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An Extension of Pochhammer's Symbol and its Application to Hypergeometric Functions, II

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Abstract. Recently we have introduced a productive form of gamma and beta functions and applied them for generalized hypergeometric series [Filomat, **31** (2017), 207–215]. In this paper, we define an additive form of gamma and beta functions and study some of their general properties in order to obtain a new extension of the Pochhammer symbol. We then apply the new symbol for introducing two different types of generalized hypergeometric functions. In other words, based on the defined additive beta function, we first introduce an extension of Gauss and confluent hypergeometric series and then, based on two additive types of the Pochhammer symbol, we introduce two extensions of generalized hypergeometric functions of any arbitrary order. The convergence of each series is studied separately and some illustrative examples are given in the sequel.

1. Introduction

The generalized hypergeometric functions ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ appear in a wide variety of mathematical and engineering sciences [1, 3, 9, 16]. For instance, there is a large set of hypergeometic-type polynomials whose variable is located in one or more of the parameters of the corresponding functions ${}_pF_q$ [7, 8]. These polynomials are of great importance in mathematics as well as in many areas of physics [5, 13, 14, 17]. It is well-known that the base of constituting ${}_pF_q$ is the gamma function and for two important cases ${}_2F_1$ and ${}_1F_1$ may also be the beta function. Hence, a natural way to extend ${}_pF_q$ is to extend these two basic functions, or even their incomplete versions, i.e., incomplete gamma and incomplete beta functions [4].

Let $\mathbb{R}$ and $\mathbb{C}$ respectively denote the sets of real and complex numbers and z be an arbitrary complex variable. The well known (Euler's) gamma function is defined, for $\mathrm{Re}(z) > 0$, as

$$\Gamma(z) = \int_0^\infty x^{z-1}\mathrm{e}^{-x}\,\mathrm{d}x, \tag{1}$$

and for $z \in \mathbb{C}\backslash\mathbb{Z}_0^-$ where $\mathbb{Z}_0^- = \{0,-1,-2,\ldots\}$ as

$$\Gamma(z) = \frac{\Gamma(z+n)}{\prod_{k=0}^{n-1}(z+k)} \quad (n \in \mathbb{N}).$$

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The limit definition of the gamma function

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{\prod_{k=0}^{n} (z+k)}, \tag{2}$$

is valid for all complex numbers except the non-positive integers. A remarkable property for the gamma function, which is provable via the limit definition (2), is

$$\overline{\Gamma(z)} = \Gamma(\bar{z}) \overset{z=p+\mathrm{i}q}{\Rightarrow} \Gamma(p+\mathrm{i}q)\Gamma(p-\mathrm{i}q) \in \mathbb{R}. \tag{3}$$

Recently in [11], we have applied this property to introduce a productive form of the gamma function as

$$\prod(p,q) = \frac{\Gamma(p+\mathrm{i}q)\Gamma(p-\mathrm{i}q)}{\Gamma(p)} \quad (p>0,\ q\in\mathbb{R}),$$

which leads to an extension of the Pochhammer symbol $(r)_k = \Gamma(r+k)/\Gamma(r)$ as [11]:

$$\frac{\prod(s+k,q)}{\prod(s,q)} = \frac{(s+\mathrm{i}q)_k(s-\mathrm{i}q)_k}{(s)_k}.$$

Now, the point is that property (3) has an additive analogue form so that we have

$$\overline{\Gamma(z)} = \Gamma(\bar{z}) \overset{z=p+\mathrm{i}q}{\Rightarrow} \Gamma(p+\mathrm{i}q)+\Gamma(p-\mathrm{i}q) \in \mathbb{R}. \tag{4}$$

This result (4) may also be extended to n complex variables. If $z_k = p_k + \mathrm{i}q_k$ $(k=1,2,\ldots,n)$ are n distinct complex numbers, then

$$A_n^* = \frac{z_1 z_2\cdots z_n + \overline{z_1 z_2\cdots z_n}}{2} \quad \text{and} \quad B_n^* = \frac{z_1 z_2\cdots z_n - \overline{z_1 z_2\cdots z_n}}{2i},$$

are always two real values, because

$$\begin{aligned} A_n^* = \frac{z_1 z_2\cdots z_n + \bar{z}_1\bar{z}_2\cdots\bar{z}_n}{2} &= \frac{1}{2}\prod_{k=1}^{n}(p_k^2+q_k^2)^{\frac{1}{2}}\left(\exp\left(\mathrm{i}\sum_{k=0}^{n}\arctan\frac{q_k}{p_k}\right)+\exp\left(-\mathrm{i}\sum_{k=0}^{n}\arctan\frac{q_k}{p_k}\right)\right) \\ &= \prod_{k=1}^{n}(p_k^2+q_k^2)^{\frac{1}{2}}\cos\left(\sum_{k=0}^{n}\arctan\frac{q_k}{p_k}\right)\in\mathbb{R}, \end{aligned} \tag{5}$$

and

$$\begin{aligned} B_n^* = \frac{z_1 z_2\cdots z_n - \bar{z}_1\bar{z}_2\cdots\bar{z}_n}{2\mathrm{i}} &= \frac{1}{2\mathrm{i}}\prod_{k=1}^{n}(p_k^2+q_k^2)^{\frac{1}{2}}\left(\exp\left(\mathrm{i}\sum_{k=0}^{n}\arctan\frac{q_k}{p_k}\right)-\exp\left(-\mathrm{i}\sum_{k=0}^{n}\arctan\frac{q_k}{p_k}\right)\right) \\ &= \prod_{k=1}^{n}(p_k^2+q_k^2)^{\frac{1}{2}}\sin\left(\sum_{k=0}^{n}\arctan\frac{q_k}{p_k}\right)\in\mathbb{R}. \end{aligned} \tag{6}$$

