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EXTREMAL PROBLEMS FOR POLYNOMIALS AND THEIR COEFFICIENTS*

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In this paper, we consider extremal problems of Markov's and Bernstein's type for certain classes of algebraic polynomials in the L^r metric, where $r \geq 1$. Under some restrictions of the class of all polynomials of degree at most n, the upper bounds for $|P^{(k)}(0)|$, which include L^2 norm of P on the real line, are investigated. In the last part of this paper, we consider some extremal problems for polynomials with prescribed zeros.

1. Extremal Problems of Markov's and Bernstein's Type

Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu$ of degree at most n, defined on the set S in the complex plane, with a given norm $\|\cdot\|$. We begin this section by considering the following extremal problem:

Determine the best constant A_n such that

$$\|P'\| \leq A_n \|P\| \quad (P \in \mathcal{P}_n) , \tag{1.1}$$

i.e.

$$A_n = \sup_{P \in \mathcal{P}_n} \frac{\|P'\|}{\|P\|} . \tag{1.2}$$

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The first result in this area was the well-known classical inequality of A. A. Markov [21], where $S = [-1, 1]$ and $||f|| = ||f||_\infty = \max_{-1\le t\le 1} |f(t)|$. Namely, Markov proved the following result:

Theorem 1.1. Let $P \in \mathcal{P}_n$, then

$$||P'||_\infty \le n^2 ||P||_\infty . \tag{1.3}$$

The equality holds only at ± 1 and only when $P(t) = cT_n(t)$, where T_n is the Chebyshev polynomial of the first kind of degree n and c is an arbitrary constant.

The best possible inequality for kth derivative was found by V. A. Markov [22].

Theorem 1.2. For each $k = 1, \dots, n$, the inequality

$$||P^{(k)}||_\infty \le T_n^{(k)} ||P||_\infty \quad (P \in \mathcal{P}_n) \tag{1.4}$$

holds. The extremal polynomial is T_n.

E. Hille, G. Szegö and J. D. Tamarkin [14] extended the previous result of A. A. Markov to L^r-norm ($r \ge 1$) on $(-1, 1)$.

Theorem 1.3. Let $r > 1$ and $P(t)$ be an arbitrary rational polynomial of degree n. Then

$$\left(\int_{-1}^{1} |P'(t)|^r \, dt\right)^{1/r} \le An^2 \left(\int_{-1}^{1} |P(t)|^r \, dt\right)^{1/r} , \tag{1.5}$$

where A is a positive constant which depends only on r, but not on P or n.

This result was also obtained by N. Bari [4] using very different methods.

The factor n^2 in (1.5) cannot be replaced by any function tending to infinity more slowly. Namely, for each n exist polynomials $P(t)$ of degree n such that the left side of (1.5) is $\le Bn^2$, where B is a constant of the same nature as A.

Under only a little restriction on the zeros of $P(z)$, M. A. Malik [20] found the following improvements of Theorem 1.3.

Theorem 1.4. Let $r > 1$ and $P(t)$ be an arbitrary rational polynomial of degree n. Let $P(z)$ have no zeros in the two circular regions

$$|z \pm a| < 1 - a \quad (0 \le a < 1) ,$$

then

$$\left(\int_{-1}^{1} |P'(t)|^r \, dt\right)^{1/r} \le Bn^{1+1/r} \left(\int_{-1}^{1} |P(t)|^r \, dt\right)^{1/r} ,$$

where B is a positive constant which depends only on r and a, but not on P or n.

Another type of these inequalities goes back to S. N. Bernstein [5]. He considered the following problem:

Let $P(z)$ be a polynomial of degree n and $|P(z)| \le 1$ in the unit disk $|z| \le 1$. Determine how large can be for $|z| \le 1$.

In other words, if we define $||f|| = \max_{|z|\le 1} |f(z)|$, this problem can be reduced to Inequality (1.1).

Theorem 1.5. Let $P \in \mathcal{P}_n$, then

$$||P'|| \le n ||P|| ,$$

with equality for $P(z) = cz^n$, $c =$ const.

Since a polynomial $P(z)$ is an analytic function, it attains its maximum absolute value for $|z| \le 1$ on the circumference $|z| = 1$, so we can put

$$||f|| = \max_{|z|=1} |f(z)| = \max_{-\pi<\theta\le\pi} |f(e^{i\theta})| .$$

Bernstein's Theorem 1.5 can be stated in several different forms. One of them is the following:

Theorem 1.6. Let $P \in \mathcal{P}_n$ and $P(t) \le 1 (-1 \le t \le 1)$, then

$$|P'(t)| \le \frac{n}{\sqrt{1-t^2}} , \quad -1 < t < 1 . \tag{1.6}$$

The equality is attained at the points $t = t_\nu = \cos\frac{(2\nu-1)\pi}{2n}$, $1 \le \nu \le n$, if and only if $P(t) = \pm T_n(t)$.

