3. F. G. Tricomi, *Integral Equations*, Interscience Publishers, Inc., New York (1957).

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ON SOME TURÁN'S EXTREMAL PROBLEMS FOR ALGEBRAIC POLYNOMIALS*

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In this survey paper, we consider extremal problems for polynomials initiated by P. Turán. We give a few classical results of Markov's type, the further generalizations and extensions, the generalizations in L^2-norm, and finally, some generalizations of extremal problems of Bernstein's and Turán type.

1. Introduction

In this survey paper, we consider some extremal problems for algebraic polynomials initiated by the very famous Hungarian mathematician Paul Turán. The first result in the theory of extremal problems for polynomials was connected with some investigations of the well-known Russian chemist Mendeleev [38]. In mathematical terms, Mendeleev's problem was: If $P(t)$ is an arbitrary quadratic polynomial defined on an interval $[a, b]$, with $\max_{t\in[a,b]} P(t) - \min_{t\in[a,b]} P(t) = L$, how large $P'(t)$ can be on $[a, b]$?

This problem can also be stated for polynomials of degree n. The problem was solved by A. A. Markov [35]. His brother V. A. Markov [36] investigated the upper bound of $|P^{(k)}(t)|$, where $k \le n$.

An analogue of Markov's theorem for the unit disk in the complex plane was investigated by S. Bernstein [8]. Markov's and Bernstein's inequalities are fundamental in proving many of the inverse theorems in polynomial ap-

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proximation theory (see Dzyadyk [15], Lorentz [30], Meinardus [37], Ivanov [26]).

There are many results on Markov's and Bernstein's inequalities, and their generalizations and extensions in various norms and restricted classes of polynomials (cf. Boas [9], Durand [13], Mamedhanov [34], Milovanović [40], Milovanović, Mitrinović, and Rassias [44], Rahman and Schmeisser [53], Rassias [55], Voronovskaja [73]). In this paper, we will give a short account of such results, including the classical, and primarily the results initiated by Turán.

2. Some Classical Results

Let $\mathcal{P}_n$ be the set of all algebraic polynomials $P(\not\equiv 0)$ of degree at most n, $W_n \subset \mathcal{P}_n$, and let $\|\cdot\|$ be a given norm. The general extremal problem of Markov's type can be stated in the following form: Determine the best constant $A_{n,k}$ such that

$$\|P^{(k)}\| \le A_{n,k}\|P\| \quad (P \in W_n)\ , \tag{2.1}$$

i.e.

$$A_{n,k} = \sup_{P\in W_n} \frac{\|P^{(k)}\|}{\|P\|}\ . \tag{2.2}$$

For $W_n = \mathcal{P}_n$, A. A. Markov [35] solved this extremal problem in the uniform norm on $[-1,1]$,

$$\|f\| = \|f\|_\infty = \max_{-1\le t\le 1} |f(t)|\ . \tag{2.3}$$

Namely, he found $A_{n,1} = n^2$. The equality in (2.1) holds only at ± 1 and only when $P(t) = cT_n(t)$, where T_n is the Chebyshev polynomial of the first kind of degree n, and c is an arbitrary constant.

His brother V. A. Markov [36] solved the corresponding problem for the kth derivative. Using a complicated variational method, V. A. Markov obtained the best constant $A_{n,k} = T_n^{(k)}(1)$, for each $k = 1,\ldots,n$. The extremal polynomial is T_n.

Schaeffer and Duffin [57] obtained an elegant proof of V. A. Markov's inequality. Namely, for $P \in \mathcal{P}_n$ and such that $\|P\|_\infty \le 1$, they proved that

$$\|P^{(k)}\|_\infty \le T_n^{(k)}(1) = \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)} \tag{2.4}$$

for $k = 1,2,\ldots,n$. The equality can occur only at $t = \pm 1$ and here only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.

Duffin and Schaeffer [12] also proved that for this inequality to hold, it is only necessary to assume that $|P(t)| \le 1$ at $n+1$ selected points in $[-1,1]$. Namely, they proved the following refinement:

Theorem 2.1. Let $P \in \mathcal{P}_n$ such that

$$|P(\cos \nu\pi/n)| \le 1 \quad (\nu = 0,1,\ldots,n)\ .$$

Then inequality (2.4) is satisfied for $k = 1,\ldots,n$. The equality occurs only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.

