Bulletin T.CLVII de l'Académie serbe des sciences et des arts − 2024
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 49

BERNSTEIN TYPE INEQUALITIES FOR ONE CLASS OF COMPLEX RATIONAL FUNCTIONS

GRADIMIR V. MILOVANOVIĆ, ABDULLAH MIR,
IDREES QASIM, SEE KEONG LEE

Dedicated to the 100th anniversary of the birth of Academician Bogoljub Stanković

(Accepted at the 9th Meeting, held on December 20, 2024)

A b s t r a c t. In this study, we establish some Bernstein-type norm estimates for a rational function and its derivative on the unit circle. The estimates involve the positioning of all the zeros of the rational function and the poles are assumed to lie outside the unit circle. Our results strengthen some existing bounds and additionally derive a polynomial inequality pertinent to the Saff conjecture. Numerical examples are included.

AMS Mathematics Subject Classification (2020): 30A10, 30C15, 30D15

Key Words: Bernstein type inequalities, rational function, algebraic polynomial, zeros and poles, finite Blaschke product, polar derivative.

1. *Introduction and preliminaries*

Inequalities and extremal problems of Markov and Bernstein type play fundamental role in proofs of many inverse theorems in approximation theory, as well as in many other mathematical areas. The first result of this type was given at the end of the 19th century by the very famous Russian academician Andrei Andreevich Markov (1856–1922), while the second important result belongs to Sergei Natanovich Bernstein (1880–1968), also a member of the Russian mathematical school, who was otherwise a student of the French Sorbonne. Over a period of more than a century, a huge number of papers have been published, so that the theory of extremal problems and inequalities of this type has developed to unimaginable proportions, including many applications.

Let $\mathcal{P}_n$ denotes the space of all algebraic polynomials of degree at most n. Then the classical basic inequalities of these types, in the uniform norm on $[-1,1]$, are

$$\|P'\|_\infty \le n^2\|P\|_\infty \quad \text{and} \quad \|\sqrt{A}P'\|_\infty \le n\|P\|_\infty \quad (A = 1-x^2), \tag{1.1}$$

respectively. The inequalities are the best possible, i.e., their constants n^2 and n are exact! There are many generalizations, extensions and refinements of (1.1) in different metrices and for various classes of functions associated with polynomials, like rational functions, functions of exponential type, etc., as well as the corresponding inequalities in the complex plane (for details see [3, 8]). Interesting cases are ones in the weighted L^r norms, in particular in L^2, due to the existence of orthogonality theory (see [7] and [4]).

Analogues of Markov's inequality for polynomials in a complex variable, with the norm on the unit disk, have also found many applications. For a complex polynomial $p \in \mathcal{P}_n$, one of the basic questions is *how large can* $|p'(z)|$ *be when* z *belongs to the unit circle* $\mathbb{T} = \{z : |z| = 1\}$?

One answer was given by M. Riesz [13], using his result for trigonometric polynomials [13] (cf. [3, p. 24, Theorem 1.3.1]), which is also related to Bernstein's inequality for algebraic polynomials (the second inequality in (1.1)). A detailed account of the various relations and priorities can be found in the recent monograph [3, §1.3]. The answer to the previous problem is well-known as *Bernstein's inequality* [1]:

$$\max_{z\in\mathbb{T}} |p'(z)| \le n \max_{z\in\mathbb{T}} |p(z)|. \tag{1.2}$$

Equality in (1.2) *holds for* $p(z) = \lambda z^n,\ \lambda \ne 0.$

Since equality in (1.2) holds if and only if p has all zeros at the origin, it is natural to expect a relationship between the bound n and the distance of the zeros of a polynomial from the origin. This fact was observed by Erdős who conjectured it and later was proved by Lax [5]: *If* $p \in \mathcal{P}_n$ *and* $p(z) \ne 0$ *in* $\mathbb{T} \cup \mathbb{D}_-$, *where* $\mathbb{D}_-$ *denotes the region outside the unit circle* $\mathbb{T}$, *then*

$$\max_{z\in\mathbb{T}} |p'(z)| \le \frac{n}{2} \max_{z\in\mathbb{T}} |p(z)|. \tag{1.3}$$

On the other hand, Turán [15] considered *a polynomial* p *having all zeros in* $\mathbb{T} \cup \mathbb{D}_-$, *where* $\mathbb{D}_- := \{z : |z| < 1\}$, and proved that

$$\max_{z\in\mathbb{T}} |p'(z)| \ge \frac{n}{2} \max_{z\in\mathbb{T}} |p(z)|. \tag{1.4}$$

In this paper we consider Bernstein type inequalities for one class of rational functions. For a complex polynomial $p \in \mathcal{P}_n$ $(n \geq 1)$, given by

$$p(z) := \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n} (z - z_j),$$

and $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, let (cf. [3, p. 59])

$$\left.\begin{aligned} &w(z) = \prod_{j=1}^{n}(z - a_j), \quad B(z) := \frac{w^*(z)}{w(z)} = \prod_{j=1}^{n} \frac{1 - \overline{a}_j z}{z - a_j}, \\ &\mathcal{R}_n := \mathcal{R}_n(a_1, \ldots, a_n) := \left\{ \frac{p(z)}{w(z)} : \ p \in \mathcal{P}_n \right\}, \end{aligned}\right\} \tag{1.5}$$

where $w^*(z) = z^n \overline{w\left(1/\overline{z}\right)}$ is the conjugate transpose (reciprocal) of $w(z)$. Then $\mathcal{R}_n$ is the set of rational functions with poles $a_1, a_2, \ldots, a_n$, at most, and a finite limit at infinity. Note that $B \in \mathcal{R}_n$ and it is known as the *finite Blaschke product*.

In our discussion, we shall assume that the poles $a_1, a_2, \ldots, a_n$ are in $\mathbb{D}_+$. For the case when all the poles are in $\mathbb{D}_-$, we can obtain analogous results with suitable modification of our methods.