In this paper, we exploit the property (4) to introduce an extension of the Pochhammer symbol in order to define the additive type of gamma, incomplete gamma, beta and incomplete beta functions and apply them for introducing two different extensions of generalized hypergeometric functions. In this sense, we first introduce an extension of Gauss and confluent hypergeometric series, which are based on the additive type beta function, and then introduce two extensions of generalized hypergeometric functions of any arbitrary order, which are based on two different generalizations of the Pochhammer symbol. The convergence of each series is studied separately and some illustrative examples are given in this sense. For this purpose, we first define an additive form of the gamma function, by referring to the property (4), as follows

$$\Gamma_c(p,q)=\frac{1}{2}\left(\Gamma(p+\mathrm{i}q)+\Gamma(p-\mathrm{i}q)\right)\quad (p>0,\ q\in\mathbb{R}). \tag{7}$$

For analogous extensions of the family of gamma functions see e.g. [2, 6, 12]. The limit definition of (7) can be derived from (2), so that we have

$$\begin{aligned}\Gamma_c(p,q)&=\frac{1}{2}\left\{\lim_{n\to\infty}\frac{n!\,n^{p+\mathrm{i}q}}{(p+\mathrm{i}q)(p+1+\mathrm{i}q)\cdots(p+n+\mathrm{i}q)}+\lim_{n\to\infty}\frac{n!\,n^{p-\mathrm{i}q}}{(p-\mathrm{i}q)(p+1-\mathrm{i}q)\cdots(p+n-\mathrm{i}q)}\right\}\\&=\lim_{n\to\infty}n!\,n^p\,\frac{\exp\Big[\mathrm{i}\Big(q\log n-\sum\limits_{k=0}^{n}\arctan\frac{q}{p+k}\Big)\Big]+\exp\Big[-\mathrm{i}\Big(q\log n-\sum\limits_{k=0}^{n}\arctan\frac{q}{p+k}\Big)\Big]}{2\prod\limits_{k=0}^{n}\left((p+k)^2+q^2\right)^{1/2}}\\&=\lim_{n\to\infty}\frac{n!\,n^p\cos\Big(q\log n-\sum\limits_{k=0}^{n}\arctan\frac{q}{p+k}\Big)}{\prod\limits_{k=0}^{n}\left((p+k)^2+q^2\right)^{1/2}}.\end{aligned} \tag{8}$$

Since $\left|\cos\Big(q\log n-\sum\limits_{k=0}^{n}\arctan\frac{q}{p+k}\Big)\right|\leqslant 1$, for any $p>0$, relation (8) implies that

$$\lim_{q\to\infty}\Gamma_c(p,q)=\lim_{q\to\infty}\frac{\Gamma(p+\mathrm{i}q)+\Gamma(p-\mathrm{i}q)}{2}=0.$$

Moreover, relations (8) and (2) show that

$$\left|\Gamma_c(p,q)\right|\leqslant\Gamma(p)\qquad (p>0,\ q\in\mathbb{R}). \tag{9}$$

In order to obtain an integral representation for $\Gamma_c(p,q)$, it is enough to consider the identity

$$\frac{1}{2}\left(x^{\mathrm{i}q}+x^{-\mathrm{i}q}\right)=\cos(q\log x),$$

and substitute it in (7) to finally obtain

$$\Gamma_c(p,q)=\int_0^\infty x^{p-1}\mathrm{e}^{-x}\cos(q\log x)\,\mathrm{d}x\qquad (p>0,\ q\in\mathbb{R}). \tag{10}$$

Remark 1.1. Since

$$\Gamma(p+\mathrm{i}q)=\left(\int_0^\infty x^{p-1}\mathrm{e}^{-x}\cos(q\log x)\,\mathrm{d}x\right)+\mathrm{i}\left(\int_0^\infty x^{p-1}\mathrm{e}^{-x}\sin(q\log x)\,\mathrm{d}x\right),$$

is valid for $p>0$ and $q\in\mathbb{R}$, there is another real function which can be defined as

$$\Gamma_s(p,q)=\int_0^\infty x^{p-1}e^{-x}\sin(q\log x)\,\mathrm{d}x=\frac{\Gamma(p+\mathrm{i}q)-\Gamma(p-\mathrm{i}q)}{2\mathrm{i}}.$$

The limit definition of this function, similar to (8), is as

$$\Gamma_s(p,q)=\lim_{n\to\infty}\frac{n!\,n^p\sin\Big(q\log n-\sum\limits_{k=0}^{n}\arctan\frac{q}{p+k}\Big)}{\prod\limits_{k=0}^{n}\left((p+k)^2+q^2\right)^{1/2}}. \tag{11}$$

It is clear that

$$\left|\Gamma_s(p,q)\right| \leqslant \Gamma(p) \quad (p>0,\ q\in\mathbb{R}),$$

and

$$\lim_{q\to\infty}\Gamma_s(p,q)=\lim_{q\to\infty}\int_0^\infty x^{p-1}\mathrm{e}^{-x}\sin(q\log x)\,\mathrm{d}x=0.$$

Remark 1.2. Some definite integrals can be computed in terms of the two real functions $\Gamma_c(a,b)$ and $\Gamma_s(a,b)$. For example, if $p,r>0$ and $q\in\mathbb{R}$, then we have

$$\int_0^\infty x^{p-1}\mathrm{e}^{-x^r}\cos(q\log x)\,\mathrm{d}x=\frac{1}{r}\Gamma_c\left(\frac{p}{r},\frac{q}{r}\right),$$

$$\int_0^\infty x^{p-1}\mathrm{e}^{-x^r}\sin(q\log x)\,\mathrm{d}x=\frac{1}{r}\Gamma_s\left(\frac{p}{r},\frac{q}{r}\right),$$

$$\int_0^\infty x^{p-1}\mathrm{e}^{-rx}\cos(q\log x)\,\mathrm{d}x=\frac{1}{r^p}\Big(\cos(q\log r)\,\Gamma_c(p,q)+\sin(q\log r)\,\Gamma_s(p,q)\Big),$$

and

$$\int_0^\infty x^{p-1}\mathrm{e}^{-rx}\sin(q\log x)\,\mathrm{d}x=\frac{1}{r^p}\Big(-\sin(q\log r)\,\Gamma_c(p,q)+\cos(q\log r)\,\Gamma_s(p,q)\Big).$$

