Positive definite solutions of some matrix equations

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Dedicated to Professor Richard Varga on the occasion of his 80th birthday

Abstract

In this paper we investigate some existence questions of positive semi-definite solutions for certain classes of matrix equations known as the generalized Lyapunov equations. We present sufficient and necessary conditions for certain equations and only sufficient for others.

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1. Introduction

Recently, Bathia and Drisi [3] studied questions related to the positive semi-definiteness of solutions of the following matrix equations:

\[ AX +XA = B, \]
\[ \quad A^2X + 2tAXA + XA^2 = B, \]
\[ \quad A^3X + t(A^2XA + AXA^2) +XA^3 = B, \]

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where $A$ is a given positive definite matrix and matrix $B$ is positive semi-definite. First equation is known to be the Lyapunov equation and has a great deal with the analysis of the stability of motion.

Second equation has been studied by Kwong [10] and he succeeded to give an answer about the existence of the positive semi-definite solutions. In [3] necessary and sufficient conditions are given for the parameter $t$ in order that Eq. (1.1) have positive semi-definite solutions, provided that $B$ is positive semi-definite. For numerous other references see [2,4–10]. There is also a strong connection between the question of positive semi-definite solutions of these equations and various inequalities involving unitarily equivalent matrix norms (see [2,6–9]).

We briefly recall that a matrix $A$ is positive definite, provided it is symmetric and for every vector $x 
eq 0$ we have $(Ax, x) > 0$. A matrix $A$ is positive semi-definite, provided it is symmetric and for every $x$ we have $(Ax, x) \geq 0$.

In this paper we investigate the existence question of positive semi-definite solutions of a general form of Eq. (1.1). Introducing characteristic polynomials for these equations, in Section 4 we present some sufficient conditions for the existence of these solutions. In the last section we present results concerning some specific equations.

2. Characteristic polynomial

We denote by $2\mathbb{N}$ and $2\mathbb{N} - 1$ sets of even and odd natural numbers. First, we prove a simple lemma to give a motivation for results of this paper.

**Lemma 2.1.** Suppose we are given matrix equation

$$
\sum_{\nu=0}^{m} a_{\nu} A^{m-\nu} X A^{\nu} = B,
$$

where $B$ is positive semi-definite, $A$ is positive definite, and $a_{\nu} = a_{m-\nu} \in \mathbb{R}$, $\nu = 0, 1, \ldots, m$, $a_{0} = a_{m} > 0$. If the function $t \mapsto \varphi_{m}(t)$, defined by

$$
\frac{1}{\varphi_{m}(t)} = \begin{cases} 
\frac{m/2-1}{2} & \sum_{v=0}^{m/2-1} a_{v} \cosh \left( \frac{m}{2} - v \right)t + \frac{1}{2}a_{m/2}, & m \in 2\mathbb{N}, \\
\sum_{v=0}^{(m-1)/2} a_{v} \cosh \left( \frac{m}{2} - v \right)t, & m \in 2\mathbb{N} - 1
\end{cases}
$$

is positive semi-definite, then Eq. (2.1) has a positive semi-definite solution. If equation has positive semi-definite solution for any positive definite matrix $A$ then function $\varphi_{m}$ is positive semi-definite.

**Proof.** Since $A$ is a positive definite matrix, its eigenvectors create a basis. Hence, we can use the system of eigenvectors as a basis in which the matrix $A$ has a diagonal form, with eigenvalues on its diagonal. Denote the eigenvalues by $\lambda_{\nu}$, $\nu = 1, \ldots, n$. Then Eq. (2.1), in the previous basis with $X = (x_{i,j})$ and $B = (b_{i,j})$, can be represented in the form

$$
\sum_{\nu=0}^{m} a_{\nu} \lambda_{i}^{m-\nu} \lambda_{j}^{\nu} x_{i,j} = b_{i,j}, \quad i, j = 1, \ldots, n,
$$
\[ x_{i,j} = \frac{b_{i,j}}{\sum_{\nu=0}^{m} a_{\nu} \lambda_{i}^{m-v} \lambda_{j}^{v}}, \quad i, j = 1, \ldots, n. \]

If we denote with \( C = (c_{i,j}) \) a matrix with entries
\[ c_{i,j} = \frac{1}{\sum_{\nu=0}^{m} a_{\nu} \lambda_{i}^{m-v} \lambda_{j}^{v}}, \quad i, j = 1, \ldots, n, \tag{2.3} \]
we can recognize that the matrix \( X \) is a direct or Schur product of the matrices \( C \) and \( B \), so that if \( C \) and \( B \) are positive semi-definite, \( X \) is also positive semi-definite.