In the L^r norm

$$\|f\|_r = \left(\int_0^{2\pi} |f(e^{i\theta})|^r \, d\theta\right)^{1/r} \quad (r \ge 1)\,,$$

it is well known that

$$\|P'\|_r \le n\|P\|_r \quad (P \in \mathcal{P}_n)\,. \tag{1.7}$$

This inequality is due to A. Zygmund [45], who proved it for all trigonometric polynomials of degree n.

N. G. de Bruijn [7] proved the following result:

Theorem 1.7. Let $P \in \mathcal{P}_n$ and $P(z) \ne 0$ in $|z| < 1$. Then for $r \ge 1$

$$\|P'\|_r \le nC_r\|P\|_r\,, \tag{1.8}$$

where

$$C_r = \left(\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\eta}|^r d\eta\right)^{-1/r}. \tag{1.9}$$

It is easily seen that the sign of equality holds if $P(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$. It can also be shown that the sign $<$ holds otherwise.

The case $r = 2$ was obtained by P. D. Lax [18], whereas $r \to \infty$ leads to the Erdős–Lax inequality:

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|\,.$$

Recently, by using an interpolation formula, A. Aziz [2] gave a new proof of Inequality (1.7) and the following generalization of Inequality (1.8).

Theorem 1.8. Let $P \in \mathcal{P}_n$ and $\min_{|z|=1} |P(z)| = m$. If $P(z) \ne 0$ for $|z| < 1$, then for every complex number β, $|\beta| \le 1$, and for $r \ge 1$,

$$\begin{aligned}&\left(\int_0^{2\pi} |P'(e^{i\theta}) + mn\beta e^{i(n-1)\theta}|^r \, d\theta\right)^{1/r}\\ &\le nC_r \left(\int_0^{2\pi} |P(e^{i\theta}) + m\beta e^{in\theta}|^r \, d\theta\right)^{1/r},\end{aligned} \tag{1.10}$$

where C_r is defined in (1.9). The result is best possible and equality in (1.10) holds for $P(z) = az^n + bk^n$, $|a| = |b|$, $k \ge 1$ and $\beta = a/|a|$.

Choosing argument of β, with $|\beta| = 1$, suitably and making r tend to infinity in (1.10), we obtain

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2}\left(\max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)|\right),$$

which is a refinement of Erdős–Lax theorem.

Some results on L^r inequalities for polynomials can be found in the paper of Q. I. Rahman and G. Schmeisser [35].

If we put

$$\begin{aligned}\|f\|_{r,\mu} &= \left(\int_{-1}^{1} |f(t)(1-t^2)^\mu|^r \, dt\right)^{1/r}, && 0 \le r < +\infty,\\ &= \sup_{-1\le t\le 1} |f(t)|(1-t^2)^\mu\,, && r = +\infty\,,\end{aligned}$$

where $r\mu > -1 (\mu \ge 0$ if $r = +\infty)$, we can consider the following general extremal problem (see S. V. Konjagin [15])

$$A_{n,k}(r,\mu;p,\nu) = \sup_{P\in\mathcal{P}_n} \frac{\|P^{(k)}\|_{p,\nu}}{\|P\|_{r,\mu}}\,. \tag{1.11}$$

So the best constant in (1.4) is $A_{n,k}(+\infty, 0; +\infty, 0)$. We note that Bernstein's inequality (1.6) can be represented in the form

$$\|P'\|_{\infty,1/2} \le n\|P\|_{\infty,0} \quad (P \in \mathcal{P}_n)\,.$$

The case $k = n$ is especially interesting. Namely, then we have the following problem: Among all polynomials of degree n, with leading coefficient unity, find the polynomial which deviates least from zero in the norm $\|\cdot\|_{r,\mu}$.

B. D. Bojanov [6] considered the case $r = +\infty$, $\mu = \nu = 0$ and $1 \le p < +\infty$. Namely, he proved the following result:

Theorem 1.9. Let $P \in \mathcal{P}_n$ and $p \in [1, +\infty)$. Then

$$\|P'\|_{p,0} \le \|T_n'\|_{p,0}\|P\|_{\infty,0}\,.$$

Equality is attained only for $P(t) = \pm T_n(t)$.