Remark 1.3. To compute $\Gamma_c(m,q)$ and $\Gamma_s(m,q)$ when $m\in\mathbb{N}$, one needs to evaluate two particular integrals

$$\Gamma_c(1,q)=\int_0^\infty \mathrm{e}^{-x}\cos(q\log x)\,\mathrm{d}x \qquad\text{and}\qquad \Gamma_s(1,q)=\int_0^\infty \mathrm{e}^{-x}\sin(q\log x)\,\mathrm{d}x,$$

with upper bounds

$$\left|\Gamma_c(1,q)\right|\leqslant 1 \quad\text{and}\quad \left|\Gamma_s(1,q)\right|\leqslant 1 \quad (q\in\mathbb{R}).$$

In other words, by recalling the fundamental recurrence relation $\Gamma(z+1)=z\,\Gamma(z)$, we have

$$\begin{aligned}\Gamma(m+\mathrm{i}q) &= (m-1+\mathrm{i}q)(m-2+\mathrm{i}q)\cdots(1+\mathrm{i}q)\,\Gamma(1+\mathrm{i}q)\\ &= \prod_{k=1}^{m-1}((m-k)^2+q^2)^{\frac12}\exp\left(i\sum_{k=1}^{m-1}\arctan\frac{q}{m-k}\right)(\Gamma_c(1,q)+\mathrm{i}\,\Gamma_s(1,q))\end{aligned}$$

and

$$\begin{aligned}\Gamma(m-\mathrm{i}q) &= (m-1-\mathrm{i}q)(m-2-\mathrm{i}q)\cdots(1-\mathrm{i}q)\,\Gamma(1-\mathrm{i}q)\\ &= \prod_{k=1}^{m-1}((m-k)^2+q^2)^{\frac12}\exp\left(-\mathrm{i}\sum_{k=1}^{m-1}\arctan\frac{q}{m-k}\right)(\Gamma_c(1,q)-\mathrm{i}\,\Gamma_s(1,q)),\end{aligned}$$

which respectively yield

$$\begin{aligned}\Gamma_c(m,q) &= \int_0^\infty x^{m-1}\mathrm{e}^{-x}\cos(q\log x)\,\mathrm{d}x=\frac12\left(\Gamma(m+\mathrm{i}q)+\Gamma(m-\mathrm{i}q)\right)\\ &= \left\{\Gamma_c(1,q)\cos\left(\sum_{k=1}^{m-1}\arctan\frac{q}{m-k}\right)-\Gamma_s(1,q)\sin\left(\sum_{k=1}^{m-1}\arctan\frac{q}{m-k}\right)\right\}\prod_{k=1}^{m-1}((m-k)^2+q^2)^{\frac12}\end{aligned}$$

and

$$\begin{aligned}\Gamma_s(m,q) &= \int_0^\infty x^{m-1}\mathrm{e}^{-x}\sin(q\log x)\,\mathrm{d}x = \frac{1}{2\mathrm{i}}\left(\Gamma(m+\mathrm{i}q)-\Gamma(m-\mathrm{i}q)\right)\\ &= \left\{\Gamma_s(1,q)\cos\left(\sum_{k=1}^{m-1}\arctan\frac{q}{m-k}\right)-\Gamma_c(1,q)\sin\left(\sum_{k=1}^{m-1}\arctan\frac{q}{m-k}\right)\right\}\prod_{k=1}^{m-1}((m-k)^2+q^2)^{\frac{1}{2}}.\end{aligned}$$

The so-called incomplete gamma functions $\gamma(z;x)$ and $\Gamma(z;x)$ defined by

$$\gamma(z;x) = \int_0^x t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad (\mathrm{Re}(z)>0,\ x\geqslant 0), \tag{12}$$

and

$$\Gamma(z;x) = \int_x^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad (x\geqslant 0\,;\ \mathrm{Re}(z)>0 \text{ when } x=0), \tag{13}$$

are known to satisfy the well-known decomposition formula

$$\gamma(z;x)+\Gamma(z;x)=\Gamma(z) \qquad (\mathrm{Re}(z)>0).$$

They play important roles in the study of the analytic solutions of a variety of problems in diverse areas of physical problems and engineering [6,12]. Now, similar to the previous cases, we can define the additive form of both functions (12) and (13) respectively as follows

$$\gamma_c(p,q;x) = \int_0^x t^{p-1}\mathrm{e}^{-t}\cos(q\log t)\,\mathrm{d}t = \frac{1}{2}\left(\gamma(p+\mathrm{i}q;x)+\gamma(p-\mathrm{i}q;x)\right) \quad (p>0,\ q\in\mathbb{R},\ x\geqslant 0), \tag{14}$$

and

$$\Gamma_c(p,q;x) = \int_x^\infty t^{p-1}\mathrm{e}^{-t}\cos(q\log t)\,\mathrm{d}t = \frac{1}{2}\left(\Gamma(p+\mathrm{i}q;x)+\Gamma(p-\mathrm{i}q;x)\right) \quad (x\geqslant 0,\ q\in\mathbb{R},\ p>0 \text{ when } x=0). \tag{15}$$

It is not difficult to verify from (14) and (15) that

$$\gamma_c(p,q;x)+\Gamma_c(p,q;x)=\Gamma_c(p,q) \quad (p>0,\ q\in\mathbb{R}).$$

Also, if $p>0$ and $q\in\mathbb{R}$, then for any $x\geqslant 0$ we have

$$\left|\gamma_c(p,q;x)\right|\leqslant\gamma(p;x) \qquad \text{and} \qquad \left|\Gamma_c(p,q;x)\right|\leqslant\Gamma(p;x).$$

