nu=j}^{(m-1)/2} a_{\frac{m-1}{2}-\nu} (-1)^{\nu} A_{2\nu+1,\nu-j}.\end{aligned}$$

This completely finishes the proof of this lemma. □

As can be seen in the denominator of the function φ_m we can recognize two polynomials in $\cosh x$.

Definition 2.1. For m even we define the characteristic polynomial Q_m for Eq. (2.1) to be

$$Q_m(\cosh t) = \frac{1}{\varphi_m(t)},$$

and for m odd we define the corresponding characteristic polynomial to be

$$Q_m(\cosh t) = \frac{1}{\cosh \frac{t}{2} \varphi_m(t)}.$$

In the next sections we are going to see a possible answer to the existence question by using zeros of the polynomial Q_m. For example, the characteristic polynomial of the third equation in (1.1) is given by $Q_3(z) = 2z + t - 1$, and for the fourth equation, $Q_4(z) = 2z^2 + tz + 2$.

3. Some Fourier transforms

First we introduce some common notation. We denote by $L^p(\mathbb{R})$, $p \geqslant 1$, a set of functions defined on the real line such that $\int_{\mathbb{R}} |f|^p \, \mathrm{d}x < +\infty$, and we denote by $C^p(\mathbb{R})$, $p \in \mathbb{N}_0$, a set of functions defined on the real line with p-th continuous derivative. Especially, we reserve $C^\infty(\mathbb{R})$ to represent the functions defined on the real line which are infinitely differentiable.

The Fourier transform $\hat{f}$ of a given function $f \in L_1(\mathbb{R})$ is defined in the following way:

$$\hat{f}(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xt} f(t) \, \mathrm{d}t,$$

and its inverse transform is given by

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}xt} \hat{f}(x) \, \mathrm{d}x \tag{3.1}$$

(cf. [1, pp. 1–2]). In the sequel we need the following results:

Lemma 3.1. *Let $f, g \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, with $\hat{f}, \hat{g} \in L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xt} f(t) g(t) \, \mathrm{d}t = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x-y)\hat{g}(y) \mathrm{d}y, \quad x \in \mathbb{R}. \tag{3.2}$$

Proof. It is easy to see that the convolution of the functions $\hat{f}$ and $\hat{g}$ belongs to $L^1(\mathbb{R}) \cap C(\mathbb{R})$, due to the fact that $\hat{f}$ and $\hat{g}$ are Fourier transforms and hence, continuous functions. We can calculate the inverse Fourier transform of the right-hand side of (3.2), to get

$$\begin{aligned} &\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx} \mathrm{d}x \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x-y)\hat{g}(y)\mathrm{d}y \\ &\quad = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ty} \hat{g}(y) \mathrm{d}y \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t(x-y)} \hat{f}(x-y) \mathrm{d}x = g(t) f(t), \end{aligned}$$

where we used the fact that f and g are continuous, hence, they satisfy the inversion formula on the whole real line. Since $f, g \in L^2(\mathbb{R})$, their product belongs to $L^1(\mathbb{R})$, which enables an application of the Fourier transform to the previous identity in order to prove this lemma. □

Lemma 3.2. *The convolution of two non-negative functions is a non-negative function, i.e., the product of two positive semi-definite functions is a positive semi-definite function.*

This is an obvious result.

Now, we are interested only in functions $t \mapsto \varphi_m(t)$, introduced in the previous section. The fact that $\mathrm{d}^k \varphi_m(t)/\mathrm{d}t^k \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $k \in \mathbb{N}_0$, has as a consequence the integrability of $\hat{\varphi}_m$ and an equality in the inversion formula (3.1) over the whole real line. Further, $(\mathrm{i}t)^k \varphi_m(t) \in L^1(\mathbb{R})$, $k \in \mathbb{N}_0$, assures that $\hat{\varphi}_m \in C^\infty(\mathbb{R})$. It is not hard to see that also $\varphi_m \in L^2(\mathbb{R})$.