Since the eigenvalues \( \lambda_{\nu}, \nu = 1, \ldots, n \), are positive, we can represent them in the form \( \lambda_{i} = e^{x_{i}}, \) where \( x_{i} \in \mathbb{R}, i = 1, \ldots, n \). Applying this to the matrix \( C \), we get for its elements and \( m \) even
\[ c_{i,j} = \frac{e^{-m/2(x_{i}+x_{j})}}{\sum_{\nu=0}^{m} a_{\nu} e^{(m/2-v)x_{i}} e^{(v-m/2)x_{j}}} = \frac{1}{2} \sum_{\nu=0}^{m/2-1} a_{\nu} \cosh \left( \frac{m}{2} - \nu \right) (x_{i} - x_{j}) + \frac{1}{2} a_{m/2}. \]

Similarly, for \( m \) odd, for elements \( c_{i,j} \) of \( C \) we get the following expression
\[ c_{i,j} = \frac{e^{-m/2(x_{i}+x_{j})}}{\sum_{\nu=0}^{m} a_{\nu} e^{(m/2-v)x_{i}} e^{(v-m/2)x_{j}}} = \frac{1}{2} \sum_{\nu=0}^{(m-1)/2} a_{\nu} \cosh \left( \frac{m}{2} - \nu \right) (x_{i} - x_{j}) \]

Two matrices \( X \) and \( Y \) are said to be congruent if there exists a non-singular matrix \( Z \) such that \( X = Z^{*} Y Z \). It is known that congruency preserves definiteness. In both cases, for even and odd \( m \), our matrix \( C \) is congruent with a matrix with elements
\[
\begin{cases}
\sum_{\nu=0}^{m/2-1} a_{\nu} \cosh \left( \frac{m}{2} - \nu \right) (x_{i} - x_{j}) + \frac{1}{2} a_{m/2}, & m \in 2\mathbb{N}, \\
\sum_{\nu=0}^{(m-1)/2} a_{\nu} \cosh \left( \frac{m}{2} - \nu \right) (x_{i} - x_{j}), & m \in 2\mathbb{N} - 1,
\end{cases}
\]
where in both cases the congruency matrix \( Z \) is a diagonal matrix with entries \( (1/\sqrt{2}) e^{-m/2x_{i}}, i = 1, \ldots, n \).

Now we introduce the function \( t \mapsto \varphi_{m}(t) \) by (2.2). Then \( \varphi_{m} \) is going to be positive semi-definite if and only if our matrix in (2.3) is positive semi-definite. \( \square \)

According to the Bochner’s theorem (see [13, p. 17,12, p. 290]) the function \( \varphi_{m} \) is positive semi-definite if and only if its Fourier transform is nonnegative on the real line. Hence, we can answer the existence question of positive semi-definite solutions of Eq. (2.1), provided we are able to answer the question whether the function \( \varphi_{m} \) is positive semi-definite, conditioned matrix \( B \) is positive semi-definite.
Next we want to show that we can express denominator of the functions $\varphi_m$ as polynomials in $\cosh(t/2)$. We have the following auxiliary result:

**Lemma 2.2.** For $n \in \mathbb{N}$, we have

$$\cosh nt = \sum_{j=0}^{[n/2]} (-1)^j A_{n,j} \cosh^{n-2j} t,$$

where

$$A_{n,j} = \sum_{v=j}^{[n/2]} \binom{n}{2v} \binom{v}{j}. \quad (2.5)$$

**Proof.** Using Moivre formula, we have

$$\cos nt = \Re((\cos t + i \sin t)^n) = \sum_{v=0}^{[n/2]} \binom{n}{2v} (-1)^v \cos^{n-2v} t \sin^{2v} t.$$

Changing $t := it$, and using $\cos it = \cosh t$, $\sin it = i \sinh t$, together with the identity $\sinh^2 t = \cosh^2 t - 1$, we get

$$\cosh nt = \sum_{v=0}^{[n/2]} \binom{n}{2v} \cosh^{n-2v} t \sum_{j=0}^{[n/2]} \binom{n}{2v} (-1)^j \cosh^{2j} t$$

$$= \sum_{j=0}^{[n/2]} (-1)^j \cosh^{n-2j} t \sum_{v=j}^{[n/2]} \binom{n}{2v} \binom{v}{j}$$

$$= \sum_{j=0}^{[n/2]} (-1)^j A_{n,j} \cosh^{n-2j} t,$$

where the coefficients $A_{n,j}$ are given by (2.5). □

Using the previous lemmas we can represent the denominator in the function $\varphi_m$ as a polynomial in $\cosh t$.