A. Lupaş [19] investigated the best constant in the following inequality

$$\|P^{(k)}\|_\infty \le A_n(k,\alpha,\beta)\|P\|_2 \quad (P \in \mathcal{P}_n)\,,$$

where

$$\|f\|_\infty = \max_{-1\le t\le 1}|f(t)| \quad \text{and} \quad \|f\|_2 = \left(\int_{-1}^{1} w(t)|f(t)|^2\,dt\right)^{1/2},$$

with Jacobi weight $w(t) = (1-t)^\alpha(1+t)^\beta$, $\alpha,\beta > -1$. So he proved the following result:

Theorem 1.10. Let $P \in \mathcal{P}_n$ and $q = \max(\alpha,\beta) \ge -1/2$. Then

$$A_n(k,\alpha,\beta) = \left(\frac{k!}{2^{2k+\alpha+\beta+1}}\sum_{i=k}^{n} C_{i,k}^{(\alpha,\beta)}\binom{i+\alpha+\beta+k}{k}\binom{i+q}{i-k}\right)^{1/2},$$

where

$$C_{i,k}^{(\alpha,\beta)} = \frac{i!(2i+\alpha+\beta+1)\Gamma(i+\alpha+\beta+k+1)}{\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}\binom{i+q}{i-k}\,.$$

Equality is attained for

$$P(t) = C\sum_{i=k}^{n} C_{i,k}^{(\alpha,\beta)}P_i^{(\alpha,\beta)}(t)\,,$$

where C is a constant and $P_i^{(\alpha,\beta)}(t)$ is the Jacobi orthogonal polynomial of degree i.

Recently, S. V. Konjagin [16] considered the extremal problem (1.11) for $r = p = 1$ and $\mu = \nu = 0$. He found an estimate for $A_{n,k} = A(1,0;1,0)$.

Theorem 1.11. There exist two constants c_1 and c_2 $(0 < c_1 < c_2 < +\infty)$ such that

$$c_1\frac{nT_n^{(k)}(1)}{(k+1)(n-k+1)} \le A_{n,k} \le c_2\frac{nT_n^{(k)}(1)}{(k+1)(n-k+1)}$$

for each $n \in \mathbb{N}$ and $k = 1,\dots,n$.

Especially important cases are $r = p = 2$ (cf. E. Schmidt [37], P. Turán [42], A. Durand [9], G. V. Milovanović [24], G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias [30], Q. I. Rahman and G. Schmeisser [34] and Th. M. Rassias [36]).

At the end of this part, we consider a restricted polynomial set. Namely, let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which are non-negative, i.e.

$$W_n = \left\{P \mid P(t) = \sum_{k=0}^{n} a_k t^k,\ a_k \ge 0 \quad (k = 0,1,\dots,n-1),\ a_n > 0\right\}.$$

We denote by $W_n^{(m-1)}$ the subset of W_n for which $a_0 = \dots = a_{m-1} = 0$ (i.e. $P(0) = \dots = P^{(m-1)}(0) = 0$).

Let $w(t) = t^\alpha e^{-t}$ $(\alpha > -1)$ be a weight function on $[0,+\infty)$. For $P \in W_n$, we define $\|P\|_r = (\int_0^\infty w(t)P(t)^r\,dt)^{1/r}$, $r \ge 1$, and consider the following extremal problem:

Determine the best constant in the inequality

$$\|P^{(m)}\|_r^r \le C_{n,r}^{(m)}\|P\|_r^r\,, \quad P \in W_n\,, \tag{1.12}$$

i.e.

$$C_{n,r}^{(m)}(\alpha) = \sup_{P\in W_n}\left(\frac{\|P^{(m)}\|_r}{\|P\|_r}\right)^r. \tag{1.13}$$

The case $r = 2$ and $m = 1$ has been recently investigated by A. K. Varma [43, 44] and G. V. Milovanović [23]. Milovanović proved the following result:

Theorem 1.12. The best constant $C_{n,2}^{(1)}(\alpha)$ defined in (1.13) is

$$C_{n,2}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{(2+\alpha)(1+\alpha)} & (-1 < \alpha \le \alpha_n)\,, \\[2ex] \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)} & (\alpha_n \le \alpha < +\infty)\,, \end{cases}$$

where

$$\alpha_n = \frac{1}{2}(n+1)^{-1}((17n^2+2n+1)^{1/2} - 3n + 1)\,.$$

An extremal problem for higher derivatives of non-negative polynomials with respect to the same weight was investigated by G. V. Milovanović and I. Ž. Milovanović [29]. A similar problem for Freud's weight function has been dealt with G. V. Milovanović and R. Ž. Djordjević [25].

A general case for $r \in \mathbb{N}$ was considered by A. Guessab, G. V. Milovanović and O. Arino [13].