In the next section we need the Fourier transform of the function

$$g(t) = \frac{1}{\cosh t - \sigma}, \quad \sigma \in \mathbb{C} \backslash [1, +\infty),$$

where $[1, +\infty)$ is excluded since for $\sigma \in [1, +\infty)$ it is clear that $g \notin L_1(\mathbb{R})$. For this Fourier transform we refer to [3], where the following results:

$$\hat{g}(x) = \int_{\mathbb{R}} \frac{e^{ixt}}{\cosh t - \sigma}\, dt = \begin{cases} \frac{2\pi \sinh(x \arccos(-\sigma))}{\sqrt{1-\sigma^2}\sinh x\pi}, & \sigma \in (-1, 1), \\ \frac{2\pi \sin(x \operatorname{arccosh}(-\sigma))}{\sqrt{\sigma^2-1}\sinh x\pi}, & \sigma < -1, \end{cases} \tag{3.3}$$

were proved. In general, for a complex σ, we have

$$\hat{g}(x) = \int_{\mathbb{R}} \frac{e^{ixt}}{\cosh t - \sigma} dt = \frac{2\pi \sinh(x \arccos(-\sigma))}{\sqrt{1-\sigma^2}\sinh x\pi}, \quad \sigma \in \mathbb{C}\backslash\mathbb{R}.$$

Also, this result can be found in [3], except the case $|\sigma| = 1, \sigma \neq \pm 1$, which can be proved using the same arguments given in [3].

Using a limiting process in (3.3) as $\sigma \to -1$, we can prove that

$$\int_{\mathbb{R}} \frac{e^{ixt}}{\cosh t + 1} dt = \frac{2\pi x}{\sinh x\pi}. \tag{3.4}$$

4. Positive semi-definite solutions

According to (3.3) and (3.4), we conclude that the function

$$g(t) = \frac{1}{\cosh t - \sigma}, \quad -1 \leqslant \sigma < 1$$

is a positive semi-definite function. This enables us to state the following result:

Theorem 4.1. *Suppose we are given Eq.* (2.1), *with a positive definite matrix* A, *with the characteristic polynomial* Q_m *which all zeros are real and contained in the interval* $[-1, 1)$. *Then the corresponding function* φ_m *is positive semi-definite*, *i.e.*, *the matrix equation* (2.1) *has a positive semi-definite solution provided* B *is positive semi-definite*. *If* λ_ν *are eigenvalues of* A, *the corresponding solution* $X = (x_{i,j})$ *is given by*

$$x_{i,j} = \frac{b_{i,j}}{\sum_{\nu=0}^{m} a_\nu \lambda_i^{m-\nu}\lambda_j^{\nu}}, \quad i, j = 1, \ldots, m.$$

Proof. Denote zeros of Q_m by σ_i, $i = 1, \ldots, [m/2]$. We distinguish two cases for our matrix equation

$$\sum_{\nu=0}^{m} a_\nu A^{m-\nu} X A^{\nu} = B.$$

Case m is even. Then

$$\varphi_m(t) = \frac{1}{Q_m(\cosh t)} = \prod_{i=1}^{m/2} \frac{1}{\cosh t - \sigma_i}.$$

Consider now the functions

$$g_1(t) = \frac{1}{\cosh t - \sigma_1}, \quad g_{j+1}(t) = \frac{g_j(t)}{\cosh t - \sigma_{j+1}}, \quad j = 1, \ldots, m/2 - 1.$$

Obviously g_j, $j = 1, \ldots, m/2$, belong to $L^2(\mathbb{R})$ and their Fourier transforms are $L^1(\mathbb{R}) \cap C(\mathbb{R})$ functions. The function g_1 is positive semi-definite according to (3.3). Assuming that g_j is positive semi-definite, according to Lemma 3.1, the function g_{j+1} has the Fourier transform which is a

convolution of the Fourier transforms of g_j and $1/(\cosh t - \sigma_j)$ and those are both non-negative. According to Lemma 3.2, the Fourier transform of g_{j+1} is also non-negative, hence, g_{j+1} is positive semi-definite. Now, by induction we conclude that $g_{m/2} = \varphi_m$ is positive semi-definite.
Case m is odd. Then

$$\varphi_m(t) = \frac{1}{\cosh \frac{t}{2} Q_m(t)} = \frac{1}{\cosh \frac{t}{2}} \prod_{i=1}^{[m/2]} \frac{1}{\cosh t - \sigma_i}.$$