For $r(z) := p(z)/w(z) \in \mathcal{R}_n$, the conjugate transpose r^* of r is defined by

$$r^*(z) = B(z)\overline{r\left(1/\overline{z}\right)}.$$

Note that if $r(z) = p(z)/w(z) \in \mathcal{R}_n$, then $r^*(z) = p^*(z)/w(z)$ and hence $r^* \in \mathcal{R}_n$.

Li, Mohapatra, and Rodriguez [6] replaced z^n by the Blaschke product $B(z)$ and extended inequalities (1.3) and (1.4) to rational functions with prescribed poles. Besides, they also proved the following results:

Theorem 1.1. *If $r \in \mathcal{R}_n$ such that $r(z) \neq 0$ for $z \in \mathbb{D}_-$, then for $z \in \mathbb{T}$,*

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{z \in \mathbb{T}} |r(z)|. \tag{1.6}$$

Equality in (1.6) *holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.*

Theorem 1.2. *If $r \in \mathcal{R}_n$ such that $r(z) \neq 0$ for $z \in \mathbb{D}_+$ and r has exactly n poles at $a_1, a_2, \ldots, a_n$, then for $z \in \mathbb{T}$,*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - (n - m)\right] |r(z)|, \tag{1.7}$$

where m is the number of zeros of r.

Since then, many results have been published on Bernstein-type inequalities for rational functions. Some of the recent papers include [9–12] and the references therein.

In [2], Giroux, Rahman and Schemeisser mentioned the following problem which they have attributed to E. B. Saff and is known as **Saff Conjecture**: *Let*

$$p(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n *having all its zeros in* $\mathrm{Re}(z) \geq 1$. *Is it true that*

$$\max_{z \in \mathbb{T}} |p'(z)| \leq \sum_{j=1}^{n} \frac{1}{1 + \mathrm{Re}(z_j)} \max_{z \in \mathbb{T}} |p(z)|?$$

For $n = 1$ and $n = 2$ the answer in the above problem is affirmative.

In the same paper, Giroux, Rahman and Schemeisser assumed polynomials which are real for real z and proved the following:

Theorem 1.3. *If the polynomial* $p(z) = \prod_{j=1}^{n}(z - z_j)$ *is real for real* z*, then*

$$\max_{z \in \mathbb{T}} |p'(z)| \leq \sum_{j=1}^{n} \frac{1}{1 + |z_j|} \max_{z \in \mathbb{T}} |p(z)| \tag{1.8}$$

provided all the zeros lie in

$$D = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0,\ |z| \geq 1\}.$$

The example $p(z) = (z + 1)(z - 3)$ shows that inequality (1.8) may not hold if the zeros are not required to lie in D.

In this paper, we establish some Bernstein-type norm estimates for a rational function and its derivative on the unit circle. These estimates take into account the locations of all the zeros of the rational function, with the poles assumed to lie outside the unit circle. The results refine existing bounds and yield a polynomial inequality connected to the Saff conjecture. Numerical examples are also included.

2. *Main results*

In this section, we present the main results. Our first result is central to this paper, and the bound in this theorem can be compared with that of Theorem 1.1.

Theorem 2.1. *If* $p(z) = \prod_{j=1}^{n}(z - z_j) \in \mathcal{P}_n$ *and* $r(z) = p(z)/w(z) \in \mathcal{R}_n$ *such that* $r(z) \neq 0$ *in* $\mathbb{D}_-$*, then for* $z \in \mathbb{T}$*,*

$$|r'(z)| \leq |B'(z)| \max_{z \in \mathbb{T}} |r(z)| - \frac{1}{2} \left[|B'(z)| + \sum_{j=1}^{n} \frac{|z_j| - 1}{|z_j| + 1} \right] |r(z)|.$$

Remark 2.1. It is worth noting that Theorem 2.1 provides a significant improvement over Theorem 1.1 when

$$M := \max_{z\in\mathbb{T}} \left|r(z)\right| < U(z) := \left[1 + \frac{1}{\left|B'(z)\right|} \sum_{j=1}^{n} \frac{|z_j|-1}{|z_j|+1}\right] |r(z)| \quad (z \in \mathbb{T}). \tag{2.1}$$

On the other hand, if this inequality is reversed, Theorem 2.1 does not yield any improvement over the bound given in Theorem 1.1. Thus, in this case we have

$$|r'(z)| \le \min\left\{\frac{1}{2}|B'(z)|M,\ |B'(z)|\left(M - \frac{1}{2}U(z)\right)\right\} \quad (z \in \mathbb{T}). \tag{2.2}$$

It can be also written in the form

$$|r'(z)| \le \frac{1}{4}|B'(z)|\left(3M - U(z) - |U(z) - M|\right) \quad (z \in \mathbb{T}).$$

If we assume one additional condition in Theorem 2.1, we obtain the following result, which provides a significantly better bound.

Theorem 2.2. *If $p(z) = \prod_{j=1}^{n}(z - z_j) \in \mathcal{P}_n$ and $r(z) = p(z)/w(z) \in \mathcal{R}_n$ such that $r(z) \neq 0$ in D_-, then for $z \in \mathbb{T}$,*

$$\max_{z\in\mathbb{T}} \left|r'(z)\right| \le \frac{1}{2}\left[\left|B'(z)\right| - \sum_{j=1}^{n} \frac{|z_j|-1}{|z_j|+1}\right] \max_{z\in\mathbb{T}} |r(z)|, \tag{2.3}$$

provided that maximum of $|r(z)|$ and $\left|r'(z)\right|$ and the minimum of $|B'(z)|$ occur at the same point on the unit circle $\mathbb{T}$.

Since $|z_j| \ge 1$, one can easily see that Theorem 2.2 gives an improvement of Theorem 1.1 under the additional hypothesis that maximum of $|r(z)|$ and $\left|r'(z)\right|$ and the minimum of $|B'(z)|$ occur at the same point on the unit circle $\mathbb{T}$.