3. Further Generalizations and Extensions

In 1970, at a conference on *Constructive Function Theory* held in Varna, Bulgaria, the late Professor Paul Turán (1910–1976) asked the following question: Given a polynomial $p_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ with real coefficients whose graph on $[-1,1]$ lies in the unit disk, how large can its derivative be on the same interval? More generally, for an arbitrary non-negative function $\varphi(x)$ on $[-1,1]$, let $\mathcal{P}(\varphi,n)$ denote the class of all polynomials p_n of degree at most n such that $|p_n(x)| \le \varphi(x)$ for $-1 \le x \le 1$. Then, how large can $|p_n^{(j)}(x_0)|$ be at a given point x_0 in $[-1,1]$ as p_n varies in $\mathcal{P}(\varphi,n)$? Such problems first appeared in approximation theory (cf. Dzyadyk [14], and Pierre and Rahman [47]) concerning converse type theorems in approximation by polynomials.

At first, for polynomials $P \in \mathcal{P}_n$ we define

$$\|P\|_* = \sup_{-1<t<1} \frac{|P(t)|}{\sqrt{1-t^2}}\ , \tag{3.1}$$

or generally,

$$\|P\|_\varphi = \sup_{-1<t<1} \frac{|P(t)|}{\varphi(t)}\ , \tag{3.2}$$

where $t \mapsto \varphi(t)$ is a non-negative function on $[-1,1]$. Also, we denote

$$\|P\| = \|P\|_\infty = \max_{-1\le t\le 1} |P(t)|\ .$$

If $||P||_* \le 1$, or $||P||_\varphi \le 1$, Turán's problem is how large can $||P^{(k)}||$ be?

In the first case (i.e. when $\varphi(t) = \sqrt{1-t^2}$) for $k = 1$ the answer was given by Rahman [50]:

Theorem 3.1. Let $P \in \mathcal{P}_n$ and $||P||_* \le 1$, then

$$||P'|| \le 2(n-1) . \tag{3.3}$$

If

$$U_n(t) = (1-t^2)^{-1/2} \sin((n+1)\arccos t) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m)!}{m!(n-2m)!} (2t)^{n-2m}$$

is the nth Chebyshev polynomial of the second kind, then

$$P(t) = (1-t^2)U_{n-2}(t)$$

satisfies the conditions of Theorem 3.1 and $|P'(\pm 1)| = 2(n-1)$. Therefore the result is best possible.

Theorem 3.2. Under the conditions of Theorem 3.1, we have

$$|P'(t)| \le (t^2(1-t^2)^{-1} + (n-1)^2)^{1/2} \quad (-1 < t < 1) . \tag{3.4}$$

If $P(t) = \gamma(1-t^2)U_{n-2}(t)$, $|\gamma| = 1$, then (3.4) is an equality at those points of the interval $(-1, 1)$, where

$$(n-1)(1-t^2)^{1/2} \tan((n-1)\arccos t) = t .$$

Let $P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu$. If we put $t = 0$, then Inequality (3.4) implies

$$|a_1| = |P'(0)| \le n-1 .$$

This inequality is sharp for odd n. Also, Rahman [50] obtained the following estimate for the coefficient a_2:

Theorem 3.3. If the polynomial $P \in \mathcal{P}_n$ satisfies $||P||_* \le 1$, then

$$|a_2| \le \frac{1}{2}((n-1)^2 + 1) . \tag{3.5}$$

For even n, the bound in (3.5) is attained when $P(t) = \gamma(1-t^2)U_{n-2}(t)$, $|\gamma| = 1$.

Rahman [51] proved the following theorem which gives a sharp estimate for each of the coefficients.

Theorem 3.4. If the polynomial $P \in \mathcal{P}_n$ satisfies $||P||_* \le 1$, then according as $n - \nu$ is even or odd, $|a_\nu|$ is bounded above by the absolute value of the coefficients of t^ν in $\gamma(1-t^2)U_{n-2}(t)$ or $\gamma(1-t^2)U_{n-3}(t)$ ($|\gamma| = 1$), respectively.