Here the proof is the same except that now we take

$$g_0(t) = \frac{1}{\cosh \frac{t}{2}}, \quad g_{j+1}(t) = \frac{g_j(t)}{\cosh t - \sigma_{j+1}}, \quad j = 0, 1, \ldots, [m/2] - 1.$$

The only missing ingredient is positive semi-definiteness of g_0. But, we have

$$\int_{\mathbb{R}} \frac{e^{ixt}}{\cosh \frac{t}{2}} dt = 2 \int_{\mathbb{R}} \frac{e^{2ixt}}{\cosh t} dt = \frac{2\pi}{\cosh \pi x},$$

using the Fourier transform given in (3.3), hence g_0 is positive semi-definite. □

In order to give further results we need the following lemma.

Lemma 4.1. *The function*

$$t \mapsto \frac{1}{(\cosh t - \sigma_1)(\cosh t - \sigma_2)} \tag{4.1}$$

for $-1 \leqslant \sigma_1 < 1$ *and* $\sigma_2 < 1$, *is positive semi-definite.*

Proof. The case $\sigma_2 \in [-1, 1)$ is covered by Theorem 4.1, so we assume now that $\sigma_2 < -1$. Using a partial fraction decomposition we have

$$\frac{1}{(\cosh t - \sigma_1)(\cosh t - \sigma_2)} = \frac{1}{\sigma_1 - \sigma_2} \left(\frac{1}{\cosh t - \sigma_1} - \frac{1}{\cosh t - \sigma_2} \right).$$

Assuming $-1 < \sigma_1$ and using (3.3), we conclude that the Fourier transform of (4.1) is given by

$$\frac{2\pi}{(\sigma_1 - \sigma_2) \sinh(x\pi)} \left(\frac{\sinh(x \arccos(-\sigma_1))}{\sqrt{1 - \sigma_1^2}} - \frac{\sin(x \operatorname{arccosh}(-\sigma_2))}{\sqrt{\sigma_2^2 - 1}} \right). \tag{4.2}$$

We are going to prove that this expression is always non negative on $\mathbb{R}$.

Fix $x \in \mathbb{R}^+$, then

$$\frac{\sinh(x \arccos(-\sigma_1))}{x \arccos(\sigma_1)}$$

is strictly increasing function in $x \arccos(-\sigma_1)$. According to the fact that $x \arccos(-\sigma_1)$ is strictly increasing function in $\sigma_1 \in (-1, 1)$, our function is strictly increasing in σ_1. Its minimum is achieved for $\sigma_1 = -1$ and its value is 1.

Now consider

$$g(\sigma_1) = \frac{\arccos(-\sigma_1)}{\sqrt{1 - \sigma_1^2}},$$

which derivative is given by

$$g'(\sigma_1) = \frac{\sigma_1 \arccos(-\sigma_1) + \sqrt{1-\sigma_1^2}}{(1-\sigma_1^2)^{3/2}}.$$

The derivative of the numerator is $\arccos(-\sigma_1) > 0$, and therefore it is an increasing function. Its value for $\sigma_1 = -1$ is 0 and for $\sigma_1 = 1$ is π. Hence $g'(\sigma_1)$ is always positive and g is increasing. The minimum value of the function g is 1 and is achieved for $\sigma_1 = -1$.

In (4.2), for fixed $x \in \mathbb{R}^+$, the term

$$\frac{\sinh(x \arccos(-\sigma_1))}{\sqrt{1-\sigma_1}}$$

has the minimum value x at $\sigma_1 = -1$.

Now, for fixed $x \in \mathbb{R}^+$, we consider the function

$$\frac{\sin(x \operatorname{arccosh}(-\sigma_2))}{x \operatorname{arccosh}(-\sigma_2)}.$$

This function has as its global maximum the value 1 at the point $\sigma_2 = -1$.