Next, we shall prove the following result:

Theorem 2.3. *If $r(z) = p(z)/w(z) \in \mathcal{R}_n$ having all m zeros in $\mathbb{T} \cup \mathbb{D}_-$ and exactly n poles at $a_1, a_2, \ldots, a_n$, then for $z \in T$,*

$$|r'(z)| \ge \frac{1}{2}\left[|B'(z)| - (n-m) + \sum_{j=1}^{m} \frac{1-|z_j|}{1+|z_j|}\right] |r(z)|. \tag{2.4}$$

Since $|z_j| \le 1$, Theorem 2.3 strengthens Theorem 1.2.

Let $a_j = \alpha > 1$ for $j = 1, 2, \ldots, n$. Then $w(z) = (z - \alpha)^n$ and $r(z) = p(z)/(z-\alpha)^n$, so that $B(z) = \left[(1-\alpha z)/(z-\alpha)\right]^n$ converges to z^n as α approaches to ∞. Also, $B'(z)$ converges to nz^{n-1} as α approaches ∞.

Using these observations in Theorem 2.3 and letting $|\alpha| \to \infty$, we get that if $p(z) = \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all zeros in $\mathbb{T} \cup \mathbb{D}_-$, then

$$\max_{z\in\mathbb{T}} |p'(z)| \geq \sum_{j=1}^{n} \frac{1}{1+|z_j|} \max_{z\in\mathbb{T}} |p(z)|. \tag{2.5}$$

Inequality (2.5) is again in connection with the Saff Conjecture and was independently proven by Giroux, Rahman and Schmeisser [2].

Similar observations can be used in Theorems 2.1 and 2.2 to obtain corresponding inequalities for polynomials.

For a complex number α and for $p \in \mathcal{P}_n$, let

$$D_\alpha p(z) := np(z) + (\alpha - z)p'(z).$$

The function $D_\alpha p(z)$ is known as the *polar derivative* of p with respect to α. It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z) \text{ uniformly for all } z \text{ with } |z| \leq R,\ R > 0.$$

Suppose that $a_j = \alpha$ for $j = 1, 2, \ldots, n$ with $|\alpha| > 1$ and p is a polynomial of degree n. Then it can be easily shown that

$$r'(z) = \frac{-D_\alpha p(z)}{(z-\alpha)^{n+1}} \qquad \text{and} \qquad B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

By using these facts in Theorem 2.3, the following result immediately follows:

Corollary 2.1. *If $p \in \mathcal{P}_n$ having all zeros in $\mathbb{T} \cup \mathbb{D}_-$, then for $|\alpha| > 1$, we have*

$$\max_{z\in\mathbb{T}} |D_\alpha p(z)| \geq \frac{|\alpha|-1}{2} \left[n + \sum_{j=1}^{n} \frac{1-|z_j|}{1+|z_j|} \right] \max_{z\in\mathbb{T}} |p(z)|, \tag{2.6}$$

which is equivalent to

$$\max_{z\in\mathbb{T}} |D_\alpha p(z)| \geq (|\alpha|-1) \sum_{j=1}^{n} \frac{1}{1+|z_j|} \max_{z\in\mathbb{T}} |p(z)|.$$

Since $|z_j| \leq 1$, Corollary 2.1 is an improvement of a result due to Shah [14].

Similar techniques can be applied to Theorems 2.1 and 2.2 to derive results for the polar derivative of a polynomial.

3. *Lemmas*

For the proofs of Theorems 2.2 and 2.3, we need the following lemmas. The first two are due to Li, Mahapatra and Rodrigues [6]. Let B and $\mathcal{R}_n$ be defined as in Equation (1.5).

Lemma 3.1. *For $a_j \in \mathbb{C}$ with $|a_j| > 1$,*

$$\frac{zB'(z)}{B(z)} = |B'(z)|, \qquad z \in \mathbb{T}.$$

Lemma 3.2. *If $r \in \mathcal{R}_n$ and $z \in \mathbb{T}$, then*

$$\left|r^{*\prime}(z)\right| + \left|r'(z)\right| \leq \left|B'(z)\right| \max_{z\in\mathbb{T}} |r(z)|.$$

Equality holds for $r(z) = uB(z)$ with $u \in \mathbb{T}$.

Lemma 3.3. *If $|w| \leq 1$ and $w \neq \mathrm{e}^{\mathrm{i}\theta}$, then*

$$\mathrm{Re}\left[\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta} - w}\right] \geq \frac{1}{1 + |w|}.$$

PROOF. Let $w = r\mathrm{e}^{\mathrm{i}\alpha}$, where $0 < r \leq 1$. Then

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta} - w}\right] &= \mathrm{Re}\left[\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta} - r\mathrm{e}^{\mathrm{i}\alpha}}\right] \\ &= \mathrm{Re}\left[\frac{1}{1 - re^{i(\alpha-\theta)}}\right] \\ &= \frac{1 - r\cos(\alpha - \theta)}{1 + r^2 - 2r\cos(\alpha - \theta)} \\ &\geq \frac{1}{1 + r}.\end{aligned}$$

The last inequality is true because

$$\begin{aligned}&\big(1 - r\cos(\alpha - \theta)\big)(1 + r) \geq 1 + r^2 - 2r\cos(\alpha - \theta) \\ &\qquad \Leftrightarrow 1 - r\cos(\alpha - \theta) + r - r^2\cos(\alpha - \theta) \geq 1 + r^2 - 2r\cos(\alpha - \theta) \\ &\qquad \Leftrightarrow -(r^2 - r)\cos(\alpha - \theta) \geq r^2 - r \\ &\qquad \Leftrightarrow \cos(\alpha - \theta) \geq -1.\end{aligned}$$

This proves the lemma. □

4. *Proof of Theorems*

Now we are ready to give the proofs of our results.