If $\varphi(t) = |t|$ and $k = 1$, Rahman [50] proved:

Theorem 3.5. Let $P \in \mathcal{P}_n$ and $|P(t)| \le |t|$ for $-1 \le t \le 1$, then

$$||P'|| \le (n-1)^2 + 1 .$$

The problem with $\varphi(t) = \sqrt{1-t^2}$ and $k = 2$ was solved by Pierre and Rahman [47] following the spirit of the variational approach of V. A. Markov [36]. However, part of Markov's reasoning is difficult to apply. Using the idea of S. N. Bernstein, B. Ya. Levin and others, Pierre and Rahman [47] proved:

Theorem 3.6. If the polynomial $P \in \mathcal{P}_n$ satisfies $||P||_* \le 1$, then

$$||P''|| \le \left| \frac{d^2}{dt^2}((1-t^2)U_{n-2}(t)) \right|_{t=\pm 1} = \frac{2}{3}(n-1)(2n^2-4n+3) .$$

For arbitrary $k \in \mathbb{N}$, the problem was also solved by Pierre and Rahman [48]:

Theorem 3.7. Let $n \ge 2$ and $P \in \mathcal{P}_n$ such that $||P||_* \le 1$. Then

$$||P^{(k)}|| \le Q_n^{(k)}(1) = \frac{k}{2k-1} \cdot \frac{2n^2-4n+k+1}{n-1} T_{n-1}^{(k-1)}(1) \tag{3.6}$$

for all $k \in \mathbb{N}$, where

$$Q_n(t) = (t^2-1)U_{n-2}(t) , \tag{3.7}$$

and U_m denotes the mth Chebyshev polynomial of the second kind. Equality in (3.6) is attained only for $P(t) = \gamma Q_n(t)$ with $|\gamma| = 1$.

Also, Pierre and Rahman [48] considered a more general case when

$$\varphi(t) = (1-t)^{\lambda/2}(1+t)^{\mu/2} , \tag{3.8}$$

where λ, μ are non-negative integers, and $P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu$ is a polynomial of degree at most n such that

$$\|P\|_\varphi \le 1 , \tag{3.9}$$

where $\|\cdot\|_\varphi$ is defined by (3.2).

Theorem 3.8. Let $\varphi(t)$ be given by (3.8). If the polynomial $P \in \mathcal{P}_n$ satisfies (3.9), then for $(\lambda+\mu)/2 \le k \le n$,

$$\|P^{(k)}\| \le \max\{\|\Lambda_n^{(k)}\|,\ \|\Lambda_{n-1}^{(k)}\|\} ,$$

where

$$\Lambda_m(t) = \begin{cases} \varphi(t) T_{m-(\lambda+\mu)/2}(t) , & (\lambda, \mu \text{ are both even}) , \\ \sqrt{1-t^2}\,\varphi(t) U_{m-1-(\lambda+\mu)/2}(t) , & (\lambda, \mu \text{ are both odd}) . \end{cases}$$

The case when $1 \le k < (\lambda+\mu)/2$, for $(\lambda+\mu)/2 > 1$, was left unresolved. An asymptotic estimate when $n \to +\infty$, for $\lambda = \mu = 2$, was recently considered by Pierre, Rahman and Schmeisser [49]. At first, they proved the following inequality:

$$P'(t)^2 + (n^2 - 4n)\left(\frac{P(t)}{1-t^2}\right)^2 \le (n-2)^2 \quad (-1 < t < 1)$$

for real-valued polynomials $P \in \mathcal{P}_n$ of a real variable, such that $\|P\|_\varphi \le 1$, where $\varphi(t) = 1 - t^2$. From this inequality, it follows that $\|P'\| \le n-2$. If one does not assume that $P(t)$ is real for real t, then for $n \ge 4$, they proved that $\|P'\| \le n - 2$. In that case, they also proved that

$$|P'(t)| \le ((n-2)^2 - (n^2-4n)t^2)^{1/2} \quad (-1 \le t \le 1) .$$

Their asymptotic result can be stated in the following form:

Theorem 3.9. For even n

$$\sup_{P \in W_n} \|P'\| = n - 2 - \frac{\pi^2}{8n} + O(n^{-2}) \quad \text{as} \quad n \to +\infty ,$$

where W_n denotes a class of polynomials $P \in \mathcal{P}_n$ for which $\|P\|_\varphi \le 1$, where $\varphi(t) = 1 - t^2$.

It is interesting to consider the case when the polynomial $P \in \mathcal{P}_n$ satisfies the condition

$$0 \le P(t) \le \varphi(t) \quad (-1 \le t \le 1) , \tag{3.10}$$

where $t \mapsto \varphi(t)$ is a non-negative function on $[-1, 1]$.