Theorem 1.13. The best constant $C_{n,r}^{(m)}(\alpha)$ defined in (1.13) is

$$C_{n,r}^{(m)}(\alpha) = \begin{cases} \dfrac{(m!)^r}{\prod_{i=1}^{mr}(i+\alpha)} & (-1 < \alpha \le \alpha_{n,r,m}), \\ \dfrac{\prod_{i=0}^{m-1}(n-i)^r}{\prod_{i=0}^{mr-1}(rn+\alpha-i)} & (\alpha_{n,r,m} \le \alpha < +\infty), \end{cases}$$

where $\alpha_{n,r,m}$ is the unique positive root of the equation

$$(m!)^r \prod_{i=0}^{mr-1}(rn+\alpha-i) = \prod_{i=1}^{mr}(i+\alpha)\prod_{i=0}^{m-1}(n-i)^r .$$

The case $r = 3$ and $m = 1$ was considered by A. Guessab and G. V. Milovanović [12]. In that case, we have

$$C_{n,3}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{(3+\alpha)(2+\alpha)(1+\alpha)} & (-1 < \alpha \le \alpha_n), \\ \dfrac{n^3}{(3n+\alpha)(3n+\alpha-1)(3n+\alpha-2)} & (\alpha_n \le \alpha < +\infty), \end{cases}$$

where α_n is the unique positive root of the equation

$$(n^2+n+1)\alpha^3 + 3(2n^2+2n-1)\alpha^2 + (11n^2-16n+2)\alpha - 3n(7n-2) = 0 .$$

Remark. The statement of Theorem 1.13 holds if W_n is the set of all algebraic polynomials $P(\not\equiv 0)$ of degree at most n (not only of exact degree n), with non-negative coefficients. In this case, for $-1 < \alpha \le \alpha_{n,r,m}$, we can see that $\tilde{P}(t) = \lambda t^m$ $(\lambda > 0)$ is an extremal polynomial.

The simplest case is $r = 1$ and $m = 1$. Then we have

$$C_{n,1}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{\alpha+1} & (-1 < \alpha \le 0), \\ \dfrac{n}{\alpha+n} & (\alpha \ge 0). \end{cases} .$$

The case $\alpha = 1$, $r = 2$ and $m = 1$ was considered by A. K. Varma [43]. Then

$$C_{n,2}^{(1)}(1) = \frac{n}{2(2n+1)} .$$

2. Estimations of the Coefficients of Polynomials

In this part, we will consider some extremal problems of the form

$$|P^{(k)}(0)| \le C_{n,k}\|P\| .$$

The first result on this subject was given by V. A. Markov [22]. Namely, if

$$\|P\| = \|P\|_\infty = \max_{-1\le x\le 1} |P(t)|$$

and $T_n(t) = \sum_{\nu=0}^{n} \gamma_{n,\nu} t^\nu$ denotes the nth Chebyshev polynomial of the first kind, Markov proved that

$$|a_k| \le \begin{cases} |\gamma_{n,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is even}, \\ |\gamma_{n-1,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is odd}. \end{cases} \tag{2.1}$$

For $k = n$, (2.1) reduces to the well-known Chebyshev inequality

$$|a_n| \le 2^{n-1}\|P\|_\infty . \tag{2.2}$$

Under the assumption that $P(1) = 0$ or $P(-1) = 0$, I. Schur [38] showed that (2.2) can be replaced by

$$|a_n| \le 2^{n-1}\left(\cos\frac{\pi}{4n}\right)^{2n}\|P\|_\infty .$$

This result was extended by Q. I. Rahman and G. Schmeisser [33] for polynomials with real coefficients, which have at most $n-1$ distinct zeros in $(-1,1)$.

In L^2 norm

$$\|P\| = \|P\|_2 = \left(\int_{-1}^{1} |P(t)|^2\,dt\right)^{1/2},$$

G. Labelle [17] proved that

$$|a_k| \le \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!}\left(k+\tfrac{1}{2}\right)^{1/2}\binom{[(n-k)/2]+k+1/2}{[(n-k)/2]}\|P\|_2$$

for all $P \in \mathcal{P}_n$ and $0 \le k \le n$, where the symbol $[x]$ denotes as usual the integral part of x. Equality in this case is attained only for the constant multiplies of the polynomial

$$\sum_{\nu=0}^{[(n-k)/2]} (-1)^\nu (4\nu+2k+1)\binom{k+\nu-1/2}{\nu} P_{k+2\nu}(t)\,,$$

where $P_m(t)$ denotes the Legendre polynomial of degree m.

Under restriction $P(1)=0$, Q. M. Tariq [40] proved that

$$|a_n| \le \frac{n}{n+1}\cdot\frac{(2n)!}{2^n(n!)^2}\left(\frac{2n+1}{2}\right)^{1/2}\|P\|_2\,,$$

with equality case

$$P(t) = P_n(t) - \frac{1}{n^2}\sum_{\nu=0}^{n-1}(2\nu+1)P_\nu(t)\,.$$

Also, for $k=n-1$, he obtained

$$|a_{n-1}| \le \frac{(n^2+2)^{1/2}}{n+1}\cdot\frac{(2n-2)!}{2^{n-1}((n-1)!)^2}\left(\frac{2n-1}{2}\right)^{1/2}\|P\|_2\,, \tag{2.3}$$

with equality case

$$P(t) = \frac{2n+1}{n^2+2}P_n(t) - P_{n-1}(t) + \frac{1}{n^2+2}\sum_{\nu=0}^{n-2}(2\nu+1)P_\nu(t)\,.$$

In the absence of the hypothesis $P(1)=0$, the factor $(n^2+2)^{1/2}/(n+1)$ appearing on the right-hand side of (2.3) is to be dropped.