In a similar way, two further incomplete functions can be defined as follows

$$\gamma_s(p,q;x) = \int_0^x t^{p-1}\mathrm{e}^{-t}\sin(q\log t)\,\mathrm{d}t = \frac{1}{2\mathrm{i}}\left(\gamma(p+\mathrm{i}q;x)-\gamma(p-\mathrm{i}q;x)\right) \quad (p>0,\ q\in\mathbb{R},\ x\geqslant 0),$$

and

$$\Gamma_s(p,q;x) = \int_x^\infty t^{p-1}\mathrm{e}^{-t}\sin(q\log t)\,\mathrm{d}t = \frac{1}{2\mathrm{i}}\left(\Gamma(p+\mathrm{i}q;x)-\Gamma(p-\mathrm{i}q;x)\right) \quad (x\geqslant 0,\ q\in\mathbb{R},\ p>0 \text{ when } x=0),$$

where

$$\gamma_s(p,q;x)+\Gamma_s(p,q;x)=\Gamma_s(p,q) \quad (p>0,\ q\in\mathbb{R}).$$

When $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, the beta function [4] has a close relationship with the gamma function as

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \mathrm{B}(y,x).$$

We can similarly apply the result (4) to define an additive form of beta function as follows

$$\begin{aligned} \mathrm{B}_c(p,q,r,s) &= \frac{1}{2}\left(\mathrm{B}(p+\mathrm{i}r, q+\mathrm{i}s) + \mathrm{B}(p-\mathrm{i}r, q-\mathrm{i}s)\right) \\ &= \frac{1}{2}\frac{\Gamma(p+q-\mathrm{i}(r+s))\Gamma(p+\mathrm{i}r)\Gamma(q+\mathrm{i}s) + \Gamma(p+q+\mathrm{i}(r+s))\Gamma(p-\mathrm{i}r)\Gamma(q-\mathrm{i}s)}{\Gamma(p+q+\mathrm{i}(r+s))\Gamma(p+q-\mathrm{i}(r+s))}. \end{aligned} \tag{16}$$

Since

$$\frac{1}{2}\left(t^{\mathrm{i}r}(1-t)^{\mathrm{i}s} + t^{-\mathrm{i}r}(1-t)^{-\mathrm{i}s}\right) = \cos\left(r\log t + s\log(1-t)\right),$$

the integral representation of (16) takes the form

$$\mathrm{B}_c(p,q,r,s) = \int_0^1 t^{p-1}(1-t)^{q-1}\cos\left(r\log t + s\log(1-t)\right) d\,\mathrm{d}t. \tag{17}$$

Also by noting that

$$\frac{1}{2\mathrm{i}}\left(t^{\mathrm{i}r}(1-t)^{\mathrm{i}s} - t^{-\mathrm{i}r}(1-t)^{-\mathrm{i}s}\right) = \sin\left(r\log t + s\log(1-t)\right),$$

a second real function corresponding to the beta function can be defined as

$$\begin{aligned} \mathrm{B}_s(p,q,r,s) &= \int_0^1 t^{p-1}(1-t)^{q-1}\sin\left(r\log t + s\log(1-t)\right)\,\mathrm{d}t \\ &= \frac{1}{2\mathrm{i}}\left(\mathrm{B}(p+\mathrm{i}r, q+\mathrm{i}s) - \mathrm{B}(p-\mathrm{i}r, q-\mathrm{i}s)\right) \\ &= \frac{1}{2\mathrm{i}}\frac{\Gamma(p+q-\mathrm{i}(r+s))\Gamma(p+\mathrm{i}r)\Gamma(q+\mathrm{i}s) - \Gamma(p+q+\mathrm{i}(r+s))\Gamma(p-\mathrm{i}r)\Gamma(q-\mathrm{i}s)}{\Gamma(p+q+\mathrm{i}(r+s))\Gamma(p+q-\mathrm{i}(r+s))}. \end{aligned} \tag{18}$$

If $p, q > 0$ in (17) and (18), then for any $r, s \in \mathbb{R}$ one can show that

$$\left|\mathrm{B}_c(p,q,r,s)\right| \leqslant \mathrm{B}(p,q) \qquad \text{and} \qquad \left|\mathrm{B}_s(p,q,r,s)\right| \leqslant \mathrm{B}(p,q).$$

The incomplete beta function [4, 16] is defined by

$$\mathrm{B}(p,q;x) = \int_0^x t^{p-1}(1-t)^{q-1}\,\mathrm{d}t \qquad (\mathrm{Re}(p) > 0,\ \ \mathrm{Re}(q) > 0,\ \ x \in [0,1]).$$

Now, by noting (17), we can define the incomplete case of the additive form of beta function as

$$\begin{aligned} \mathrm{B}_c(p,q,r,s;x) &= \int_0^x t^{p-1}(1-t)^{q-1}\cos\left(r\log t + s\log(1-t)\right)\,\mathrm{d}t \\ &= \frac{1}{2}\left(\mathrm{B}(p+\mathrm{i}r, q+\mathrm{i}s; x) + \mathrm{B}(p-\mathrm{i}r, q-\mathrm{i}s; x)\right). \end{aligned}$$

Again, if $p, q > 0$ and $r, s \in \mathbb{R}$, then for any $x \in [0,1]$ we have

$$\left|\mathrm{B}_c(p,q,r,s;x)\right| \leqslant \mathrm{B}(p,q;x).$$

Finally, according to (4), a further type of additive incomplete beta function can be defined as

$$\begin{aligned}\mathrm{B}_s(p,q,r,s;x) &= \int_0^x t^{p-1}(1-t)^{q-1}\sin\left(r\log t + s\log(1-t)\right)\,\mathrm{d}t\\ &= \frac{1}{2\mathrm{i}}\left(\mathrm{B}(p+\mathrm{i}r,q+\mathrm{i}s;x)-\mathrm{B}(p-\mathrm{i}r,q-\mathrm{i}s;x)\right).\end{aligned}$$