For the function

$$g(\sigma_2) = \frac{\operatorname{arccosh}(-\sigma_2)}{\sqrt{\sigma_2^2 - 1}},$$

we have

$$g'(\sigma_2) = \frac{-\sqrt{\sigma_2^2-1} - \sigma_2 \operatorname{arccosh}(-\sigma_2)}{(\sigma_2^2-1)^{3/2}}.$$

Since the derivative of the numerator of g' is $-\operatorname{arccosh}(-\sigma_2)$, we conclude that it is decreasing, with value 0 at $\sigma_2 = -1$. It follows that $g'(\sigma_2)$ is always positive, which shows g is increasing with the maximum value 1 at $\sigma_2 = -1$.

In total, for fixed $x \in \mathbb{R}^+$, we have that

$$\frac{\sin(x \operatorname{arccosh}(-\sigma_2))}{\sqrt{\sigma_2^2-1}}$$

has as its maximum value x at $\sigma_2 = -1$.

Putting all together, for $x \in \mathbb{R}^+$, $\sigma_1 \in (-1, 1)$ and $\sigma_2 < -1$, we have

$$\frac{\sinh(x \arccos(-\sigma_1))}{\sqrt{1-\sigma_1^2}} - \frac{\sin(x \operatorname{arccosh}(-\sigma_2))}{\sqrt{\sigma_2^2-1}} > x - x = 0.$$

By the continuity argument, for the Fourier transform (4.2) at $x = 0$, we find that

$$\frac{2}{\sigma_1 - \sigma_2}\left(\frac{\arccos(-\sigma_1)}{\sqrt{1-\sigma_1^2}} - \frac{\operatorname{arccosh}(-\sigma_2)}{\sqrt{\sigma_2^2-1}}\right) \geqslant 0.$$

This means that our function (4.1) is positive semi-definite.

In the case $\sigma_1 = -1$, the Fourier transform of the function (4.1) is given by

$$-\frac{2\pi x}{(\sigma_2+1)\sinh \pi x}\left(1 - \frac{\sin(x \operatorname{arccosh}(-\sigma_2))}{x \operatorname{arccosh}(-\sigma_2)} \frac{\operatorname{arccosh}(-\sigma_2)}{\sqrt{\sigma_2^2-1}}\right).$$

It is easily seen that

$$\left|\frac{\sin(x\,\mathrm{arccosh}(-\sigma_2))}{x\,\mathrm{arccosh}(-\sigma_2)}\frac{\mathrm{arccosh}(-\sigma_2)}{\sqrt{\sigma_2^2-1}}\right|\leqslant 1,$$

and we have finished the proof. □

The next lemma shows that essentially the function (4.1) is positive semi-definite only for $-1\leqslant\sigma_1,\sigma_2<1$ or $-1\leq\sigma_1<1$ and $\sigma_2<-1$.

Lemma 4.2. *The function* (4.1) *is not positive semi-definite for* $\sigma_1,\sigma_2<-1$ *or* $\sigma_1=\overline{\sigma}_2\in\mathbb{C}\backslash\mathbb{R}$.

Proof. It is easy to see that for $\sigma_1,\sigma_2<-1$, the function (4.1) cannot be positive semi-definite, because its Fourier transform, for $\sigma_1\neq\sigma_2$, is given by

$$\frac{2\pi}{(\sigma_1-\sigma_2)\sinh x\pi}\left(\frac{\sin(x\,\mathrm{arccosh}(-\sigma_1))}{\sqrt{\sigma_1^2-1}}-\frac{\sin(x\,\mathrm{arccosh}(-\sigma_2))}{\sqrt{\sigma_2^2-1}}\right) \tag{4.3}$$

and must have at least one point where both terms in (4.3) are negative.