Here, we will consider more general problem including L^2-norm of polynomials with respect to a non-negative measure on the real line $\mathbb{R}$ and using some restricted polynomial classes.

Let $d\lambda(t)$ be a given non-negative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k\,d\lambda(t)$, $k = 0,1,\ldots$, exist and are finite, and $\mu_0 > 0$. In that case, there exists a unique set of orthonormal polynomials $\pi(\cdot) = \pi(\cdot\,; d\lambda)$, $k=0,1,\ldots$, defined by

$$\begin{aligned}&\pi_k(t) = b_k t^k + c_k t^{k-1} + \text{lower degree terms}\,, \quad b_k > 0\,,\\ &(\pi_k,\pi_m) = \delta_{km}\,, \quad k,m \ge 0\,,\end{aligned} \tag{2.4}$$

where

$$(f,g) = \int_{\mathbb{R}} f(t)\overline{g(t)}\,d\lambda(t) \quad (f,g \in L^2(\mathbb{R}))\,.$$

For $P \in \mathcal{P}_n$, we define

$$\|P\| = \sqrt{(P,P)} = \left(\int_{\mathbb{R}} |P(t)|^2\,d\lambda(t)\right)^{1/2}. \tag{2.5}$$

The polynomial $P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu \in \mathcal{P}_n$ can be represented in the form

$$P(t) = \sum_{\nu=0}^{n} \alpha_\nu \pi_\nu(t)\,,$$

where

$$\alpha_\nu = (P,\pi_\nu)\,, \quad \nu = 0,1,\ldots,n\,.$$

We note that

$$a_n = \alpha_n b_n\,, \quad a_{n-1} = \alpha_n c_n + \alpha_{n-1}b_{n-1}\,.$$

Since

$$\|P\| = \left(\sum_{\nu=0}^{n}|\alpha_\nu|^2\right)^{1/2} \ge |\alpha_n|\,,$$

we have a simple estimate $|a_n| \le b_n\|P\|$. This inequality can be improved for some restricted classes of polynomials. Because of that, we consider a linear bounded functional $L\colon \mathcal{P}_n \to \mathbb{R}$, such that

$$M = \sum_{\nu=0}^{n} |L\pi_\nu|^2 > 0\,, \tag{2.6}$$

and a subset of $\mathcal{P}_n$ defined by

$$W_n = \{P \in \mathcal{P}_n \mid LP = 0\,,\ \mathrm{dg}\, P = n\}\,.$$

Using a method given by A. Giroux and Q. I. Rahman [11] (see also Q. M. Tariq [40]), we can prove the following result:

Theorem 2.1. If $P \in W_n$ and $\gamma_0, \gamma_1, \ldots, \gamma_n$ are non-negative numbers such that $\gamma_\mu > \gamma_\nu$ for $\nu = 0, 1, \ldots, \mu-1, \mu+1, \ldots, n$, then

$$\sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 \le (\gamma_\mu - \gamma) \sum_{\nu=0}^{n} |\alpha_\nu|^2\,, \tag{2.7}$$

where γ is the unique root of the equation

$$\sum_{\nu=0}^{n} \frac{|L\pi_\nu|^2}{\gamma_\mu - \gamma_\nu - \gamma} = 0 \tag{2.8}$$

in the interval $(0, \Gamma)$, where

$$\Gamma = \min_{\substack{0 \le \nu \le n \\ \nu \ne \mu}} (\gamma_\mu - \gamma_\nu)\,.$$

Proof. Since

$$\begin{aligned}
\sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 &= \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} (\gamma_\mu - \gamma_\nu)|\alpha_\nu|^2 \\
&= \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} (\gamma_\mu - \gamma_\nu - \gamma)|\alpha_\nu|^2 - \gamma \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} |\alpha_\nu|^2\,,
\end{aligned}$$

starting from $\sum_{\nu=0}^{n} \alpha_\nu L\pi_\nu = 0$, we have

$$\begin{aligned}
|\alpha_\mu L\pi_\mu|^2 &= \left| \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} \alpha_\nu L\pi_\nu \right|^2 \\
&\le \left(\sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} |\alpha_\nu||L\pi_\nu|(\gamma_\mu - \gamma_\nu - \gamma)^{1/2}(\gamma_\mu - \gamma_\nu - \gamma)^{-1/2} \right)^2 \\
&\le \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} |\alpha_\nu|^2(\gamma_\mu - \gamma_\nu - \gamma) \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} \frac{|L\pi_\nu|^2}{\gamma_\mu - \gamma_\nu - \gamma}\,.
\end{aligned}$$