2. Two additive types of generalized hypergeometric functions

One of the main reasons for introducing and developing the generalized hypergeometric series is that many special functions [4, 8, 10] can be represented in terms of them and their initial properties can be directly found via the initial properties of hypergeometric functions. Also, they appear as solutions of many important ordinary differential equations [8, 10, 13]. The generalized hypergeometric function

$$ {}_pF_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p\\ b_1,\ b_2,\ \ldots & b_q\end{matrix}\ \middle|\ z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\,\frac{z^k}{k!}, \tag{19}$$

in which $(r)_k = \prod_{j=0}^{k-1}(r+j) = \Gamma(r+k)/\Gamma(r)$ denotes the Pochhammer symbol [4] and z may be a complex variable is indeed a Taylor series expansion for a function, say f, as $\sum_{k=0}^{\infty} c_k^* z^k$ with $c_k^* = f^{(k)}(0)/k!$ for which the ratio of successive terms can be written as

$$\frac{c_{k+1}^*}{c_k^*} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}.$$

According to the ratio test [4], the series (19) is convergent for any $p \leqslant q+1$. In fact, it converges in $|z|<1$ for $p=q+1$, converges everywhere for $p<q+1$ and converges nowhere ($z\neq 0$) for $p>q+1$. Moreover, for $p=q+1$ it absolutely converges for $|z|=1$ if the condition

$$A^* = \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) > 0,$$

holds and is conditionally convergent for $|z|=1$ and $\neq 1$ if $-1 < A^* \leqslant 0$ and is finally divergent for $|z|=1$ and $z\neq 1$ if $A^*\leqslant -1$. There are two important cases of the series (19) arising in many physical problems [3, 7, 11, 13]. The first case is the Gauss hypergeometric function convergent in $|z|\leqslant 1$ that is denoted by

$$y = {}_2F_1\left(\begin{matrix} a,\ b\\ c\end{matrix}\ \middle|\ z\right) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\,\frac{z^k}{k!},$$

and satisfies the differential equation

$$z\,(z-1)\,y'' + ((a+b+1)z-c)\,y' + ab\,y = 0. \tag{20}$$

Particular choices of the parameters in the linearly independent solutions of the differential equation (20) yield 24 special cases. The ${}_2F_1$ can be given an integral representation as

$$ {}_2F_1\left(\begin{matrix} a,\ b\\ c\end{matrix}\ \middle|\ z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t \quad (\mathrm{Re}\,c > \mathrm{Re}\,b > 0;\ \left|\arg(1-z)\right| < \pi). \tag{21}$$

By using a series expansion of $(1-tz)^{-a}$ in (21), one can also write the ${}_2F_1$ in terms of the beta function as

$$ {}_2F_1\left(\begin{matrix} a,\ b\\ c\end{matrix}\ \middle|\ z\right) = \sum_{k=0}^{\infty}(a)_k\frac{\mathrm{B}(b+k,c-b)}{\mathrm{B}(b,c-b)}\,\frac{z^k}{k!}. \tag{22}$$

The second case, which converges everywhere, is the confluent hypergeometric function

$$y = {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!},$$

as a basis solution of the differential equation

$$z\, y'' + (c - z)\, y' - b\, y = 0,$$

which is a degenerate form of equation (20) where two of the three regular singularities merge into an irregular singularity. The ${}_1F_1$ has an integral form as

$${}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \mathrm{e}^{zt}\, \mathrm{d}t \qquad (\mathrm{Re}\, c > \mathrm{Re}\, b > 0;\ \left|\arg(1-z)\right| < \pi),$$

and can be represented in terms of the beta function as

$${}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{\mathrm{B}(b+k, c-b)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}.$$

Due to the form of their differential equations, most of the special functions of mathematical physics may be obtained from ${}_2F_1$ and ${}_1F_1$ by special choices of the parameters. There have been some extensions of these functions in the literature [15]. In three next sections, we introduce an extension of ${}_2F_1$ and ${}_1F_1$ which are based on the additive form of beta function (17) and in the sequel we introduce two extensions of generalized hypergeometric series of arbitrary order which are based on two new different definitions of the Pochhammer symbol by adding an extra parameter. The convergence problem of each introduced series is separately studied.

2.1. An extension of ${}_2F_1$ and ${}_1F_1$ based on the integral (17)

By noting the relations (17) and (22), the proposed extension of ${}_2F_1$ can be considered as

$${}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| z; (r,s) \right) = \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}_c(b+k, c-b, r, s)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}, \tag{23}$$

which reduces to the same as ${}_2F_1$ when $r = s = 0$. Since we deal with complex variables, all parameters defined in (23) are considered real. By recalling this condition, if $c > b > 0$ and c is not a negative integer or zero, then the integral representation of (23) will be derived by (17) as follows

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| z; (r,s) \right) &= \sum_{k=0}^{\infty} (a)_k \frac{\int_0^1 t^{b+k-1}(1-t)^{c-b-1} \cos\left(r \log t + s \log(1-t)\right) \mathrm{d}t}{\mathrm{B}(b, c-b)} \frac{z^k}{k!} \\ &= \frac{1}{\mathrm{B}(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \cos\left(r \log t + s \log(1-t)\right) \left(\sum_{k=0}^{\infty} (a)_k \frac{(zt)^k}{k!} \right) \mathrm{d}t \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \cos\left(r \log t + s \log(1-t)\right) \mathrm{d}t. \end{aligned} \tag{24}$$

Since $\left|\cos\left(r \log t + s \log(1-t)\right)\right| \leqslant 1$ in (24), according to the well-known Abel theorem, the series (23) converges in the circle $|z| \leqslant 1$. Similarly, for the extension of ${}_1F_1$ we can define

$${}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z; (r,s) \right) = \sum_{k=0}^{\infty} \frac{\mathrm{B}_c(b+k, c-b, r, s)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!},$$

which would have an integral representation as

$$
\begin{aligned}
{}_1F_1\left(\begin{array}{c} b \\ c \end{array} \,\middle|\, z;(r,s)\right) &= \frac{1}{\mathrm{B}(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\cos\left(r\log t + s\log(1-t)\right)\left(\sum_{k=0}^{\infty}\frac{(zt)^k}{k!}\right)\mathrm{d}t \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\mathrm{e}^{zt}\cos\left(r\log t + s\log(1-t)\right)\,\mathrm{d}t\,,
\end{aligned}
$$

and is convergent everywhere provided that $c > b > 0$ and c is not a negative integer or zero.