We consider now the Fourier transform of the function (4.1) in the case $\sigma_1=\sigma_2=\sigma<-1$, by a limit process when $\sigma_1\to\sigma_2=\sigma<-1$, i.e.,

$$\begin{aligned}&\lim_{\sigma_1\to\sigma}\frac{2\pi}{(\sigma_1-\sigma)\sinh x\pi}\left(\frac{\sin(x\,\mathrm{arccosh}(-\sigma_1))}{\sqrt{\sigma_1^2-1}}-\frac{\sin(x\,\mathrm{arccosh}(-\sigma))}{\sqrt{\sigma^2-1}}\right)\\&\quad=-\frac{2\pi}{\sinh x\pi}\frac{\sqrt{\sigma^2-1}x\cos(x\,\mathrm{arccosh}(-\sigma))+\sigma\sin(x\,\mathrm{arccosh}(-\sigma))}{(\sigma^2-1)^{3/2}}.\end{aligned}$$

It is clear that this transform is negative for

$$x\in\left(\frac{(4k-1)\pi}{2\,\mathrm{arccosh}(-\sigma)},\frac{2k\pi}{\mathrm{arccosh}(-\sigma)}\right),\quad k\in\mathbb{N}.$$

Consider now the case $\sigma=\sigma_1=\overline{\sigma_2}\notin\mathbb{R}$, with $\sigma=a+\mathrm{i}b$, $b>0$. Then we have

$$\begin{aligned}\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}xt}}{(\cosh t-\sigma)(\cosh t-\bar{\sigma})}\,\mathrm{d}t&=\frac{2\pi}{\sinh x\pi}\frac{\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\varphi/2}\sinh(x\arccos(-\sigma)))}{b\sqrt{|1-\sigma^2|}}\\&=\frac{2\pi}{b\sinh x\pi\sqrt{|1-\sigma^2|}}\Big(\cos\frac{\varphi}{2}\sin\beta x\cosh\alpha x\\&\quad-\sin\frac{\varphi}{2}\cos\beta x\sinh\alpha x\Big),\end{aligned}$$

where we denoted by φ the argument of the complex number $1-\sigma^2$ and $\alpha+\mathrm{i}\beta=\arccos(-\sigma)$. Depending on the sign of $\sin(\varphi/2)$ and $\cos(\varphi/2)$, we can always choose x such that both terms in the final expression are negative. We conclude that in the case $\sigma_1=\overline{\sigma_2}$ the function (4.1) cannot be positive semi-definite. □

The previous two lemmas give as an opportunity to give a stronger result than the one obtained in [3].

Theorem 4.2. *Let B be a positive semi-definite matrix and let A be a positive definite matrix. Then the equation*

$$A^4X - 2tA^3XA + 2(2u+1)A^2XA^2 - 2tAXA^3 + XA^4 = B$$

with $(t,u) \in \mathbb{R}^2$, *has a positive semi-definite solution if and only if* $(t,u) \in D \subset \mathbb{R}^2$, *where the domain D is determined by*

$$(t < -2 \wedge u - t + 1 > 0 \wedge u + t + 1 \geqslant 0) \vee (t \geqslant -2 \wedge t^2/4 - u \geqslant 0 \wedge u - t + 1 > 0).$$

Proof. All we need to do is to construct the corresponding characteristic polynomial of the mentioned equation. Using the equality (2.6) and Definition 2.1, we have

$$Q_4(z) = 2(z^2 - tz + u).$$

Using Lemmas 4.1 and 4.2, the function φ_4 is positive semi-definite if and only if zeros σ_1 and σ_2 of the polynomial Q_4 are $\sigma_1 \in [-1,1)$ and $\sigma_2 \in (-\infty,1)$. Thus, the function $\varphi_4(x)$ is positive semi-definite on the set

$$\{(t,u) = (\sigma_1+\sigma_2, \sigma_1\sigma_2) | \sigma_1 \in [-1,1), \sigma_2 \in (-\infty,1)\}.$$

Since

$$\sigma_{1,2} = \frac{t}{2} \pm \sqrt{\frac{t^2}{4} - u},$$

we get the system of inequalities

$$-1 \leqslant \frac{t}{2} + \sqrt{\frac{t^2}{4} - u} < 1, \quad \frac{t}{2} - \sqrt{\frac{t^2}{4} - u} < 1,$$

which solution is exactly given in the statement of the theorem. □

The domain D from this theorem is presented in Fig. 4.1 for $t \geqslant -4$.