Since γ is the unique root of Equation (2.8), we find that

$$-\sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} |\alpha_\nu|^2(\gamma_\mu - \gamma_\nu - \gamma) \le \frac{-|\alpha_\mu|^2 |L\pi_\mu|^2}{\displaystyle\sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} \frac{|L\pi_\nu|^2}{\gamma_\mu - \gamma_\nu - \gamma}} = -\gamma |\alpha_\mu|^2\,,$$

wherefrom

$$\sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 \le \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \gamma|\alpha_\mu|^2 - \gamma \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} |\alpha_\nu|^2\,,$$

i.e. (2.7). □

Using this result, G. V. Milovanović and L. Z. Marinković [26] proved the following theorem:

Theorem 2.2. If $P \in W_n$ then

$$|a_n| \le b_n \sqrt{1 - \frac{1}{M} |L\pi_n|^2} \|P\|\,, \tag{2.9}$$

where M is given by (2.3). Inequality (2.9) is sharp and becomes an equality if and only if $P(t)$ is a constant multiple of the polynomial

$$\pi_n(t) - \frac{L\pi_n}{M - |L\pi_n|^2} \sum_{\nu=0}^{n-1} \overline{(L\pi_\nu)}\pi_\nu(t)\,.$$

Also, they considered several special cases with respect to the measure $d\lambda(t)$ and the functional L.

3. An Extremal Problem for Polynomials with Prescribed Zeros on a Circle

There are various ways in which we can introduce norm $\|\cdot\|$ in the linear space of all algebraic polynomials of degree at most n. Given β, let W_n^β denote the subspace consisting of those polynomials which vanish at β. J. D. Donaldson and Q. I. Rahman [8] stated the following question:

How large can $\|P(z)/(z-\beta)\|$ be if P belongs to W_n^β and $\|P\|=1$? In the mentioned paper, they solved this problem when

$$\|P\| = \max_{|z|\le 1} |P(z)| \quad \text{and} \quad \|P\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2\, d\theta\right)^{1/2}.$$

Thus, J. D. Donaldson and Q. I. Rahman [8] proved the following inequality:

$$\max_{|z|=1} |P(z)/(z-\beta)| \le (n+1)/2\,,$$

if $P \in W_n^\beta$ and $\max_{|z|\le 1}|P(z)| \le 1$.

In the case when $\beta = 1$, Q. I. Rahman and Q. G. Mohammad [32] found that

$$\max_{|z|\le 1} |P(z)/(z-1)| \le n/2\,.$$

Also, Donaldson and Rahman [8] proved the following inequality:

$$\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^2 d\theta \ge \left(1+\beta^2-2\beta\cos\frac{\pi}{n+1}\right)\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta\,, \tag{3.1}$$

for all polynomials of the form $P(z) = (z-\beta)(x_1 + x_2 z + \cdots + x_n z^{n-1})$, $\beta \ge 0$, where $x_1, x_2, \ldots, x_n$ are given real numbers.

The opposite inequality of (3.1), i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^2 d\theta \le \left(1+\beta^2+2\beta\cos\frac{\pi}{n+1}\right)\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta\,,$$

was proved by I. Ž. Milovanović [31].

A. Aziz [1] estimated

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta$$

in terms of the maximum of $|P(z_k)|$, where z_k, $k=1,\ldots,n$ are the zeros of z^n+1. Namely, he proved the following results:

Theorem 3.1. If $P \in \mathcal{P}_n$ and $P(\beta)=0$, where β is an arbitrary non-negative real number then

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \le \frac{1+\beta^2+\cdots+\beta^{2(n-1)}}{(1+\beta^n)^2}\max_{1\le k\le n}|P(z_k)|^2\,, \tag{3.2}$$

where $z_1,\ldots,z_n$ are the zeros of z^n+1. The result is best possible and equality in (3.2) holds for $P(z) = z^n - \beta^n$.

If we take min instead of max on the right side in (3.2), then an opposite inequality holds.

Let $W_{n,m}^\beta$ $(1 \le m \le n,\ \beta \ge 0)$ be the set of all algebraic polynomials of degree $\le n$, with all real coefficients and m prescribed zeros in the points $\sqrt[m]{\beta}\exp(i2k\pi/m)$, $k=0,1,\ldots,m-1$.