2.2. An extension of generalized hypergeometric functions based on integral (10)

Since the Pochhammer symbol is defined as $(r)_k = \Gamma(r+k)/\Gamma(r)$, one can extend it using the additive form of gamma function as follows

$$
[r;q]_k = \frac{\Gamma_c(r+k,q)}{\Gamma(r)} = \frac{\Gamma(r+k+\mathrm{i}q)+\Gamma(r+k-\mathrm{i}q)}{2\,\Gamma(r)} \quad (r>0,\ q\in\mathbb{R}). \tag{25}
$$

By noting the definition (25), an extension of ${}_2F_1$ can now be considered as

$$
\begin{aligned}
{}_2F_1\left(\begin{array}{c} [a;\lambda_1 q],\ [b;\lambda_2 q] \\ c \end{array} \,\middle|\, z\right) &= \sum_{k=0}^{\infty}\frac{[a;\lambda_1 q]_k[b;\lambda_2 q]_k}{(c)_k}\frac{z^k}{k!} \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma_c(a+k,\lambda_1 q)\,\Gamma_c(b+k,\lambda_2 q)}{\Gamma(c+k)}\frac{z^k}{k!}\,,
\end{aligned} \tag{26}
$$

in which $a,b,c>0$ and $q,\lambda_1,\lambda_2\in\mathbb{R}$. According to (9), since

$$
\left|\Gamma_c(a+k,\lambda_1 q)\,\Gamma_c(b+k,\lambda_2 q)\right| \leqslant \Gamma(a+k)\Gamma(b+k),
$$

the series (26) would be convergent in $|z|\leqslant 1$ if c is not a negative integer or zero. For example, for $(\lambda_1,\lambda_2)=(1,0)$, the integral representation of (26) is reduced to

$$
\begin{aligned}
{}_2F_1\left(\begin{array}{c} [a;q],\ [b;0] \\ c \end{array} \,\middle|\, z\right) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma_c(a+k,q)\,\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!} \\
&= \frac{1}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_k}\left(\int_0^{\infty}x^{a+k-1}\mathrm{e}^{-x}\cos(q\log x)\,\mathrm{d}x\right)\frac{z^k}{k!} \\
&= \frac{1}{\Gamma(a)}\int_0^{\infty}x^{a-1}\mathrm{e}^{-x}\cos(q\log x)\left(\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_k}\frac{(xz)^k}{k!}\right)\mathrm{d}x \\
&= \frac{1}{\Gamma(a)}\int_0^{\infty}x^{a-1}\mathrm{e}^{-x}\cos(q\log x)\,{}_1F_1\left(\begin{array}{c} b \\ c \end{array} \,\middle|\, xz\right)\mathrm{d}x\,.
\end{aligned}
$$

Similarly, one can extend ${}_1F_1$ as

$$
{}_1F_1\left(\begin{array}{c} [b;q] \\ c \end{array} \,\middle|\, z\right) = \sum_{k=0}^{\infty}\frac{[b;q]_k}{(c)_k}\frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma_c(b+k,q)}{\Gamma(c+k)}\frac{z^k}{k!}\,, \tag{27}
$$

whose integral representation

$$
\begin{aligned}
{}_1F_1\left(\begin{array}{c} [b;q] \\ c \end{array} \,\middle|\, z\right) &= \frac{1}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{1}{(c)_k}\left(\int_0^{\infty}x^{b+k-1}\mathrm{e}^{-x}\cos(q\log x)\,\mathrm{d}x\right)\frac{z^k}{k!} \\
&= \frac{1}{\Gamma(b)}\int_0^{\infty}x^{b-1}\mathrm{e}^{-x}\cos(q\log x)\left(\sum_{k=0}^{\infty}\frac{1}{(c)_k}\frac{(xz)^k}{k!}\right)\mathrm{d}x \\
&= \frac{1}{\Gamma(b)}\int_0^{\infty}x^{b-1}\mathrm{e}^{-x}\cos(q\log x)\,{}_0F_1\left(\begin{array}{c} - \\ c \end{array} \,\middle|\, xz\right)\mathrm{d}x,
\end{aligned}
$$

is convergent everywhere when $b > 0$ and c is not a negative integer or zero. By noting the definition of the two particular cases (26) and (27), it is a good position now to consider the extension problem for any general case (p, q). Let $q, \{b_k\}_{k=1}^q, \{\lambda_k\}_{k=1}^p \in \mathbb{R}$ and $\{a_k\}_{k=1}^p > 0$ and then define

$$ {}_pF_q\left(\begin{array}{cccc} [a_1;\lambda_1 q], & [a_2;\lambda_2 q], & \ldots & [a_p;\lambda_p q] \\ b_1, & b_2, & \ldots & b_q \end{array} \middle| \, z\right) = \sum_{k=0}^{\infty} \frac{[a_1;\lambda_1 q]_k [a_2;\lambda_2 q]_k \ldots [a_p;\lambda_p q]_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{z^k}{k!}. \tag{28} $$

Since

$$ \left| [a_1;\lambda_1 q]_k [a_2;\lambda_2 q]_k \cdots [a_p;\lambda_p q]_k \right| \leqslant (a_1)_k (a_2)_k \cdots (a_p)_k, $$

the series (28) is convergent whenever the corresponding series (19) is convergent and $\{b_k\}_{k=1}^q$ are not negative integers or zero. Let us consider some illustrative examples.