**Lemma 2.3.** Let $m$ be an even number. Then

$$\varphi_m(t) = \frac{1}{\sum_{j=0}^{(m-1)/2} (-1)^j \cosh^{2j} t \sum_{v=j}^{(m-1)/2} \frac{a_{m/2-v}}{1+\delta_{v,0}} (-1)^v A_{2v,v-j}}. \quad (2.6)$$

In the case $m$ is an odd integer, we have

$$\varphi_m(t) = \frac{1}{\cosh^{\frac{m-1}{2}} \sum_{j=0}^{(m-1)/2} (-1)^j \cosh^{2j} \sum_{v=j}^{(m-1)/2} \frac{a_{m-1-v}}{2} (-1)^v A_{2v+1,v-j}}.$$
Proof. The proof can be given using equality (2.4). According to (2.2) and (2.4), for $m$ even we get
\[
\frac{1}{\varphi_m(t)} = \sum_{v=0}^{m/2} \frac{a_{\frac{m}{2}-v}}{1 + \delta_{v,0}} \cosh vt
\]
\[
= \sum_{v=0}^{m/2} \frac{a_{\frac{m}{2}-v}}{1 + \delta_{v,0}} \sum_{j=0}^{v} (-1)^{v-j} A_{2v,v-j} \cosh^{2j} \frac{t}{2}
\]
\[
= \sum_{j=0}^{m/2} \frac{(-1)^j}{2^j} \sum_{\ell=0}^{j} \binom{j}{\ell} \cosh^\ell \sum_{v=j}^{m/2} \frac{a_{\frac{m}{2}-v}}{1 + \delta_{v,0}} (-1)^v A_{2v,v-j}
\]
\[
= \sum_{\ell=0}^{m/2} \cosh^\ell \sum_{j=0}^{m/2} \frac{(-1)^j}{2^j} \sum_{v=j}^{m/2} \frac{a_{\frac{m}{2}-v}}{1 + \delta_{v,0}} (-1)^v A_{2v,v-j}.
\]
Similarly, for odd $m$, we obtain
\[
\frac{1}{\varphi_m(t)} = \sum_{v=0}^{(m-1)/2} a_{\frac{m-1}{2}-v} \cosh(2v+1) \frac{t}{2}
\]
\[
= \sum_{v=0}^{(m-1)/2} a_{\frac{m-1}{2}-v} \sum_{j=0}^{v} (-1)^{v-j} \cosh^{2j+1} \frac{t}{2} A_{2v+1,nu-j}
\]
\[
= \cosh \frac{x}{2} \sum_{j=0}^{(m-1)/2} (-1)^j \cosh^{2j} \frac{t}{2} \sum_{v=j}^{(m-1)/2} \frac{a_{\frac{m-1}{2}-v}}{1 + \delta_{v,0}} (-1)^v A_{2v+1,v-j}
\]
\[
= \cosh \frac{x}{2} \sum_{\ell=0}^{(m-1)/2} \cosh^\ell \sum_{j=\ell}^{(m-1)/2} \frac{(-1)^j}{2^j} \sum_{v=j}^{(m-1)/2} \frac{a_{\frac{m-1}{2}-v}}{1 + \delta_{v,0}} (-1)^v A_{2v+1,v-j}.
\]
This completely finishes the proof of this lemma. □

As can be seen in the denominator of the function $\varphi_m$ we can recognize two polynomials in $\cosh x$.

Definition 2.1. For $m$ even we define the characteristic polynomial $Q_m$ for Eq. (2.1) to be
\[
Q_m(\cosh t) = \frac{1}{\varphi_m(t)},
\]
and for $m$ odd we define the corresponding characteristic polynomial to be
\[
Q_m(\cosh t) = \frac{1}{\cosh \frac{t}{2} \varphi_m(t)}.
\]

In the next sections we are going to see a possible answer to the existence question by using zeros of the polynomial $Q_m$. For example, the characteristic polynomial of the third equation in (1.1) is given by $Q_3(z) = 2z + t - 1$, and for the fourth equation, $Q_4(z) = 2z^2 + tz + 2$. 
3. Some Fourier transforms

First we introduce some common notation. We denote by $L^p(\mathbb{R})$, $p \geq 1$, a set of functions defined on the real line such that $\int_{\mathbb{R}} |f|^p \, dx < +\infty$, and we denote by $C^p(\mathbb{R})$, $p \in \mathbb{N}_0$, a set of functions defined on the real line with $p$-th continuous derivative. Especially, we reserve $C^\infty(\mathbb{R})$ to represent the functions defined on the real line which are infinitely differentiable.

The Fourier transform $\hat{f}$ of a given function $f \in L^1(\mathbb{R})$ is defined in the following way:

$$\hat{f}(x) = \int_{\mathbb{R}} e^{ixt} f(t) \, dt,$$

and its inverse transform is given by

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixt} \hat{f}(x) \, dx \quad \text{(3.1)}$$

(cf. [1, pp. 1–2]). In the sequel we need the following results:

Lemma 3.1. Let $f, g \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, with $\hat{f}, \hat{g} \in L^1(\mathbb{R})$. Then

$$\int_{\mathbb{R}} e^{ixt} f(t)g(t) \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x-y)\hat{g}(y) \, dy, \quad x \in \mathbb{R}. \quad \text{(3.2)}$$

Proof. It is easy to see that the convolution of the functions $\hat{f}$ and $\hat{g}$ belongs to $L^1(\mathbb{R}) \cap C(\mathbb{R})$, due to the fact that $\hat{f}$ and $\hat{g}$ are Fourier transforms and hence, continuous functions. We can calculate the inverse Fourier transform of the right-hand side of (3.2), to get

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixt} \hat{f}(x) \, dx \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x-y)\hat{g}(y) \, dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\hat{g}(y)\, dy} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it(x-y)} \hat{f}(x) \, dx = g(t)f(t),$$

where we used the fact that $f$ and $g$ are continuous, hence, they satisfy the inversion formula on the whole real line. Since $f, g \in L^2(\mathbb{R})$, their product belongs to $L^1(\mathbb{R})$, which enables an application of the Fourier transform to the previous identity in order to prove this lemma. □

Lemma 3.2. The convolution of two non-negative functions is a non-negative function, i.e., the product of two positive semi-definite functions is a positive semi-definite function.