In this section, we will determine

$$A_{n,m}(\beta) = \min\frac{\|P\|_C^2}{\|P\|_L^2} \quad \text{and} \quad B_{n,m}(\beta) = \max\frac{\|P\|_C^2}{\|P\|_L^2}\,, \tag{3.3}$$

when $P \in W_{n,m}^\beta$, where

$$\|P\|_C^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^2 d\theta \quad \text{and} \quad \|P\|_L^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{im\theta}-\beta}\right|^2 d\theta\,.$$

Namely, we will prove the following result:

Theorem 3.2. The best constants $A_{n,m}(\beta)$ and $B_{n,m}(\beta)$ defined in (3.3) are

$$A_{n,m}(\beta) = 1-2\beta\cos\frac{\pi}{r+1}+\beta^2 \quad \text{and} \quad B_{n,m}(\beta) = 1+2\beta\cos\frac{\pi}{r+1}+\beta^2\,,$$

where $r = [n/m]$.

First, we will prove the following auxiliary result:

Lemma 3.3. For a given sequence of real numbers $x_1, x_2, \ldots, x_N$, where $N = mr - q$, with $r = [(N-1)/m]+1$ and $0 \le q \le m-1$, the following inequalities

$$-B_r\sum_{i=1}^{N} x_i^2 \le \sum_{i=1}^{N-m} x_i x_{i+m} \le B_r \sum_{i=1}^{N} x_i^2 \tag{3.4}$$

hold, where $B_r = \cos\frac{\pi}{r+1}$ is the best constant.

The equality in the right (left) inequality (3.4) exists if and only if

$$x_{km+i} = C_i \sin\frac{(k+1)\pi}{r+1} \quad \left(x_{km+i} = (-1)^k C_i \sin\frac{(k+1)\pi}{r+1}\right)\,, \tag{3.5}$$

where $C_1, \ldots, C_{m-q}$ are arbitrary constants and $C_{m-q+1} = \ldots = C_m = 0$. The index i takes the values $i = 1, \ldots, m$, when $k = 0, 1, \ldots, r-2$, and $i = 1, \ldots, m-q$, when $k = r-1$.

Proof. Let X be an N-dimensional Euclidean space with inner product $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N} x_i y_i$, where $\mathbf{x} = [x_1, \ldots, x_N]^T$ and $\mathbf{y} = [y_1, \ldots, y_N]^T$. We define a symmetric matrix of the order N, $H_N = [h_{ij}]_N$, where

$$\begin{aligned} &h_{i,i} = 1, \quad i = 1, \ldots, N\,, \\ &h_{i,i+m} = h_{i+m,i} = -1/2, \quad i = 1, \ldots, N-m\,, \\ &h_{i,j} = 0, \quad \text{otherwise}\,, \end{aligned}$$

and consider the corresponding quadratic form

$$(H_N \mathbf{x}, \mathbf{x}) = \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N-m} x_i x_{i+m}\,.$$

At first, we note that

$$\lambda_1(H_N)(\mathbf{x}, \mathbf{x}) \le (H_N \mathbf{x}, \mathbf{x}) \le \lambda_N(H_N)(\mathbf{x}, \mathbf{x})\,, \tag{3.6}$$

where $\lambda_1(H_N)$ and $\lambda_N(H_N)$ are minimal and maximal eigenvalues of the matrix H_N, respectively.

In order to determine these eigenvalues, we define a tridiagonal symmetric matrix A_r of the order r, with unit diagonal and $-1/2$ as subdiagonal elements, and a sequence $\{Q_k(t)\}$ of polynomials using the recurrence relation

$$xQ_{k-1}(t) = (-1/2)Q_k(t) + Q_{k-1}(t) + (-1/2)Q_{k-2}(t), \quad k = 1, 2, \ldots\,, \tag{3.7}$$

with $Q_0(t) = Q_0 \ne 0$ and $Q_{-1}(t) = 0$. If we take $Q_0 = 1$, we can find $Q_k(t) = \sin(k+1)\theta / \sin\theta$ $(0 < x < 2)$, where $e^{i\theta} = 1 - x + i\sqrt{2x - x^2}$. The eigenvalues of the matrix A_r are the zeros of the polynomial $Q_r(t)$, i.e.

$$\lambda_k(A_r) = 2\sin^2 \frac{k\pi}{2(r+1)}, \quad k = 1, \ldots, r,$$

because (see G. V. Milovanović and I. Ž. Milovanović [27, 28])

$$\lambda \mathbf{v} = A_r \mathbf{v} + \frac{1}{2} Q_r(\lambda)\mathbf{e},$$

where $\mathbf{v} = \mathbf{v}(\lambda) = [Q_0(\lambda), \ldots, Q_{r-1}(\lambda)]^T$ and $\mathbf{e}$ is the last coordinate vector. The corresponding eigenvectors are $\mathbf{v}(\lambda_k(A_r))$, $k = 1, \ldots, r$. Note that

$$2\sin^2 \frac{\pi}{2(r+1)} = \lambda_1(A_r) < \lambda_2(A_r) < \cdots < \lambda_r(A_r) = 2\cos^2 \frac{\pi}{2(r+1)}\,.$$