Pierre, Rahman and Schmeisser [49] considered such case when $\varphi(t) = \sqrt{1-t^2}$.

Theorem 3.10. If $P \in \mathcal{P}_n$ and satisfies (3.10), where $\varphi(t) = \sqrt{1-t^2}$, then

$$\|P\| \le \frac{1}{n-1} \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{n-1} \left(\sin\frac{(2\nu-1)\pi}{4(n-1)}\right)^{-2} .$$

Lachance [27] studied pointwise and uniform bounds for the derivatives of real polynomials $P \in \mathcal{P}_n$ such that $\|P\|_\varphi \le 1$, in the case when $\varphi(t) = (1-t^2)^{-\lambda/2}$ and λ is a fixed positive integer. We denote such a class of polynomials by $\mathcal{P}_n(\lambda)$.

Theorem 3.11. Let $P \in \mathcal{P}_n(\lambda)$. Then

$$|P^{(k)}(t)| \le 2^k (n+\lambda)^k (1-t^2)^{-(\lambda+k)/2} \quad (-1 < t < 1) ,$$

$$\|P^{(k)}\| \le 2^k (n+\lambda)^k \binom{n+\lambda}{\lambda+k} \sim \frac{2^k}{(\lambda+k)!}\, n^{\lambda+2k} ,$$

for $k = 0, 1, \ldots, n$.

Using this theorem and properties of the *constrained Chebyshev polynomial*

$$t \mapsto T_{n,\lambda}(t) = Q_{n,\lambda}(t)/E_n(\lambda) ,$$

where $Q_{n,\lambda}(t)$ is defined as a unique monic extremal polynomial of precise degree n for which

$$\|(1-t^2)^{\lambda/2} Q_{n,\lambda}(t)\| = E_n(\lambda) ,$$

where

$$E_n(\lambda) = \min_{g \in \mathcal{P}_{n-1}} \|(1-t^2)^{\lambda/2}(t^n - g(t))\| ,$$

Lachance was able to give a few interesting applications.

The polynomial $T_{n,\lambda}$ is characterized by the existence of a unique set of $n+1$ points $\xi_\nu = \xi_\nu^{(n)}(\lambda)$ satisfying

$$-a_n(\lambda) \le \xi_0 < \xi_1 < \cdots < \xi_n \le a_n(\lambda) ,$$

where

$$a_n(\lambda) = \left(1 - \left(\frac{\lambda}{n+\lambda}\right)^2\right)^{1/2} , \tag{3.11}$$

on which

$$T_{n,\lambda}(\xi_\nu) = (-1)^{n-\nu}(1-\xi_\nu^2)^{-\lambda/2} \quad (\nu = 0, 1, \ldots, n) .$$

Since the function φ is even, the nodes ξ_ν are symmetrically distributed with respect to the origin, i.e. $\xi_\nu = -\xi_{n-\nu}$ for each ν. The following property of the constrained Chebyshev polynomials was proved by Lachance, Saff and Varga [28]:

Theorem 3.12. If $P \in \mathcal{P}_n(\lambda)$, then for $t < \xi_0$ and for $t > \xi_n$, we have

$$|P(t)| \le |T_{n,\lambda}(t)| .$$

For the polynomial P constrained by the zeros at each endpoint of $[-1, 1]$ its absolute maximum is achieved only in a smaller subinterval (Lachance, Saff and Varga [28]):

Theorem 3.13. Let $P \in \mathcal{P}_n(\lambda)$. Then

$$\|(1-t^2)^{\lambda/2}P(t)\| = \max_{|t|\le a_n(\lambda)} |(1-t^2)^{\lambda/2}P(t)| ,$$

where $a_n(\lambda)$ is defined by (3.11).

The following theorem of Lachance [27] is a corollary of Theorem 3.11 together with properties of the constrained Chebyshev polynomials:

Theorem 3.14. Let $P \in \mathcal{P}_n(\lambda)$. Then

$$\|P^{(k)}\| \le 2^k(n+\lambda)^k T_{n-k,\lambda+k}(1) ,$$

for each $k = 0, 1, \ldots, n$.