This is an obvious result.

Now, we are interested only in functions $t \mapsto \varphi_m(t)$, introduced in the previous section. The fact that $d^k\varphi_m(t)/dt^k \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $k \in \mathbb{N}_0$, has as a consequence the integrability of $\hat{\varphi}_m$ and an equality in the inversion formula (3.1) over the whole real line. Further, $(it)^k \varphi_m(t) \in L^1(\mathbb{R})$, $k \in \mathbb{N}_0$, assures that $\hat{\varphi}_m \in C^\infty(\mathbb{R})$. It is not hard to see that also $\varphi_m \in L^2(\mathbb{R})$.

In the next section we need the Fourier transform of the function

$$g(t) = \frac{1}{\cosh t - \sigma}, \quad \sigma \in \mathbb{C}\backslash[1, +\infty),$$

where $[1, +\infty)$ is excluded since for $\sigma \in [1, +\infty)$ it is clear that $g \notin L^1(\mathbb{R})$. For this Fourier transform we refer to [3], where the following results:
\[
\hat{g}(x) = \int_{\mathbb{R}} \frac{e^{ixt}}{\cosh t - \sigma} \, dt = \begin{cases} 
\frac{2\pi \sinh(x \arccos(-\sigma))}{\sqrt{1 - \sigma^2} \sinh x \pi}, & \sigma \in (-1, 1), \\
\frac{2\pi \sin(x \arccosh(\sigma))}{\sqrt{\sigma^2 - 1} \sinh x \pi}, & \sigma < -1,
\end{cases}
\tag{3.3}
\]

were proved. In general, for a complex \(\sigma\), we have
\[
\hat{g}(x) = \int_{\mathbb{R}} \frac{e^{ixt}}{\cosh t - \sigma} \, dt = \begin{cases} 
\frac{2\pi \sinh(x \arccos(\sigma))}{\sqrt{1 - \sigma^2} \sinh x \pi}, & \sigma \in \mathbb{C} \setminus \mathbb{R}.
\end{cases}
\]

Also, this result can be found in [3], except the case \(|\sigma| = 1, \sigma \neq \pm 1\), which can be proved using the same arguments given in [3].

Using a limiting process in (3.3) as \(\sigma \to -1\), we can prove that
\[
\int_{\mathbb{R}} \frac{e^{ixt}}{\cosh t + 1} \, dt = \frac{2\pi x}{\sinh x \pi}.
\tag{3.4}
\]

4. Positive semi-definite solutions

According to (3.3) and (3.4), we conclude that the function
\[
g(t) = \frac{1}{\cosh t - \sigma}, \quad -1 \leq \sigma < 1
\]
is a positive semi-definite function. This enables us to state the following result:

**Theorem 4.1.** Suppose we are given Eq. (2.1), with a positive definite matrix \(A\), with the characteristic polynomial \(Q_m\) which all zeros are real and contained in the interval \([-1, 1]\). Then the corresponding function \(\varphi_m\) is positive semi-definite, i.e., the matrix equation (2.1) has a positive semi-definite solution provided \(B\) is positive semi-definite. If \(\lambda_v\) are eigenvalues of \(A\), the corresponding solution \(X = (x_{i,j})\) is given by
\[
x_{i,j} = \frac{b_{i,j}}{\sum_{v=0}^{m} a_v \lambda_i^{m-v} \lambda_j^v}, \quad i, j = 1, \ldots, m.
\]

**Proof.** Denote zeros of \(Q_m\) by \(\sigma_i, i = 1, \ldots, [m/2]\). We distinguish two cases for our matrix equation
\[
\sum_{v=0}^{m} a_v A^{m-v} X A^v = B.
\]

Case \(m\) is even. Then
\[
\varphi_m(t) = \frac{1}{Q_m(\cosh t)} = \prod_{i=1}^{m/2} \frac{1}{\cosh t - \sigma_i}.
\]

Consider now the functions
\[
g_1(t) = \frac{1}{\cosh t - \sigma_1}, \quad g_{j+1}(t) = \frac{g_j(t)}{\cosh t - \sigma_{j+1}}, \quad j = 1, \ldots, m/2 - 1.
\]

Obviously \(g_j, j = 1, \ldots, m/2\), belong to \(L^2(\mathbb{R})\) and their Fourier transforms are \(L^1(\mathbb{R}) \cap C(\mathbb{R})\) functions. The function \(g_1\) is positive semi-definite according to (3.3). Assuming that \(g_j\) is positive semi-definite, according to Lemma 3.1, the function \(g_{j+1}\) has the Fourier transform which is a
convolution of the Fourier transforms of \( g_j \) and \( 1/(\cosh t - \sigma_j) \) and those are both non-negative. According to Lemma 3.2, the Fourier transform of \( g_{j+1} \) is also non-negative, hence, \( g_{j+1} \) is positive semi-definite. Now, by induction we conclude that \( g_{m/2} = \varphi_m \) is positive semi-definite.