In order to be able to express an influence of the factor $\cosh t/2$ in the function φ_m for m odd, we have the following result:

Lemma 4.3. *The function*

$$\varphi_3(t) = \frac{1}{\cosh\frac{t}{2}(\cosh t - \sigma)}, \quad \sigma \in (-\infty, 1)$$

is positive semi-definite.

Proof. We have

$$\int_{\mathbb{R}} \frac{e^{ixt}}{\cosh\frac{t}{2}(\cosh t - \sigma)}\, dt = \int_{\mathbb{R}} \frac{e^{2ixt}}{\cosh t\left(\cosh^2 t - \frac{1+\sigma}{2}\right)}\, dt.$$

For $\sigma \in [-1,1)$, it is clear that $(1+\sigma)/2 \in [0,1)$, so that, according to Lemma 3.2, the corresponding Fourier transform is a non-negative function on $\mathbb{R}$ and φ_3 is positive semi-definite.

For $\sigma \in (-\infty,-1)$, we have $(1+\sigma)/2 \in (-\infty,0)$, so that we denote $a^2 = -(1+\sigma)/2$. Finally, we end-up in

$$\int_{\mathbb{R}} \frac{e^{2ixt}}{\cosh t(\cosh^2 t + a^2)}\, dt,$$

which is non-negative function according to Proposition 4.1 form [3]. □

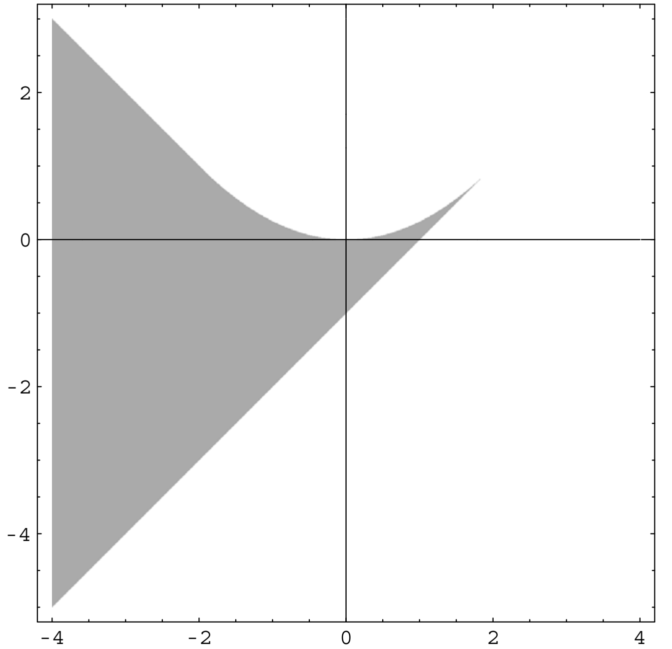

Fig. 4.1. The domain D from Theorem 4.2 for $t \geqslant -4$.

Now, we are able the state the main result of the paper.

Theorem 4.3. *Suppose we are given Eq.* (2.1), *with a positive definite matrix* A, *with the characteristic polynomial* Q_m *which has* k_1 *real zeros contained in the interval* $[-1, 1)$ *and* k_2 *zeros smaller than* -1, *with* $k_1 \geqslant k_2$, *for m even*, *and* $k_1 + 1 \geqslant k_2$, *for m odd*, *where* $k_1 + k_2 = [m/2]$. *Then the corresponding function* φ_m *is positive semi-definite*, *i.e.*, *the matrix equation* (2.1) *has a positive semi-definite solution*, *provided B is positive semi-definite*. *If* λ_ν *are eigenvalues of* A, *the solution* $X = (x_{i,j})$ *is given by*

$$x_{i,j} = \frac{b_{i,j}}{\sum_{\nu=0}^{m} a_\nu \lambda_i^{m-\nu} \lambda_j^{\nu}}, \quad i, j = 1, \ldots, m.$$