PROOF OF THEOREM 2.1. Since

$$r^*(z) = \frac{p^*(z)}{w(z)} = \prod_{j=1}^{n} \frac{1-\overline{z_j}z}{w(z)},$$

a logarithmic differentiation gives

$$\frac{zr^{*\prime}(z)}{r^*(z)} = \sum_{j=1}^{n} \frac{z}{z-w_j} - \frac{zw'(z)}{w(z)},$$

where $w_j = 1/\overline{z_j}$, $j = 1, 2, \ldots, n$, and $|w_j| = 1/|z_j| \le 1$. Hence, for points $\mathrm{e}^{\mathrm{i}\theta}$, $0 \le \theta < 2\pi$, other than the zeros of $r^*(z)$, we have

$$\mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}\theta} r^{*\prime}(\mathrm{e}^{\mathrm{i}\theta})}{r^*(\mathrm{e}^{\mathrm{i}\theta})}\right) = \mathrm{Re}\left(\sum_{j=1}^{n} \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta} - w_j}\right) - \mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}\theta} w'(\mathrm{e}^{\mathrm{i}\theta})}{w(\mathrm{e}^{\mathrm{i}\,\theta})}\right). \tag{4.1}$$

Since

$$B(z) = \frac{w^*(z)}{w(z)},$$

the logarithmic differentiation gives

$$\frac{zB'(z)}{B(z)} = \frac{zw^{*\prime}(z)}{w^*(z)} - \frac{zw'(z)}{w(z)}.$$

By using Lemma 3.1, we have

$$\mathrm{Re}\, \frac{zw^{*\prime}(z)}{w^*(z)} - \mathrm{Re}\, \frac{zw'(z)}{w(z)} = |B'(z)|. \tag{4.2}$$

On the other hand, since $1/\overline{z} = z$ for $z \in \mathbb{T}$ and

$$w^{*\prime}(z) = nz^{n-1}\overline{w\left(1/\overline{z}\right)} - z^{n-2}\overline{w'\left(1/\overline{z}\right)},$$

it can be easily shown that for $z \in \mathbb{T}$,

$$\mathrm{Re}\, \frac{zw^{*\prime}(z)}{w^*(z)} = n - \mathrm{Re}\, \frac{zw'(z)}{w(z)}. \tag{4.3}$$

Equations (4.2) and (4.3) thus give, for $z \in \mathbb{T}$,

$$\mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}. \tag{4.4}$$

Therefore, from Eq. (4.1), (4.3) and Lemma 3.3, we have

$$\left|\frac{e^{i\theta}r^{*\prime}(e^{i\theta})}{r^*(e^{i\theta})}\right| \geq Re\left(\frac{e^{i\theta}r^{*\prime}(e^{i\theta})}{r^*(e^{i\theta})}\right) \geq \sum_{j=1}^{n}\frac{1}{1+|w_j|} - \frac{n-|B'(e^{i\theta})|}{2}.$$

Equivalently,

$$|r^{*\prime}(e^{i\theta})| \geq \left[\sum_{j=1}^{n}\frac{1}{1+|w_j|} - \frac{n-|B'(e^{i\theta})|}{2}\right]|r^*(e^{i\theta})|, \tag{4.5}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$ other than the zeros of $r^*(z)$. Since (4.5) is trivially true for points $e^{i\theta}$, $0 \leq \theta < 2\pi$ that are the zeros of $r^*(z)$, it follows that

$$|r^{*\prime}(z)| \geq \left[\sum_{j=1}^{n}\frac{1}{1+|w_j|} - \frac{n-|B'(z)|}{2}\right]|r^*(z)|, \qquad \text{for } z \in \mathbb{T}. \tag{4.6}$$

But for $z \in \mathbb{T}$,

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\overline{z}}\right)} = B(z)\overline{r}\left(\frac{1}{z}\right)$$

implies

$$r^{*\prime}(z) = B'(z)\overline{r}\left(\frac{1}{z}\right) - \frac{B(z)}{z^2}\overline{r}'\left(\frac{1}{z}\right),$$

i.e.,

$$\begin{aligned}|r^{*\prime}(z)| &= \left|zB'(z)\overline{r(z)} - B(z)\overline{zr'(z)}\right| \\ &= \left|\frac{zB'(z)}{B(z)}\overline{r(z)} - \overline{zr'(z)}\right|.\end{aligned}$$

It follows by using Lemma 3.1 that for $z \in \mathbb{T}$,

$$\begin{aligned}|r^{*\prime}(z)| &= \left||B'(z)|\overline{r(z)} - \overline{zr'(z)}\right| \\ &= \left||B'(z)|r(z) - zr'(z)\right| \\ &= \left|\frac{zB'(z)}{B(z)}r(z) - zr'(z)\right| \\ &= \left|B'(z)r(z) - r'(z)B(z)\right|.\end{aligned} \tag{4.7}$$

Note that for $z \in \mathbb{T}$, $|r^*(z)| = |r(z)|$. Hence from inequality (4.6), for $z \in \mathbb{T}$ we have

$$|B'(z)r(z) - r'(z)B(z)| \geq \left[\sum_{j=1}^{n} \frac{1}{1+|w_j|} - \frac{n-|B'(z)|}{2}\right] |r(z)|. \tag{4.8}$$

Using equation (4.7) and (4.8) in Lemma 3.2, we get for $z \in \mathbb{T}$,

$$|r'(z)| \leq |B'(z)| \max_{z\in\mathbb{T}} |r(z)| - \frac{1}{2}\left[|B'(z)| + \sum_{j=1}^{n} \frac{|z_j|-1}{|z_j|+1}\right] |r(z)|. \qquad \square$$