In the same paper, Lachance considered *incomplete polynomials* introduced by Lorentz [31]:

Theorem 3.15. For each pair of integers $m \ge 0$, $s \ge 1$ let $q(t)$ be a polynomial in $\mathcal{P}_m$. If $|t^s q(t)| \le 1$ for $0 \le t \le 1$, then

$$|t^s q'(t)| \le 2(s+m)(t(1-t))^{-1/2} \quad (0 < t < 1)$$

and

$$\max_{0\le t\le 1} |q(t)| \le T_{2m,2s}(1) .$$

Goetgheluck [19] considered a class of polynomials $P \in \mathcal{P}_n$ such that

$$|P(t)|m(t) \le |\phi(t)| \quad (-1 \le t \le 1) ,$$

where

$$m(t) = |t-a_1|^{\alpha_1} \cdots |t-a_p|^{\alpha_p} \quad (-1 \le a_1 < \cdots < a_p \le 1) ,$$

$$\phi(t) = (t-b_1)\cdots(t-b_q)u(t) \quad (b_1, \ldots, b_q \in [-1, 1]) ,$$

u is a function in $C^s[-1, 1]$ not identically equal to zero over $[-1, 1]$, and s a positive integer.

Theorem 3.16. Let $1 \le k \le s$ and the above conditions on P be satisfied. Then, there exists a constant A, such that the following inequality:

$$\left|P^{(k)}(t)|m(t) \le A\sum_{\nu=0}^{k}(C(n,t))^\nu|\phi^{(k-\nu)}(t)\right| \quad (-1 \le t \le 1)$$

holds, where

$$C(n,t) = n\left(\sqrt{1-t^2} + \frac{1}{n}\right)^{-1} .$$

Also, under the same conditions, there exists a constant B such that

$$|P^{(k)}(t)| \le \begin{cases} Bn^{k+\alpha_\nu} , & \text{if } |a_\nu| \ne 1 , \\ Bn^{2k+2\alpha_\nu} , & \text{if } |a_\nu| = 1 , \end{cases}$$

for $t \in J_\nu = \left[\frac{1}{2}(a_{\nu-1}+a_\nu), \frac{1}{2}(a_\nu+a_{\nu+1})\right]$ $(\nu = 1, \ldots, p)$, where $a_0 = -1$ and $a_{p+1} = 1$.

In the previous section, we mentioned refinement of Markov inequalities. Now, we will consider some refinement of the above inequalities given in Theorems 3.1, 3.6 and 3.7. Namely, we can ask whether or not it is enough to require that $|P(t)| \le \sqrt{1-t^2}$ holds only at $n+1$ selected points in $[-1,1]$ in order for (3.6) to hold. This type of problem has been investigated by Rahman and Schmeisser [54].

Theorem 3.17. Given any infinite triangular matrix

$$\begin{bmatrix} \tau_0^{(0)} & & & \\ \tau_0^{(1)} & \tau_1^{(1)} & & \\ \vdots & & \ddots & \\ \tau_0^{(n)} & \tau_1^{(n)} & & \tau_n^{(n)} \\ \vdots & & & & \ddots \end{bmatrix},$$

with nodes

$$-1 \le \tau_0^{(n)} < \tau_1^{(n)} < \cdots < \tau_n^{(n)} \le 1 \quad (n \in \mathbb{N}),$$

there always exists a sequence of polynomials $p_n \in \mathcal{P}_n$ and a sequence of points $x_n \in [-1,1]$ such that

$$|p_n(\tau_\nu^{(n)})| \le \Big(1-(\tau_\nu^{(n)})^2\Big)^{1/2} \quad (\nu = 0,1,\dots,n)$$

and

$$|p_n'(x_n)| \ge \frac{2}{\pi}\Big(1-o(1)\Big) n \log n, \quad n \to +\infty .$$

Theorem 3.18. Let

$$\xi_0 = -1, \quad \xi_\nu = \cos\left(\frac{2\nu-1}{n-1}\cdot\frac{\pi}{2}\right), \quad \nu = 1,\dots,n-1, \quad \xi_n = 1 .$$

If $P \in \mathcal{P}_n$ such that

$$|P(\xi_\nu)| \le (1-\xi_\nu^2)^{1/2} \quad (\nu = 0,1,\dots,n),$$

then

$$\|P^{(k)}\| \le Q_n^{(k)}(1) \quad (k = 2,3,\dots), \tag{3.12}$$

where Q_n is defined by (3.7), and

$$\|P'\| \le (n-1)\left(\frac{2}{\pi}\log(n-1)+3\right) = \frac{2}{\pi}\Big(1+o(1)\Big) n \log n \quad (n \to +\infty) .$$

Further, in (3.12) equality holds only if $P(t) = \gamma Q_n(t)$, where $|\gamma| = 1$.