Example 2.1. By noting that ${}_1F_0\left(\begin{array}{c} a \\ - \end{array} \middle| \, z\right) = (1-z)^{-a}$, the series (28) is simplified as

$$ \begin{aligned} {}_1F_0\left(\begin{array}{c} [a;q] \\ - \end{array} \middle| \, z\right) &= \sum_{k=0}^{\infty} \frac{\Gamma(a + \mathrm{i}q + k) + \Gamma(a - \mathrm{i}q + k)}{2\Gamma(a)} \frac{z^k}{k!} \\ &= \frac{1}{2\Gamma(a)} \left(\Gamma(a+\mathrm{i}q)(1-z)^{-a-\mathrm{i}q} + \Gamma(a-\mathrm{i}q)(1-z)^{-a+\mathrm{i}q}\right) \\ &= (1-z)^{-a}\left(\frac{\Gamma_c(a,q)}{\Gamma(a)} \cos(q\log(1-z)) - \frac{\Gamma_s(a,q)}{\Gamma(a)} \sin(q\log(1-z))\right). \end{aligned} $$

Example 2.2. By noting that ${}_2F_1\left(\begin{array}{c} 1,\ 1 \\ 2 \end{array} \middle| \, z\right) = -\dfrac{\log(1-z)}{z}$, the series (28) is simplified as

$$ \begin{aligned} {}_2F_1\left(\begin{array}{c} [1;\lambda_1 q],\ [1;\lambda_2 q] \\ 2 \end{array} \middle| \, z\right) &= \frac{\Gamma(1+\mathrm{i}\lambda_1 q)\Gamma(1+\mathrm{i}\lambda_2 q)}{4} {}_2F_1\left(\begin{array}{c} 1+\mathrm{i}\lambda_1 q,\ 1+\mathrm{i}\lambda_2 q \\ 2 \end{array} \middle| \, z\right) \\ &+ \frac{\Gamma(1+\mathrm{i}\lambda_1 q)\Gamma(1-\mathrm{i}\lambda_2 q)}{4} {}_2F_1\left(\begin{array}{c} 1+\mathrm{i}\lambda_1 q,\ 1-\mathrm{i}\lambda_2 q \\ 2 \end{array} \middle| \, z\right) \\ &+ \frac{\Gamma(1-\mathrm{i}\lambda_1 q)\Gamma(1+\mathrm{i}\lambda_2 q)}{4} {}_2F_1\left(\begin{array}{c} 1-\mathrm{i}\lambda_1 q,\ 1+\mathrm{i}\lambda_2 q \\ 2 \end{array} \middle| \, z\right) \\ &+ \frac{\Gamma(1-\mathrm{i}\lambda_1 q)\Gamma(1-\mathrm{i}\lambda_2 q)}{4} {}_2F_1\left(\begin{array}{c} 1-\mathrm{i}\lambda_1 q,\ 1-\mathrm{i}\lambda_2 q \\ 2 \end{array} \middle| \, z\right) \end{aligned} \tag{29} $$

For instance, if $(\lambda_1, \lambda_2) = (0, 1)$ in (29), then by noting that

$$ {}_2F_1\left(\begin{array}{c} 1,\ a \\ 2 \end{array} \middle| \, z\right) = -\frac{(1-z)^{1-a}}{(1-a)\,z} \quad (a \neq 1), $$

it is reduced to

$$ \begin{aligned} {}_2F_1\left(\begin{array}{c} 1,\ [1;q] \\ 2 \end{array} \middle| \, z\right) &= \frac{1}{qz}\left(\frac{\Gamma(1+\mathrm{i}q)(1-z)^{-\mathrm{i}q} - \Gamma(1-\mathrm{i}q)(1-z)^{\mathrm{i}q}}{2i}\right) \\ &= \frac{\Gamma_s(1,q)\cos(q\log(1-z)) - \Gamma_c(1,q)\ \sin(q\log(1-z))}{q\,z}. \end{aligned} $$

In this sense, it is interesting to know that

$$ \lim_{q\to 0} \frac{\Gamma_s(1,q)\cos(q\log(1-z)) - \Gamma_c(1,q)\ \sin(q\log(1-z))}{q\,z} = -\frac{\log(1-z)}{z}. $$

2.3. Another extension of generalized hypergeometric functions based on relations (5) and (6)

If in relations (5) and (6) we take $z_k = p + k - 1 + \mathrm{i}q$ where $p, q \in \mathbb{R}$, then they respectively change to

$$\frac{(p+\mathrm{i}q)_n + (p-\mathrm{i}q)_n}{2} = \prod_{k=1}^{n} \left((p+k-1)^2 + q^2\right)^{\frac{1}{2}} \cos\left(\sum_{k=1}^{n} \arctan\frac{q}{p+k-1}\right) \in \mathbb{R}, \tag{30}$$

and

$$\frac{(p+\mathrm{i}q)_n - (p-\mathrm{i}q)_n}{2i} = \prod_{k=1}^{n} \left((p+k-1)^2 + q^2\right)^{\frac{1}{2}} \sin\left(\sum_{k=1}^{n} \arctan\frac{q}{p+k-1}\right) \in \mathbb{R}.$$

Relation (30) is clearly a real extension of Pochhammer symbol for $q = 0$. Hence, we can here define another extension of the hypergeometric functions (19) as

$${}_pF_q\left(\begin{matrix} \{a_1;\lambda_1 q\}, & \{a_2;\lambda_2 q\}, & \dots & \{a_p;\lambda_p q\} \\ b_1, & b_2, & \dots & b_q \end{matrix} \;\middle|\; z \right) = \sum_{k=0}^{\infty} \frac{\{a_1;\lambda_1 q\}_k \{a_2;\lambda_2 q\}_k \cdots \{a_p;\lambda_p q\}_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{31}$$

where

$$\{a;\lambda q\}_k = \frac{(a+\mathrm{i}\lambda q)_k + (a-\mathrm{i}\lambda q)_k}{2} = \frac{1}{2}\left(\frac{\Gamma(a+k+\mathrm{i}\lambda q)}{\Gamma(a+\mathrm{i}\lambda q)} + \frac{\Gamma(a+k-\mathrm{i}\lambda q)}{\Gamma(a-\mathrm{i}\lambda q)}\right),$$

is an additive form for the Pochhammer symbol and $\{b_k\}_{k=1}^{q}$, $\{a_k, \lambda_k\}_{k=1}^{p}$ and q are all real parameters. Note that the numerator term of the fraction (31) is simplified as

$$\{a_1;\lambda_1 q\}_n \cdots \{a_p;\lambda_p q\}_n = \prod_{k=1}^{p} \frac{(a_k+\mathrm{i}\lambda_k q)_n + (a_k-\mathrm{i}\lambda_k q)_n}{2} = \frac{1}{2^p}\sum_{j=1}^{2^p} (A_{1,n,j})(A_{2,n,j}) \cdots (A_{p,n,j}),$$

where $\{A_{k,n,j}\}_{k=1}^{p}$ are specific values in terms of $\{a_j, \lambda_j\}_{j=1}^{p}$, q and $\mathrm{i} = \sqrt{-1}$. This means that the right hand side of (31) is indeed a combination of the sum of at most 2^p hypergeometric functions of the same order (p, q), see also [10]. Hence, the convergence radius of (31) would directly depend on the convergence radius of (19), as the following examples show.