PROOF OF THEOREM 2.2. Since by hypothesis, $|r(z)|$ and $|r'(z)|$ become maximum and $|B'(z)|$ becomes minimum at the same point on $\mathbb{T}$, if

$$|r(\mathrm{e}^{\mathrm{i}\alpha})| = \max_{z\in\mathbb{T}} |r(z)|,$$

then

$$|r'(\mathrm{e}^{\mathrm{i}\alpha})| = \max_{z\in\mathbb{T}} |r'(z)|$$

and

$$|B'(\mathrm{e}^{\mathrm{i}\alpha})| = \min_{z\in\mathbb{T}} |B'(z)|.$$

Therefore, from inequality (4.8), we have for $z = \mathrm{e}^{\mathrm{i}\alpha}$,

$$\left|B'(\mathrm{e}^{\mathrm{i}\alpha})r(\mathrm{e}^{\mathrm{i}\alpha}) - r'(\mathrm{e}^{\mathrm{i}\alpha})B(\mathrm{e}^{\mathrm{i}\alpha})\right| \geq \left[\sum_{j=1}^{n} \frac{1}{1+|w_j|} - \frac{n-|B'(\mathrm{e}^{\mathrm{i}\alpha})|}{2}\right] |r(\mathrm{e}^{\mathrm{i}\alpha})|. \tag{4.9}$$

By using Eq. (4.7) in Lemma 3.2, it follows that

$$|r'(\mathrm{e}^{\mathrm{i}\alpha})| + |B'(\mathrm{e}^{\mathrm{i}\alpha})r(\mathrm{e}^{\mathrm{i}\alpha}) - r'(\mathrm{e}^{\mathrm{i}\alpha})B(\mathrm{e}^{\mathrm{i}\alpha})| \leq |B'(\mathrm{e}^{\mathrm{i}\alpha})||r(\mathrm{e}^{\mathrm{i}\alpha})|.$$

Hence from inequality (4.9), we get

$$\left[\sum_{j=1}^{n} \frac{1}{1+|w_j|} - \frac{n-|B'(\mathrm{e}^{\mathrm{i}\alpha})|}{2}\right] |r(\mathrm{e}^{\mathrm{i}\alpha})| \leq |B'(\mathrm{e}^{\mathrm{i}\alpha})||r(\mathrm{e}^{\mathrm{i}\alpha})| - |r'(\mathrm{e}^{\mathrm{i}\alpha})|.$$

This implies,

$$\begin{aligned} |r'(\mathrm{e}^{\mathrm{i}\alpha})| &\leq |B'(\mathrm{e}^{\mathrm{i}\alpha})||r(\mathrm{e}^{\mathrm{i}\alpha})| - \left[\sum_{j=1}^{n} \frac{1}{1+|w_j|} - \frac{n-|B'(\mathrm{e}^{\mathrm{i}\alpha})|}{2}\right] |r(\mathrm{e}^{\mathrm{i}\alpha})| \\ &= \left[\frac{1}{2}|B'(\mathrm{e}^{\mathrm{i}\alpha})| - \sum_{j=1}^{n} \frac{|z_j|-1}{2(|z_j|+1)}\right] |r(\mathrm{e}^{\mathrm{i}\alpha})|, \end{aligned}$$

i.e.,

$$\max_{z\in\mathbb{T}}|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \sum_{j=1}^{n}\frac{|z_j|-1}{|z_j|+1}\right]\max_{z\in\mathbb{T}}|r(z)| \quad (z\in\mathbb{T}).$$

This proves the theorem completely. □

PROOF OF THEOREM 2.3. Since r has m zeros and all lie in $\mathbb{T}\cup\mathbb{D}_-$, we can write

$$r(z) = \frac{1}{w(z)}\prod_{j=1}^{m}(z-z_j),$$

where $|z_j|\leq 1$, for $j=1,2,\ldots,m$. Logarithmic differentiation gives

$$\frac{zr'(z)}{r(z)} = \sum_{j=1}^{m}\frac{z}{z-z_j} - \frac{zw'(z)}{w(z)}.$$

This gives for $z=\mathrm{e}^{\mathrm{i}\theta}$, $0\leq\theta\leq 2\pi$,

$$\operatorname{Re}\left[\mathrm{e}^{\mathrm{i}\theta}\frac{r'(\mathrm{e}^{\mathrm{i}\theta})}{r(\mathrm{e}^{\mathrm{i}\theta})}\right] = \operatorname{Re}\left[\sum_{j=1}^{m}\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-z_j}\right] - \operatorname{Re}\left[\frac{\mathrm{e}^{\mathrm{i}\theta}w'(\mathrm{e}^{\mathrm{i}\theta})}{w(\mathrm{e}^{\mathrm{i}\theta})}\right].$$

Lemma 3.3 and Eq. (4.4) give

$$\operatorname{Re}\left(\frac{\mathrm{e}^{\mathrm{i}\theta}r'(\mathrm{e}^{\mathrm{i}\theta})}{r(\mathrm{e}^{\mathrm{i}\theta})}\right) \geq \sum_{j=1}^{m}\frac{1}{1+|z_j|} - \frac{n-|B'(\mathrm{e}^{\mathrm{i}\theta})|}{2}.$$

Hence, for $z\in\mathbb{T}$,

$$\begin{aligned}\left|\frac{r'(z)}{r(z)}\right| &\geq \operatorname{Re}\left[\frac{zr'(z)}{r(z)}\right] \geq \sum_{j=1}^{m}\frac{1}{1+|z_j|} - \frac{n-|B'(z)|}{2}\\ &\geq \sum_{j=1}^{m}\frac{1}{1+|z_j|} - \left(\sum_{j=1}^{n}\frac{1}{2}\right) + \frac{|B'(z)|}{2}\\ &\geq \sum_{j=1}^{m}\left(\frac{1}{1+|z_j|}-\frac{1}{2}\right) - \frac{(n-m)}{2} + \frac{|B'(z)|}{2}.\end{aligned}$$

Therefore, for $z\in\mathbb{T}$,

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| - (n-m) + \sum_{j=1}^{m}\frac{1-|z_j|}{1+|z_j|}\right]|r(z)|.$$