We can see that the answer to the question is positive for $k \ge 2$ and negative for $k = 1$. If E is any closed subset of $[-1,1]$ which does not contain all of the points ξ_ν $(\nu = 0,1,\dots,n)$, then there is a polynomial $p \in \mathcal{P}_n$ such that

$$|p(t)| \le (1-t^2)^{1/2} \quad \text{on } E$$

for which (3.12) is not satisfied.

If the polynomial $P \in \mathcal{P}_n$ of the above theorem has real coefficients, then (3.12) can be extended to the inequality

$$|P^{(k)}(t+is)| \le |Q_n^{(k)}(1+is)| \quad (-1 \le t \le 1,\ -\infty < s < +\infty)$$

for $k = 2,3,\dots$.

4. Extremal Problems in L^2-Norm

In the L^2-metric, we give first the following result of Schmidt [58] and Turán [64]:

Theorem 4.1. (i) Let $(a,b) = (-\infty, +\infty)$ and $\|f\|^2 = \int_{-\infty}^{\infty} e^{-t^2} f(t)^2\,dt$. Then the best constant in (2.2) is $A_{n,1} = \sqrt{2n}$. An extremal polynomial is Hermite's polynomial H_n;
(ii) Let $(a,b) = (0, +\infty)$ and $\|f\|^2 = \int_0^{\infty} e^{-t} f(t)^2\,dt$. Then

$$A_{n,1} = \left(2\sin\frac{\pi}{4n+2}\right)^{-1} .$$

The extremal polynomial is

$$P(t) = \sum_{\nu=1}^{n} \sin\frac{\nu\pi}{2n+1} L_\nu(t) ,$$

where L_ν is Laguerre polynomial.

Theorem 4.1, in this form, was formulated by Turán [64]. Schmidt [58] proved only

$$A_{n,1} = \frac{2n+1}{\pi}\left(\frac{\pi^2}{24(2n+1)^2} + \frac{R}{(2n+1)^4}\right)^{-1},$$

where $-8/3 < R < 4/3$.

The case of L^2-metric with an arbitrary weight function $w\colon (a,b) \to \mathbb{R}_+$ $(-\infty \le a < b \le +\infty)$ for which all moments are finite was considered recently by Mirsky [46], Dörfler [10, 11] and Milovanović [40].

Using the Turán method, Milovanović [40] showed that the exact constant in (2.1) can be found as the maximal eigenvalue of a matrix of Gram's type. Namely, he considered a more general case with a given non-negative measure $d\lambda(t)$ on the real line $\mathbb{R}$, with compact or infinite support, for which all moments

$$\mu_\nu = \int_{\mathbb{R}} t^\nu \, d\lambda(t), \quad \nu = 0, 1, \dots ,$$

exist and are finite, and $\mu_0 > 0$. Then there exists a unique set of orthonormal polynomials $\pi_\nu(\cdot) = \pi_\nu(\cdot; d\lambda)$, $\nu = 0, 1, \dots$, defined by

$$\pi_\nu(t) = a_\nu t^\nu + \text{ lower degree terms}, \quad a_\nu > 0 ,$$

$$\int_{\mathbb{R}} \pi_\nu(t)\pi_\mu(t)\, d\lambda(t) = \delta_{\nu\mu}, \quad \nu, \mu \ge 0 . \tag{4.1}$$

For each polynomial $P \in \mathcal{P}_n$, with complex coefficients, we define

$$\|P\| = \left(\int_{\mathbb{R}} |P(t)|^2 \, d\lambda(t)\right)^{1/2}$$

and consider the extremal problem

$$A_{n,k} = A_{n,k}(d\lambda) = \sup_{P \in \mathcal{P}_n} \frac{\|P^{(k)}\|}{\|P\|} \quad (1 \le k \le n) . \tag{4.2}$$

Theorem 4.2. The best constant $A_{n,k}$ defined in (4.2) is given by

$$A_{n,k} = (\lambda_{\max}(B_{n,k}))^{1/2},$$

where $\lambda_{\max}(B_{n,k})$ is the maximal eigenvalue of the matrix $B_{n,k} = [b_{i,j}^{(k)}]_{k \le i,j \le n}$, whose elements are given by

$$b_{i,j}^{(k)} = \int_{\mathbb{R}} \pi_i^{(k)}(t)\pi_j^{(k)}(t)\, d\lambda(t), \quad k \le i, j \le n .$$

An extremal polynomial is

$$P^*(t) = \sum_{\nu=k}^{n} c_\nu \pi_\nu(t) ,$$

where $[c_k, c_{k+1}, \dots , c_n]^T$ is an eigenvector of the matrix $B_{n,k}$ corresponding to the eigenvalue $\lambda_{\max}(B_{n,k})$.