**Case m is odd.**

Then

\[
\varphi_m(t) = \frac{1}{\cosh \frac{t}{2}} Q_m(t) = \frac{1}{\cosh \frac{t}{2}} \prod_{i=1}^{[m/2]} \frac{1}{\cosh t - \sigma_i}.
\]

Here the proof is the same except that now we take

\[
g_0(t) = \frac{1}{\cosh \frac{t}{2}}, \quad g_{j+1}(t) = \frac{g_j(t)}{\cosh t - \sigma_{j+1}}, \quad j = 0, 1, \ldots, [m/2] - 1.
\]

The only missing ingredient is positive semi-definiteness of \( g_0 \). But, we have

\[
\int_{\mathbb{R}} \frac{e^{ixt}}{\cosh \frac{t}{2}} \, dt = 2 \int_{\mathbb{R}} \frac{e^{2ixt}}{\cosh t} \, dt = \frac{2\pi}{\cosh \pi x},
\]

using the Fourier transform given in (3.3), hence \( g_0 \) is positive semi-definite. \( \Box \)

In order to give further results we need the following lemma.

**Lemma 4.1.** The function

\[
t \mapsto \frac{1}{(\cosh t - \sigma_1)(\cosh t - \sigma_2)} \tag{4.1}
\]

for \(-1 \leq \sigma_1 < 1 \) and \( \sigma_2 < 1 \), is positive semi-definite.

**Proof.** The case \( \sigma_2 \in [-1, 1) \) is covered by Theorem 4.1, so we assume now that \( \sigma_2 < -1 \). Using a partial fraction decomposition we have

\[
\frac{1}{(\cosh t - \sigma_1)(\cosh t - \sigma_2)} = \frac{1}{\sigma_1 - \sigma_2} \left( \frac{1}{\cosh t - \sigma_1} - \frac{1}{\cosh t - \sigma_2} \right).
\]

Assuming \(-1 < \sigma_1 \) and using (3.3), we conclude that the Fourier transform of (4.1) is given by

\[
\frac{2\pi}{(\sigma_1 - \sigma_2) \sinh(x\pi)} \left( \frac{\sinh(x \arccos(-\sigma_1))}{\sqrt{1 - \sigma_1^2}} - \frac{\sin(x \arccosh(-\sigma_2))}{\sqrt{\sigma_2^2 - 1}} \right). \tag{4.2}
\]

We are going to prove that this expression is always non-negative on \( \mathbb{R} \).

Fix \( x \in \mathbb{R}^+ \), then

\[
\frac{\sinh(x \arccos(-\sigma_1))}{x \arccos(\sigma_1)}
\]

is strictly increasing function in \( x \arccos(-\sigma_1) \). According to the fact that \( x \arccos(-\sigma_1) \) is strictly increasing function in \( \sigma_1 \in (-1, 1) \), our function is strictly increasing in \( \sigma_1 \). Its minimum is achieved for \( \sigma_1 = -1 \) and its value is 1.

Now consider

\[
g(\sigma_1) = \frac{\arccos(-\sigma_1)}{\sqrt{1 - \sigma_1^2}},
\]
which derivative is given by
\[ g'(\sigma_1) = \frac{\sigma_1 \arccos(-\sigma_1) + \sqrt{1 - \sigma_1^2}}{(1 - \sigma_1^2)^{3/2}}. \]

The derivative of the numerator is \( \arccos(-\sigma_1) > 0 \), and therefore it is an increasing function. Its value for \( \sigma_1 = -1 \) is 0 and for \( \sigma_1 = 1 \) is \( \pi \). Hence \( g'(\sigma_1) \) is always positive and \( g \) is increasing.

The minimum value of the function \( g \) is 1 and is achieved for \( \sigma_1 = -1 \).

In (4.2), for fixed \( x \in \mathbb{R}^+ \), the term
\[ \frac{\sinh(x \arccos(-\sigma_1))}{\sqrt{1 - \sigma_1}} \]
has the minimum value \( x \) at \( \sigma_1 = -1 \).

Now, for fixed \( x \in \mathbb{R}^+ \), we consider the function
\[ \frac{\sin(x \arccosh(-\sigma_2))}{x \arccosh(-\sigma_2)}. \]
This function has as its global maximum the value 1 at the point \( \sigma_2 = -1 \).