This completes the proof of Theorem 2.3. □

5. *Numerical experiments*

In numerical computation of $r(z) \in \mathcal{R}_n$ and the corresponding Blaschke product $B(z)$ we use (1.5), as well as the following formulas for their derivatives

$$r'(z) = r(z)\sum_{j=1}^{n} \frac{z_j - a_j}{(z-b_j)(z-a_j)} \quad \text{and} \quad B'(z) = B(z)\sum_{j=1}^{n} \frac{|a_j|^2 - 1}{(z-a_j)(1-\overline{a}_j z)}.$$

The bounds of $|r'(\mathrm{e}^{\mathrm{i}\theta})| = L(\theta)$, when $\theta \in [0, 2\pi)$, in Theorems 1.1 and 2.1, will be denoted by

$$G_1(\theta) = \frac{M}{2}|B_n'(\mathrm{e}^{\mathrm{i}\theta})| \quad \text{and} \quad G_2(\theta) = |B_n'(\mathrm{e}^{\mathrm{i}\theta})|\left(M - \frac{1}{2}U(\mathrm{e}^{\mathrm{i}\theta})\right),$$

respectively, where M and U are defined in (2.1). The constant M can be obtained by applying the WOLFRAM MATHEMATICA function `MaxValue` to the real function

$$\theta \mapsto |r(\mathrm{e}^{\mathrm{i}\theta})| = \prod_{j=1}^{n} \sqrt{\frac{|b_j|^2 + 1 - 2|b_j|\cos(\theta - \arg b_j)}{|a_j|^2 + 1 - 2|a_j|\cos(\theta - \arg a_j)}}$$

on the interval $[0, 2\pi)$.

All computations and graphics in the following examples were performed in WOLFRAM MATHEMATICA, Ver. 14.2, on MacOS Sequola 15.5.

Example 5.1. We take the following complex numbers

$$z_1 = 10\,\mathrm{e}^{\mathrm{i}\pi/3}, \quad z_2 = 11\mathrm{i}, \quad z_3 = 8\,\mathrm{e}^{-\mathrm{i}\pi/3}, \quad z_4 = 14\,\mathrm{e}^{\mathrm{i}5\pi/4}$$

and

$$a_1 = 5, \quad a_2 = 6\,\mathrm{e}^{\mathrm{i}\pi/8}, \quad a_3 = 7\,\mathrm{e}^{\mathrm{i}\pi/3}, \quad a_4 = 8\,\mathrm{e}^{\mathrm{i}\pi} = -8,$$

as zeros and poles, respectively, for the rational functions $r(z) = r_n(z) \in \mathcal{R}_n$, so that

$$\begin{aligned}
r_1(z) &= \frac{z - 10\mathrm{e}^{\mathrm{i}\pi/3}}{z-5},\\
r_2(z) &= \frac{z - 10\mathrm{e}^{\mathrm{i}\pi/3}}{z-5}\cdot\frac{z-11\mathrm{i}}{z-6\mathrm{e}^{\mathrm{i}\pi/8}},\\
r_3(z) &= \frac{z - 10\mathrm{e}^{\mathrm{i}\pi/3}}{z-5}\cdot\frac{z-11\mathrm{i}}{z-6\mathrm{e}^{\mathrm{i}\pi/8}}\cdot\frac{z-18\,\mathrm{e}^{-\mathrm{i}\pi/3}}{z-7\,\mathrm{e}^{\mathrm{i}\pi/3}},\\
r_4(z) &= \frac{z - 10\mathrm{e}^{\mathrm{i}\pi/3}}{z-5}\cdot\frac{z-11\mathrm{i}}{z-6\mathrm{e}^{\mathrm{i}\pi/8}}\cdot\frac{z-18\,\mathrm{e}^{-\mathrm{i}\pi/3}}{z-7\,\mathrm{e}^{\mathrm{i}\pi/3}}\cdot\frac{z-14\,\mathrm{e}^{\mathrm{i}5\pi/4}}{z+8}.
\end{aligned}$$

We note that all these rational functions satisfy the conditions of Theorems 1.1 and 2.1. Moreover, from Figures 1–4 (left), we see that the condition (2.1) holds for each $n \in \{1,2,3,4\}$. For the corresponding maximal values on the unit circle $\mathbb{T}$, $M = \max_{0 \le \theta < 2\pi} |(r_n(\mathrm{e}^{\mathrm{i}\theta})|$, we obtained the following constants 2.42366, 5.22463, 6.11864, 10.2169, respectively for $n = 1,2,3,4$ (displayed as dashed lines). That means Theorem 2.1 gives better bounds than Theorem 1.1 in all these cases (see the corresponding graphics on the right side in Figures 1–4).

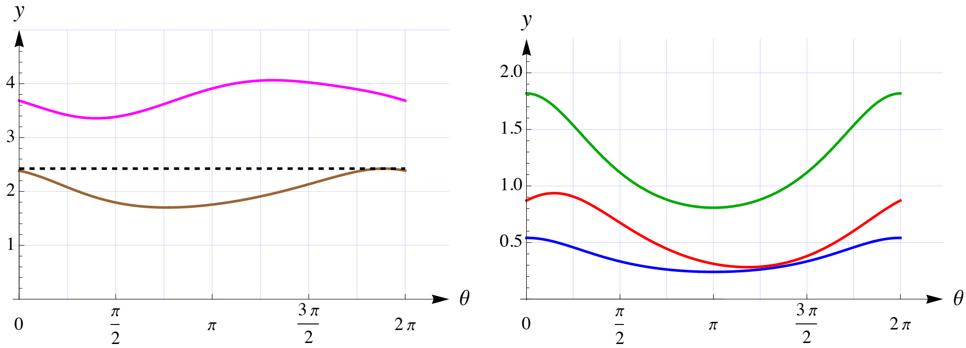

Figure 1: Curves for $n = 1$: (left) $y = U\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (magenta), $y = |r(\mathrm{e}^{\mathrm{i}\theta})|$ (brown); (right) $y = L(\theta)$ (blue), $y = G_1(\theta)$ (green), $y = G_2(\theta)$ (red)