An alternative result is the following theorem (see Milovanović [40]):

Theorem 4.3. The best constant $A_{n,k}$ defined in (4.2) is equal to the spectral norm of one triangular matrix $Q_{n,k}^T$, $Q_{n,k} = [q_{ij}^{(k)}]_{k \le i,j \le n}$ $(q_{i,j}^{(k)} = 0 \Leftarrow i > j)$, i.e.

$$A_{n,k} = \sigma(Q_{n,k}^T) = (\lambda_{\max}(Q_{n,k}Q_{n,k}^T))^{1/2} , \tag{4.3}$$

where the elements $q_{ij}^{(k)}$ are given by the following inner product:

$$q_{ij}^{(k)} = (\pi_j^{(k)}, \pi_{i-k}) \quad (k \le i, j \le n) .$$

Alternatively, (4.3) can be expressed in the form

$$A_{n,k} = (\lambda_{\min}(C_{n,k}))^{-1/2} , \tag{4.4}$$

where $C_{n,k} = (Q_{n,k}Q_{n,k}^T)^{-1}$.

To prove this theorem, it is enough to consider only a real polynomial set $\mathcal{P}_n$. Let $P \in \mathcal{P}_n$ and $\pi_j^{(k)}(t) = \sum_{i=k}^{n} q_{ij}^{(k)} \pi_{i-k}(t)$, where $q_{ij}^{(k)} = (\pi_j^{(k)}, \pi_{i-k})$. Then

$$P^{(k)}(t) = \sum_{j=k}^{n} c_j \sum_{i=k}^{j} q_{ij}^{(k)} \pi_{i-k}(t) = \sum_{i=k}^{n}\left(\sum_{j=k}^{n} c_j q_{ij}^{(k)}\right)\pi_{i-k}(t)$$

and

$$\|P^{(k)}\|^2 = \sum_{i=k}^{n}\left(\sum_{j=i}^{n} c_j q_{ij}^{(k)}\right)^2 = \sum_{i=k}^{n} y_i^2 ,$$

where

$$y_i = \sum_{j=i}^{n} c_j q_{ij}^{(k)}, \quad i = k, \dots, n . \tag{4.5}$$

Let $\mathbf{c} = [c_k, \dots, c_n]^T$, $\mathbf{y} = [y_k, \dots, y_n]^T$, and $Q_{n,k} = [q_{ij}^{(k)}]_{k \le i,j \le n}$. Since $\mathbf{y} = Q_{n,k}\mathbf{c}$, it follows that

$$\frac{\|P^{(k)}\|^2}{\|P\|^2} \le \frac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{c}, \mathbf{c} \rangle} = \frac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle (Q_{n,k} Q_{n,k}^T)^{-1} \mathbf{y}, \mathbf{y} \rangle} .$$

Thus (4.3) and (4.4) hold.

Now, we will consider a few special measures.

1° $d\lambda(t) = e^{-t^2} dt$, $-\infty < t < +\infty$. Here we have

$$\pi_\nu(t) = \hat{H}_\nu(t) = (\sqrt{\pi} 2^\nu \nu!)^{-1/2} H_\nu(t) ,$$

where H_ν is a Hermite polynomial of degree ν. Since

$$H'_\nu(t) = 2\nu H_{\nu-1}(t) \quad \text{and} \quad \hat{H}'_\nu(t) = \sqrt{2\nu}\, \hat{H}_{\nu-1}(t) ,$$

we have

$$\hat{H}_\nu^{(k)}(t) = \sqrt{2\nu}\sqrt{2(\nu-1)} \cdots \sqrt{2(\nu-k+1)}\, \hat{H}_{\nu-k}(t) = \sqrt{2^k k! \binom{\nu}{k}}\, \hat{H}_{\nu-k}(t)$$

and

$$b_{ij}^{(k)} = 2^k k! \binom{i}{k} \delta_{ij}, \quad k \le i, j \le n .$$

Thus, we find $\lambda_{\max}(B_{n,k}) = 2^k k! \binom{n}{k}$ and $A_{n,k} = 2^{k/2}\sqrt{n!/(n-k)!}$.