For the function
\[ g(\sigma_2) = \frac{\arccosh(-\sigma_2)}{\sqrt{\sigma_2^2 - 1}}, \]
we have
\[ g'(\sigma_2) = \frac{-\sqrt{\sigma_2^2 - 1} - \sigma_2 \arccosh(-\sigma_2)}{(\sigma_2^2 - 1)^{3/2}}. \]

Since the derivative of the numerator of \( g' \) is \( -\arccosh(-\sigma_2) \), we conclude that it is decreasing, with value 0 at \( \sigma_2 = -1 \). It follows that \( g'(\sigma_2) \) is always positive, which shows \( g \) is increasing with the maximum value 1 at \( \sigma_2 = -1 \).

In total, for fixed \( x \in \mathbb{R}^+ \), we have that
\[ \frac{\sin(x \arccosh(-\sigma_2))}{\sqrt{\sigma_2^2 - 1}} \]
has as its maximum value \( x \) at \( \sigma_2 = -1 \).

Putting all together, for \( x \in \mathbb{R}^+ \), \( \sigma_1 \in (-1, 1) \) and \( \sigma_2 < -1 \), we have
\[ \frac{\sinh(x \arccos(-\sigma_1))}{\sqrt{1 - \sigma_1^2}} \frac{\sin(x \arccosh(-\sigma_2))}{\sqrt{\sigma_2^2 - 1}} > x - x = 0. \]

By the continuity argument, for the Fourier transform (4.2) at \( x = 0 \), we find that
\[ \frac{2}{\sigma_1 - \sigma_2} \left( \frac{\arccos(-\sigma_1)}{\sqrt{1 - \sigma_1^2}} - \frac{\arccosh(-\sigma_2)}{\sqrt{\sigma_2^2 - 1}} \right) \geq 0. \]

This means that our function (4.1) is positive semi-definite.

In the case \( \sigma_1 = -1 \), the Fourier transform of the function (4.1) is given by
\[ -\frac{2\pi x}{(\sigma_2 + 1) \sinh \pi x} \left( 1 - \frac{\sin(x \arccosh(-\sigma_2))}{x \arccosh(-\sigma_2)} \frac{\arccosh(-\sigma_2)}{\sqrt{\sigma_2^2 - 1}} \right). \]
It is easily seen that
\[
\left|\frac{\sin(x \arccosh(-\sigma_2)) \arccosh(-\sigma_2)}{x \arccosh(-\sigma_2)} \frac{\arccosh(-\sigma_2)}{\sqrt{\sigma_2^2 - 1}}\right| \leq 1,
\]
and we have finished the proof. □

The next lemma shows that essentially the function (4.1) is positive semi-definite only for 
\(-1 \leq \sigma_1, \sigma_2 < 1\) or \(-1 \leq \sigma_1 < 1\) and \(\sigma_2 < -1\).

**Lemma 4.2.** The function (4.1) is not positive semi-definite for \(\sigma_1, \sigma_2 < -1\) or \(\sigma_1 \sigma_2 \in \mathbb{C} \setminus \mathbb{R}\).

**Proof.** It is easy to see that for \(\sigma_1, \sigma_2 < -1\), the function (4.1) cannot be positive semi-definite, because its Fourier transform, for \(\sigma_1 = \sigma_2\), is given by
\[
\lim_{\sigma_1 \rightarrow \sigma} \frac{2\pi}{(\sigma_1 - \sigma) \sinh x\pi} \left(\frac{\sin(x \arccosh(-\sigma_1))}{\sqrt{\sigma_1^2 - 1}} - \frac{\sin(x \arccosh(-\sigma_2))}{\sqrt{\sigma_2^2 - 1}}\right).
\]
and must have at least one point where both terms in (4.3) are negative.