Example 2.3. Let $(p, q) = (2, 1)$. Then (31) is reduced to

$$\begin{aligned} 4\,{}_2F_1\left(\begin{matrix} \{a;\lambda_1 q\}, & \{b;\lambda_2 q\} \\ & c \end{matrix} \;\middle|\; z \right) &= 4\sum_{k=0}^{\infty} \frac{\{a;\lambda_1 q\}_k \{b;\lambda_2 q\}_k}{(c)_k} \frac{z^k}{k!} = {}_2F_1\left(\begin{matrix} a+\mathrm{i}\lambda_1 q, & b+\mathrm{i}\lambda_2 q \\ & c \end{matrix} \;\middle|\; z \right) \\ + {}_2F_1\left(\begin{matrix} a+\mathrm{i}\lambda_1 q, & b-\mathrm{i}\lambda_2 q \\ & c \end{matrix} \;\middle|\; z \right) &+ {}_2F_1\left(\begin{matrix} a-\mathrm{i}\lambda_1 q, & b+\mathrm{i}\lambda_2 q \\ & c \end{matrix} \;\middle|\; z \right) + {}_2F_1\left(\begin{matrix} a-\mathrm{i}\lambda_1 q, & b-\mathrm{i}\lambda_2 q \\ & c \end{matrix} \;\middle|\; z \right). \end{aligned} \tag{32}$$

As we observe, the right hand side of (32) consists of four hypergeometric functions of the same order . So, the convergence radius of the left hand side of (32) must be $|z| \leqslant 1$ provided that $c > b > 0$, $a, q \in \mathbb{R}$ and c is not a negative integer or zero. As a very particular case, let $(\lambda_1, \lambda_2) = (1, 0)$ in (32). Then it changes to

$${}_2F_1\left(\begin{matrix} \{a;q\}, & b \\ & c \end{matrix} \;\middle|\; z \right) = \frac{1}{2}\left({}_2F_1\left(\begin{matrix} a+\mathrm{i}q, & b \\ & c \end{matrix} \;\middle|\; z \right) + {}_2F_1\left(\begin{matrix} a-\mathrm{i}q, & b \\ & c \end{matrix} \;\middle|\; z \right) \right),$$

having the integral representation

$$\begin{aligned} {}_2F_1\left(\begin{matrix} \{a;q\}, & b \\ & c \end{matrix} \;\middle|\; z \right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{(1-tz)^{-a-\mathrm{i}q} + (1-tz)^{-a+\mathrm{i}q}}{2}\,\mathrm{d}t \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\cos\left(q\log(1-tz)\right)\,\mathrm{d}t. \end{aligned}$$

Example 2.4. In comparison to Example 2.1, the generalized function (31) takes the form

$$
\begin{aligned}
{}_1F_0\left(\begin{array}{c}\{a;q\}\\ -\end{array}\,\middle|\, z\right) &= \sum_{k=0}^{\infty}\frac{(a+\mathrm{i}q)_k+(a-\mathrm{i}q)_k}{2}\,\frac{z^k}{k!}\\
&= \frac{1}{2}\left((1-z)^{-a-\mathrm{i}q}+(1-z)^{-a+\mathrm{i}q}\right)\\
&= (1-z)^{-a}\cos(q\log(1-z))\,.
\end{aligned}
$$

Example 2.5. In comparison to Example 2.2, the generalized function (31) takes the form

$$
\begin{aligned}
&4\,{}_2F_1\left(\begin{array}{c}\{1;\lambda_1 q\},\ \{1;\lambda_2 q\}\\ 2\end{array}\,\middle|\, z\right) = 4\sum_{k=0}^{\infty}\frac{\{1;\lambda_1 q\}_k\{1;\lambda_2 q\}_k}{(2)_k}\,\frac{z^k}{k!} = {}_2F_1\left(\begin{array}{c}1+\mathrm{i}\lambda_1 q,\ 1+\mathrm{i}\lambda_2 q\\ 2\end{array}\,\middle|\, z\right)\\
&+{}_2F_1\left(\begin{array}{c}1+\mathrm{i}\lambda_1 q,\ 1-\mathrm{i}\lambda_2 q\\ 2\end{array}\,\middle|\, z\right)+{}_2F_1\left(\begin{array}{c}1-\mathrm{i}\lambda_1 q,\ 1+\mathrm{i}\lambda_2 q\\ 2\end{array}\,\middle|\, z\right)+{}_2F_1\left(\begin{array}{c}1-\mathrm{i}\lambda_1 q,\ 1-\mathrm{i}\lambda_2 q\\ 2\end{array}\,\middle|\, z\right).
\end{aligned}
\tag{33}
$$

For instance, if $(\lambda_1, \lambda_2) = (0, 1)$ then (33) is simplified as

$$
{}_2F_1\left(\begin{array}{c}1,\ \{1;q\}\\ 2\end{array}\,\middle|\, z\right) = \frac{-1}{qz}\left(\frac{(1-z)^{\mathrm{i}q}-(1-z)^{-\mathrm{i}q}}{2\mathrm{i}}\right) = -\frac{\sin(q\log(1-z))}{q\,z},
$$

so that

$$
\lim_{q\to 0}\frac{\sin(q\log(1-z))}{q\,z} = \frac{\log(1-z)}{z}.
$$

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