Now, we define m sequences $\{Q_k^{(i)}(t)\}$, $i = 1, \ldots, m$, using the same recurrence relation (3.7), where only the constants $Q_0^{(i)}$ may differ, and a vector in X by

$$\mathbf{w} = \mathbf{w}(\lambda) = [Q_0^{(1)}(\lambda), \ldots, Q_0^{(m)}(\lambda), \ldots, Q_r^{(1)}(\lambda), \ldots, Q_r^{(m)}(\lambda)]^T\,.$$

Then we have

$$\lambda \mathbf{w} = H_N \mathbf{w} + \frac{1}{2} \mathbf{z}_q\,,$$

where

$$\mathbf{z}_q = [0, \ldots, 0, Q_{r-1}^{(m-q+1)}(\lambda), \ldots, Q_{r-1}^{(m)}(\lambda), Q_r^{(1)}(\lambda), \ldots, Q_r^{(m-q)}(\lambda)]^T\,.$$

The first $N - m$ coordinates in $\mathbf{z}_q$ are equal to zero.

From this, we can conclude that the matrix H_N has the following eigenvalues

$$\lambda_k(A_r) \text{ of multiple } m - q \quad (k = 1, \ldots, r)$$

and

$$\lambda_k(A_{r-1}) \text{ of multiple } q \quad (k = 1, \ldots, r-1)\,.$$

For $q = 0$ (i.e. $N = mr$) the eigenvalues are only $A_k(A_r)$, $k = 1, \ldots, r$, of multiple m.

Since $\lambda_1(A_r) < \lambda_1(A_{r-1})$ and $\lambda_r(A_r) > \lambda_{r-1}(A_{r-1})$, we have $\lambda_1(H_N) = \lambda_1(A_r)$ and $\lambda_N(H_N) = \lambda_r(A_r)$. Then Inequalities (3.6) reduce to (3.4), where we have the equality for eigenvectors corresponding to eigenvalues $\lambda_1(A_r)$ and $\lambda_r(A_r)$, i.e. for $\mathbf{x} = \mathbf{w}(\lambda_1(A_r))$ and $\mathbf{x} = \mathbf{w}(\lambda_r(A_r))$, with $Q_0^{(i)} = C_i$ $(i = 1, \ldots, m)$, where C_i $(i = 1, \ldots, m-q)$ are arbitrary constants and $C_{m-q+1} = \ldots = C_m = 0$. The last statement is equivalent to (3.5). □

Remark 3.1. Inequalities (3.4) are in connection with extremal properties of non-negative trigonometric polynomials considered by G. Szegö [39], and

E. Egerváry and O. Szász [10]. In the mentioned papers, we can find another proof of the best constant B_r in (3.4).

Remark 3.2. The inequalities (3.4) can be represented in the form

$$\left|\sum_{i=1}^{N-m} x_i x_{i+m}\right| \le \cos\frac{\pi}{r+1}\sum_{i=1}^{N} x_i^2 .$$

For $m = 1$, this inequality reduces to an inequality proved in the monograph of V. M. Tihomirov [41, pp. 113–115] (see, also G. V. Milovanović and I. Ž. Milovanović [28], and A. G. Babenko [3]).

Now, we will give the proof of Theorem 3.2.

Proof of Theorem 3.2. Let $P \in W_{n,m}^{\beta}$, i.e.

$$P(z) = (z^m - \beta)(x_1 + x_2 z + \cdots + x_{n-m+1} z^{n-m}) .$$

Then we have

$$\|P\|_C^2 = (1+\beta^2)\sum_{i=1}^{N} x_i^2 - 2\beta \sum_{i=1}^{N-m} x_i x_{i+m}$$

and

$$\|P\|_L^2 = \sum_{i=1}^{N} x_i^2 ,$$

where $N = n - m + 1$. Using Lemma 3.3, we obtain the best constants $A_{n,m}(\beta)$ and $B_{n,m}(\beta)$, where $r = [(n-m)/m] + 1 = [n/m]$.

The corresponding extremal polynomial in the "minimum problem" ("maximum problem") in (3.3) is

$$P(z) = (z^m - \beta)\sum_{i=1}^{N} x_i z^{i-1} ,$$

where x_i is given by (3.5). □

Remark 3.3. For $m = 1$, Theorem 3.2 reduces to the results given by J. D. Donaldson and Q. I. Rahman [8] and I. Ž. Milovanović [31].

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