We consider now the Fourier transform of the function (4.1) in the case \(\sigma_1 = \sigma_2 = \sigma < -1\), by a limit process when \(\sigma_1 \rightarrow \sigma_2 = \sigma < -1\), i.e.,
\[
\lim_{\sigma_1 \rightarrow \sigma} \frac{2\pi}{(\sigma_1 - \sigma) \sinh x\pi} \left(\frac{\sin(x \arccosh(-\sigma_1))}{\sqrt{\sigma_1^2 - 1}} - \frac{\sin(x \arccosh(-\sigma_2))}{\sqrt{\sigma_2^2 - 1}}\right) = -\frac{2\pi}{\sinh x\pi} \frac{\sqrt{\sigma_2^2 - 1} x \cos(x \arccosh(-\sigma)) + \sigma \sin(x \arccosh(-\sigma))}{(\sigma^2 - 1)^{3/2}}.
\]
It is clear that this transform is negative for
\[
x \in \left(\frac{(4k - 1)\pi}{2 \arccosh(-\sigma)}, \frac{2k\pi}{\arccosh(-\sigma)}\right), \quad k \in \mathbb{N}.
\]
Consider now the case \(\sigma = \sigma_1 = \sigma_2 \notin \mathbb{R}\), with \(\sigma = a + ib, b > 0\). Then we have
\[
\int_{\mathbb{R}} \frac{e^{ixt}}{(\cosh t - \sigma)(\cosh t - \overline{\sigma})} \ dt = \frac{2\pi}{\sinh x\pi} \frac{\text{Im}(e^{-i\varphi/2} \sinh(x \arccos(-\sigma))))}{b\sqrt{|1 - \sigma^2|}} = \frac{2\pi}{b \sinh x\pi \sqrt{|1 - \sigma^2|}} \left(\frac{\cos \varphi}{2} \sin \beta x \cosh \alpha x - \sin \frac{\varphi}{2} \cos \beta x \sinh \alpha x\right),
\]
where we denoted by \(\varphi\) the argument of the complex number \(1 - \sigma^2\) and \(\alpha + i\beta = \arccos(-\sigma)\). Depending on the sign of \(\sin(\varphi/2)\) and \(\cos(\varphi/2)\), we can always choose \(x\) such that both terms in the final expression are negative. We conclude that in the case \(\sigma_1 = \sigma_2\) the function (4.1) cannot be positive semi-definite. □

The previous two lemmas give as an opportunity to give a stronger result than the one obtained in [3].
Theorem 4.2. Let $B$ be a positive semi-definite matrix and let $A$ be a positive definite matrix. Then the equation

$$A^4X - 2tA^3XA + 2(2u + 1)A^2XA^2 - 2tAXA^3 + XA^4 = B$$

with $(t, u) \in \mathbb{R}^2$, has a positive semi-definite solution if and only if $(t, u) \in D \subset \mathbb{R}^2$, where the domain $D$ is determined by

$$(t < -2 \land u - t + 1 > 0 \land u + t + 1 \geq 0) \lor (t \geq -2 \land t^2/4 - u \geq 0 \land u - t + 1 > 0).$$

Proof. All we need to do is to construct the corresponding characteristic polynomial of the mentioned equation. Using the equality (2.6) and Definition 2.1, we have

$$Q_4(z) = 2(z^2 - tz + u).$$

Using Lemmas 4.1 and 4.2, the function $\varphi_4$ is positive semi-definite if and only if zeros $\sigma_1$ and $\sigma_2$ of the polynomial $Q_4$ are $\sigma_1 \in [-1, 1)$ and $\sigma_2 \in (-\infty, 1)$. Thus, the function $\varphi_4(x)$ is positive semi-definite on the set

$$\{(t, u) = (\sigma_1 + \sigma_2, \sigma_1\sigma_2) | \sigma_1 \in [-1, 1), \sigma_2 \in (-\infty, 1)\}.$$

Since

$$\sigma_{1,2} = \frac{t}{2} \pm \sqrt{\frac{t^2}{4} - u},$$

we get the system of inequalities

$$-1 \leq \frac{t}{2} + \sqrt{\frac{t^2}{4} - u} < 1, \quad \frac{t}{2} - \sqrt{\frac{t^2}{4} - u} < 1,$$

which solution is exactly given in the statement of the theorem. □

The domain $D$ from this theorem is presented in Fig. 4.1 for $t \geq -4$.

In order to be able to express an influence of the factor $\cosh t/2$ in the function $\varphi_m$ for $m$ odd, we have the following result:

Lemma 4.3. The function

$$\varphi_3(t) = \frac{1}{\cosh \frac{t}{2}(\cosh t - \sigma)}, \quad \sigma \in (-\infty, 1)$$

is positive semi-definite.

Proof. We have

$$\int_{\mathbb{R}} \frac{e^{ixt}}{\cosh \frac{t}{2}(\cosh t - \sigma)} \, dt = \int_{\mathbb{R}} \frac{e^{2ixt}}{\cosh t \left(\cosh^2 t - \frac{1+\sigma}{2}\right)} \, dt.$$

For $\sigma \in [-1, 1)$, it is clear that $(1 + \sigma)/2 \in [0, 1)$, so that, according to Lemma 3.2, the corresponding Fourier transform is a non-negative function on $\mathbb{R}$ and $\varphi_3$ is positive semi-definite.

For $\sigma \in (-\infty, -1)$, we have $(1 + \sigma)/2 \in (-\infty, 0)$, so that we denote $a^2 = -(1 + \sigma)/2$. Finally, we end-up in

$$\int_{\mathbb{R}} \frac{e^{2ixt}}{\cosh t (\cosh^2 t + a^2)} \, dt,$$

which is non-negative function according to Proposition 4.1 form [3]. □
Now, we are able to state the main result of the paper.

**Theorem 4.3.** Suppose we are given Eq. (2.1), with a positive definite matrix $A$, with the characteristic polynomial $Q_m$ which has $k_1$ real zeros contained in the interval $[-1, 1)$ and $k_2$ zeros smaller than $-1$, with $k_1 \geq k_2$, for $m$ even, and $k_1 + 1 \geq k_2$, for $m$ odd, where $k_1 + k_2 = \lceil m/2 \rceil$.