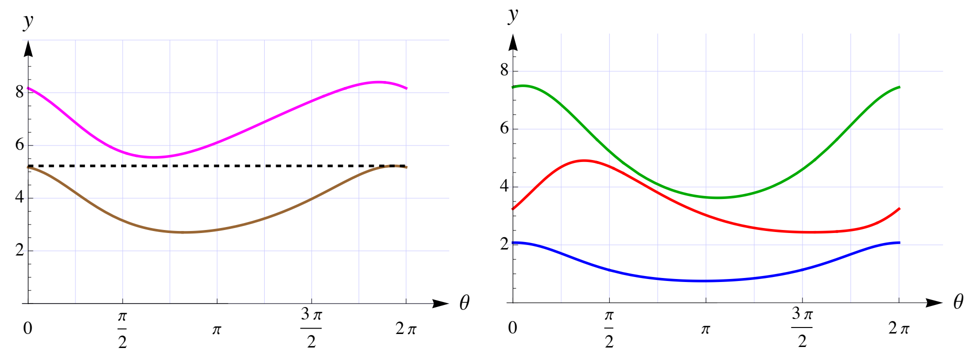

Figure 2: Curves for $n = 2$: (left) $y = U\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (magenta), $y = |r(\mathrm{e}^{\mathrm{i}\theta})|$ (brown); (right) $y = L(\theta)$ (blue), $y = G_1(\theta)$ (green), $y = G_2(\theta)$ (red)

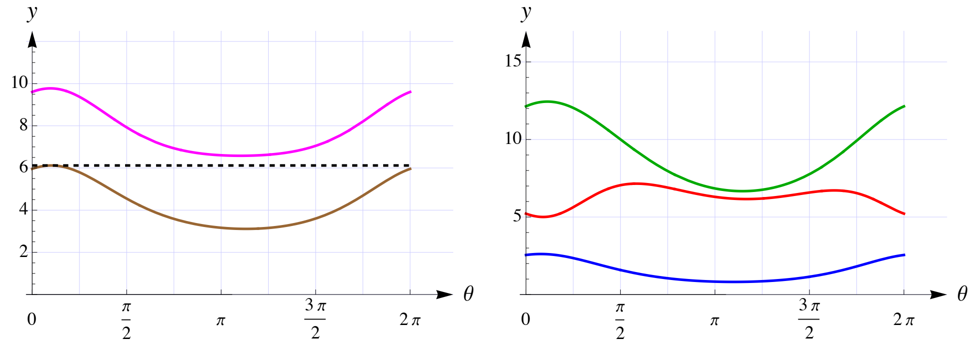

Figure 3: Curves for $n = 3$: (left) $y = U\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (magenta), $y = |r(\mathrm{e}^{\mathrm{i}\theta})|$ (brown); (right) $y = L(\theta)$ (blue), $y = G_1(\theta)$ (green), $y = G_2(\theta)$ (red)

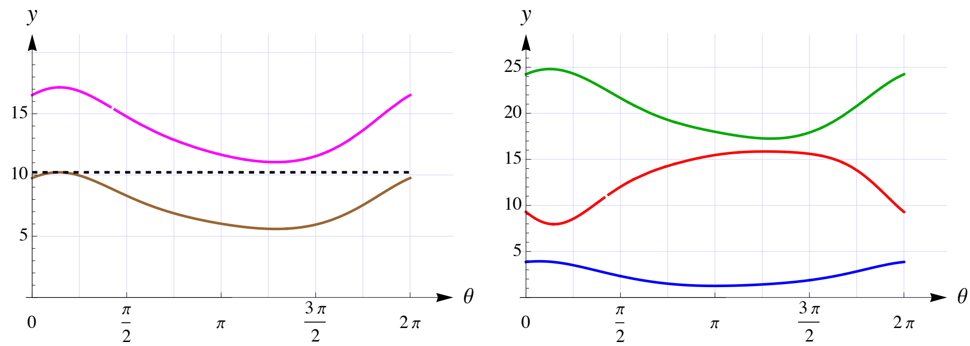

Figure 4: Curves for $n = 4$: (left) $y = U\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (magenta), $y = |r(\mathrm{e}^{\mathrm{i}\theta})|$ (brown); (right) $y = L(\theta)$ (blue), $y = G_1(\theta)$ (green), $y = G_2(\theta)$ (red)

Example 5.2. Now we consider the rational function, with the following zeros and poles,

$$z_1 = 6\,\mathrm{e}^{\mathrm{i}3\pi/4}, \quad z_2 = 8\,\mathrm{e}^{-\mathrm{i}\pi/3}, \quad z_3 = 4\,\mathrm{e}^{\mathrm{i}\pi/4}$$

and

$$a_1 = 2\,\mathrm{e}^{\mathrm{i}\pi/8}, \quad a_2 = 3\,\mathrm{e}^{-\mathrm{i}\pi/6}, \quad a_3 = 4\,\mathrm{e}^{\mathrm{i}\pi},$$

respectively, i.e.,

$$r(z) = \frac{\left(z - 6\,\mathrm{e}^{\mathrm{i}3\pi/4}\right)\left(z - 8\,\mathrm{e}^{-\mathrm{i}\pi/3}\right)\left(z - 4\,\mathrm{e}^{\mathrm{i}\pi/4}\right)}{\left(z - 2\,\mathrm{e}^{\mathrm{i}\pi/8}\right)\left(z - 3\,\mathrm{e}^{-\mathrm{i}\pi/6}\right)(z + 4)}. \tag{5.1}$$

Since in this case

$$M = \max_{0 \le \theta < 2\pi} |(r_n(\mathrm{e}^{\mathrm{i}\theta})| = 13.7226,$$