Proof. Consider first the case when m is even. Then, we can group the zeros of Q_m according to the following $x_i \in [-1, 1)$, $y_i \in (-\infty, 1)$, $i = 1, \ldots, k_2$, and $x_i \in [-1, 1)$, $i = k_2 + 1, \ldots, k_1$. According to the fact that the convolution is commutative and associative and using Lemma 4.1 we conclude that the Fourier transforms of

$$\frac{1}{(\cosh t - x_i)(\cosh t - y_i)}, \quad i = 1, \ldots, k_2$$

are non-negative functions, and therefore the functions itself are positive semi-definite. According to Lemma 3.2, the Fourier transform of the function

$$\prod_{i=1}^{k_2} \frac{1}{(\cosh t - x_i)(\cosh t - y_i)}$$

is non-negative and the function itself is positive semi-definite. Finally, if we include the part

$$\prod_{i=k_2+1}^{k_1} \frac{1}{\cosh t - x_i},$$

which Fourier transform is a non-negative function, we have that the function φ_m is positive semi-definite.

Now for m odd, we group the zeros according to the following $y_0 \in (-\infty, 1)$, $x_1 \in [-1, 1)$ and $y_i \in (-\infty, 1)$, $i = 1, \ldots, k_2 - 1$, and $x_i \in [-1, 1)$, $i = k_2, \ldots, k_1$. Using Lemma 4.3, we have that

$$\frac{1}{\cosh \frac{t}{2}(\cosh t - y_0)}$$

is positive semi-definite. Also all functions

$$\frac{1}{(\cosh t - x_i)(\cosh t - y_i)}, \quad i = 1, \ldots, k_2 - 1,$$

are positive semi-definite according to Lemma 4.1. Finally, the functions

$$\frac{1}{\cosh t - x_i}, \quad i = k_2, \ldots, k_1,$$

are also positive semi-definite and, therefore, the function φ_m is positive semi-definite. □

5. Examples

Using the previous considerations we are able to give sufficient conditions for the existence of positive semi-definite solutions of some equations with higher order.

Theorem 5.1. *If* $t \in (-6, 10]$, *then the equation*

$$A^5X + 5A^4XA + tA^3XA^2 + tA^2XA^3 + 5AXA^4 + XA^5 = B$$

has a positive semi-definite solution, provided B is positive semi-definite.

Proof. In this case we have for the characteristic polynomial

$$Q_5(z) = 4\left(z^2 + 2z + \frac{t-6}{4}\right).$$

According to Theorem 4.3, the equation has positive semi-definite solutions, provided the polynomial Q_5 has zeros $\sigma_1 \in [-1, 1)$ and $\sigma_1 \in (-\infty, 1)$. Using Viète formulas we have

$$-2 = \sigma_1 + \sigma_2, \quad \frac{t-6}{4} = \sigma_1\sigma_2,$$

from which we deduce $t = 6 - 4\sigma_1(2 + \sigma_1)$, $\sigma_1 \in [-1, 1)$, so that we have $t \in (-6, 10]$. □

Theorem 5.2. *If* $t \in (-\infty, -11) \cup [5, +\infty)$, *then the equation*

$$A^5X + tA^4XA + 10A^3XA^2 + 10A^2XA^3 + tAXA^4 + XA^5 = B$$

has a positive semi-definite solution, provided B is positive semi-definite.

Proof. The characteristic polynomial is

$$Q_5(z) = 4\left(z^2 + \frac{t-1}{2}z + \frac{9-t}{4}\right).$$

Using Viète formulas and Theorem 4.3, we have a positive semi-definite solution provided

$$-\frac{t-1}{2} = \sigma_1 + \sigma_2, \quad \frac{9-t}{4} = \sigma_1\sigma_2,$$

with $\sigma_1 \in [-1, 1)$ and $\sigma_2 \in (-\infty, 1)$. Solving this system of equations we get $t = -(4\sigma_1^2 - 2\sigma_1 + 9)/(2\sigma_1 - 1)$, $\sigma_1 \in [-1, 1)$.

Therefore, $t \in (-\infty, -11) \cup [5, +\infty)$. □

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