Then the corresponding function $\varphi_m$ is positive semi-definite, i.e., the matrix equation (2.1) has a positive semi-definite solution, provided $B$ is positive semi-definite. If $\lambda_v$ are eigenvalues of $A$, the solution $X = (x_{i,j})$ is given by

$$x_{i,j} = \frac{b_{i,j}}{\sum_{v=0}^{m} a_v \lambda_i^{m-v} \lambda_j^v}, \quad i, j = 1, \ldots, m.$$  

**Proof.** Consider first the case when $m$ is even. Then, we can group the zeros of $Q_m$ according to the following $x_i \in [-1, 1)$, $y_i \in (-\infty, 1)$, $i = 1, \ldots, k_2$, and $x_i \in [-1, 1)$, $i = k_2 + 1, \ldots, k_1$.

According to the fact that the convolution is commutative and associative and using Lemma 4.1 we conclude that the Fourier transforms of

$$\frac{1}{(\cosh t - x_i)(\cosh t - y_i)} , \quad i = 1, \ldots, k_2$$

are non-negative functions, and therefore the functions itself are positive semi-definite. According to Lemma 3.2, the Fourier transform of the function

$$\prod_{i=1}^{k_2} \frac{1}{(\cosh t - x_i)(\cosh t - y_i)}$$
is non-negative and the function itself is positive semi-definite. Finally, if we include the part
\[
\prod_{i=k_2+1}^{k_1} \frac{1}{\cosh t - x_i},
\]
which Fourier transform is a non-negative function, we have that the function \( \varphi_m \) is positive semi-definite.

Now for \( m \) odd, we group the zeros according to the following \( y_0 \in (\infty, 1), x_1 \in [-1, 1) \) and \( y_i \in (-\infty, 1), i = 1, \ldots, k_2 - 1, \) and \( x_i \in [-1, 1), i = k_2, \ldots, k_1. \) Using Lemma 4.3, we have that
\[
\frac{1}{\cosh \frac{t}{2} (\cosh t - y_0)}
\]
is positive semi-definite. Also all functions
\[
\frac{1}{(\cosh t - x_i)(\cosh t - y_i)}, \quad i = 1, \ldots, k_2 - 1,
\]
are positive semi-definite according to Lemma 4.1. Finally, the functions
\[
\frac{1}{\cosh t - x_i}, \quad i = k_2, \ldots, k_1,
\]
are also positive semi-definite and, therefore, the function \( \varphi_m \) is positive semi-definite. 

5. Examples

Using the previous considerations we are able to give sufficient conditions for the existence of positive semi-definite solutions of some equations with higher order.

**Theorem 5.1.** If \( t \in (-6, 10], \) then the equation
\[
A^5 X + 5A^4 X A + tA^3 X A^2 + tA^2 X A^3 + 5A X A^4 + X A^5 = B
\]
has a positive semi-definite solution, provided \( B \) is positive semi-definite.

**Proof.** In this case we have for the characteristic polynomial
\[
Q_5(z) = 4 \left( z^2 + 2z + \frac{t - 6}{4} \right).
\]
According to Theorem 4.3, the equation has positive semi-definite solutions, provided the polynomial \( Q_5 \) has zeros \( \sigma_1 \in [-1, 1) \) and \( \sigma_1 \in (-\infty, 1). \) Using Viète formulas we have
\[
-2 = \sigma_1 + \sigma_2, \quad \frac{t - 6}{4} = \sigma_1 \sigma_2,
\]
from which we deduce \( t = 6 - 4\sigma_1(2 + \sigma_1), \sigma_1 \in [-1, 1), \) so that we have \( t \in (-6, 10]. \)

**Theorem 5.2.** If \( t \in (-\infty, -11) \cup [5, +\infty), \) then the equation
\[
A^5 X + tA^4 X A + 10A^3 X A^2 + 10A^2 X A^3 + t A X A^4 + X A^5 = B
\]
has a positive semi-definite solution, provided \( B \) is positive semi-definite.
Proof. The characteristic polynomial is

\[ Q_5(z) = 4 \left( z^2 + \frac{t - 1}{2} z + \frac{9 - t}{4} \right). \]

Using Viète formulas and Theorem 4.3, we have a positive semi-definite solution provided

\[-\frac{t - 1}{2} = \sigma_1 + \sigma_2, \quad \frac{9 - t}{4} = \sigma_1 \sigma_2,\]

with \( \sigma_1 \in [-1, 1) \) and \( \sigma_2 \in (-\infty, 1) \). Solving this system of equations we get \( t = -(4\sigma_1^2 - 2\sigma_1 + 9)/(2\sigma_1 - 1), \sigma_1 \in [-1, 1) \).

Therefore, \( t \in (-\infty, -11) \cup [5, +\infty) \). □

References