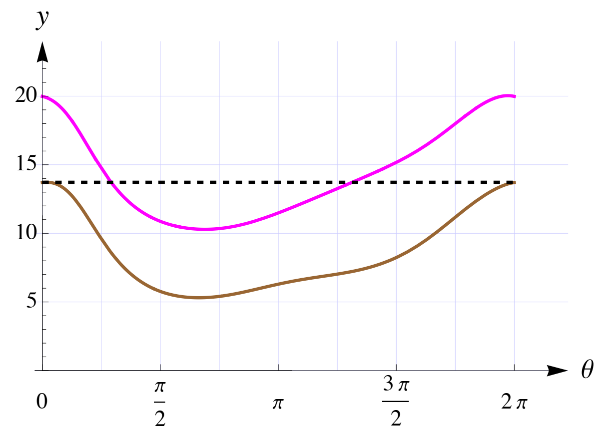

Figure 5: Case $n = 3$: Curves $y = U\left(e^{i\theta}\right)$ (magenta), and $y = |r(e^{i\theta})|$ (brown) and the constant M (dashed line)

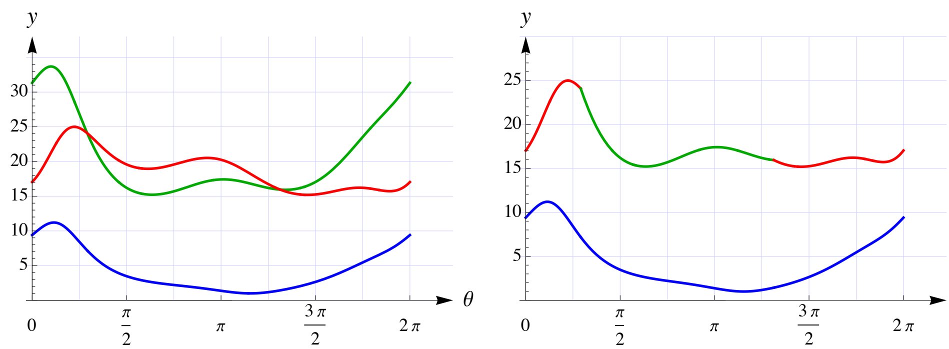

Figure 6: Curves for $n = 3$: $y = L(\theta)$ (blue) and $y = \min\{G_1(\theta), G_2(\theta)\}$ (red-green-red)

the condition $M < U(z)$ is not satisfied for all $z \in \mathbb{T}$ (see Fig. 5), and we can take the estimate (2.2) as the best option in this case. In Fig. 6 we present bounds obtained by Theorem 1.1 (green line) and Theorem 2.1 (red line). Their intersection is for $\theta_1 \approx 0.915509$ and $\theta_2 \approx 4.1263$. The best possible bound is given in Fig. 6 (right), as a curve (red-green-red line)

$$y = \min\{G_1(\theta), G_2(t)\} = \frac{1}{4}|B'(z)|\left(3M - U(z) - |U(z) - M|\right),$$

where $z = e^{i\theta}$, $0 \leq \theta < 2\pi$.

Acknowledgement. The work of the first author was supported by the Serbian Academy of Sciences and Arts (Φ-96)

REFERENCES

[1] S. N. Bernstein, *Sur l'ordre de la meilleure approximation des functions continues par des polynômes de degré donné*, Mem. Acad. R. Belg. **4**(1912), 1–103.

[2] A. Giroux, Q. I. Rahman, G. Schmeisser, *On Bernstein's inequality*, Canad. J. Math. **31** (2) (1979), 347–353.

[3] R. B. Gardner, N. K. Govil, G. V. Milovanović, *Extremal Problems and Inequalities of Markov-Bernstein Type for Algebraic Polynomials*, Mathematical Analysis and Its Applications. Elsevier/Academic Press, London, 2022.

[4] A. Guessab, G. V. Milovanović, *Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials*, *J. Math. Anal. Appl.* **182** (1994), 244–249.

[5] P. D. Lax, *Proof of a conjecture of P. Erdős on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.

[6] X. Li, R. N. Mohapatra, R. S. Rodgriguez, *Bernstein inequalities for rational functions with prescribed poles*, J. London Math. Soc. **51** (1995), 523–531.

[7] G. V. Milovanović, *On the Markov extremal problem in the L^2-norm with the classical weight functions*, J. Nonlinear Convex Anal. **23** (6) (2022), 1179–1212.

[8] G. V. Milovanović, D. D. Mitrinović, Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific Publ. Co., Singapore – New Jersey – London – Hong Kong, 1994.

[9] A. Mir, T. Fayaz, *Inequalities for a rational function with prescribed poles*, Siberian Math. J. **65** (2024), 899–908.

[10] A. Mir, S. Hans, *Inequalities concerning rational functions in the complex domain*, Siberian Math. J. **63** (2022), 1012–1022.

[11] I. Qasim, A. Liman, *Bernstein type inequalities for rational functions*, Indian J. Pure Appl. Math. **46** (2015), 337–348.

[12] I. Qasim, *Inequalities for rational function functions in the complex plane*, Filomat (to appear).

[13] M. Riesz, *Eine trigonometrische interpolation formel und einige Ungleichung für Polynome*, Jahresber. Dtsch. Math.-Ver. **23** (1914) 354–368.

[14] W. M. Shah, *A generalization of a theorem of Paul Turán*, J. Ramanujan Math. Soc. **11** (1) (1996), 67–72.

[15] P. Turán, *Über die Ableitung von Polynomen*, Compositio Mathematica **7** (1939), 89–95.

Serbian Academy of Sciences and Arts
Mathematical Institute
11000 Beograd, Serbia
&
University of Niš
Faculty of Sciences and Mathematics
18000 Niš, Serbia
e-mail: gvm@mi.sanu.ac.rs (G.V. Milovanović)

University of Kashmir
Department of Mathematics
Srinagar, Jammu and Kashmir, 190006
India
e-mail: mabdullah_mir@uok.edu.in (A. Mir)

National Institute of Technology Srinagar
Department of Mathematics
Jammu and Kashmir, 190006
India
e-mail: idreesf3@nitsri.net (I. Qasim)

Universiti Sains Malaysia
School of Mathematical Sciences
11800 USM, Penang
Malaysia
e-mail: sklee@usm.my (S.K. Lee)