Also, this result can be found in the unpublished Ph.D. Thesis of Shampine [59] (see also Shampine [61]). For $k = 1$, this result reduces to the assertion (i) in Theorem 4.1.

2° $d\lambda(t) = t^s e^{-t} dt$, $0 < t < +\infty$. Here we have the generalized Laguerre case with

$$\pi_\nu(t) = \hat{L}_\nu^s(t) = \sqrt{\nu!/\Gamma(\nu+s+1)} \sum_{i=0}^{\nu} (-1)^{\nu-i} \binom{\nu+s}{\nu-i} \frac{t^i}{i!} ,$$

where Γ is the gamma function.

First, we consider the simplest case where $k = 1$. Since

$$\frac{d}{dt}\hat{L}_j^s(t) = \sum_{i=1}^{j} q_{ij}^{(1)} \hat{L}_{i-1}^s(t), \quad q_{ij}^{(1)} = -\sqrt{\frac{j!}{\Gamma(j+s+1)}} \cdot \sqrt{\frac{\Gamma(i+s)}{(i-1)!}} ,$$

from the equalities (4.5) it follows that

$$c_i = y_{i+1} - \sqrt{\frac{i+s}{i}}\, y_i, \quad i = 1, \dots, n,$$

where we put $y_{n+1} = 0$. The elements $p_{ij}^{(1)}$ of the matrix $P_{n,1} = Q_{n,1}^{-1}$ are

$$p_{ij}^{(1)} = -\sqrt{1 + \frac{s}{i}}, \quad i = 1, \dots, n; \quad p_{i,i+1}^{(1)} = 1, \quad i = 1, \dots, n-1;$$
$$p_{ij}^{(1)} = 0, \quad \text{otherwise},$$

so that

$$C_{n,1} = P_{n,1}^T P_{n,1} = -\begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & O \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ O & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix} = -J_n ,$$

where

$$\alpha_0 = -(1+s), \quad \alpha_\nu = -\left(2 + \frac{s}{\nu+1}\right), \quad \beta_\nu = 1 + \frac{s}{\nu}, \quad \nu = 1, \dots, n-1 .$$

We see that J_n is the Jacobi matrix for monic orthogonal polynomials $\{Q_\nu\}$, which satisfy the following three-term recurrence relation:

$$Q_{k+1}(t) = (t - \alpha_k) Q_k(t) - \beta_k Q_{k-1}(t), \quad k = 0, 1, 2, \dots$$
$$Q_{-1}(t) = 0, \quad Q_0(t) = 1 .$$

The eigenvalues of $C_{n,1}$ are $\lambda_\nu = -t_\nu$, where $Q_n(t_\nu) = 0$, $\nu = 1, \dots, n$.

The standard Laguerre case ($s = 0$) can be exactly solved. In fact, for $t = 2(z-1)$ and $-1 \le z \le 1$, we have

$$Q_\nu(t) = \cos(2\nu+1)\frac{\theta}{2} \Big/ \cos\frac{\theta}{2}, \quad z = \cos\theta .$$

The eigenvalues of the matrix $C_{n,1}$ are

$$\lambda_\nu = -t_\nu = 4\sin^2\frac{(2\nu-1)\pi}{2(2n+1)}, \quad \nu = 1,\dots,n.$$

Since $\lambda_{\min}(C_{n,1}) = \lambda_1$, we obtain $A_{n,1} = \left(2\sin\frac{\pi}{2(2n+1)}\right)^{-1}$. This is Turán's result (Theorem 4.1 (ii)).

Now, we consider the case when $k = 2$ and $s = 0$. First, we note that

$$\frac{d^k}{dt^k}\hat{L}_j(t) = (-1)^k \sum_{i=k}^{j} \binom{j-i+k-1}{k-1} \hat{L}_{i-k}(t)\ .$$

The formulas (4.5), for $k = 2$, become

$$y_i = \sum_{j=i}^{n}(j-i+1)c_j, \quad i = 2,\dots,n.$$

Since $\Delta^2 y_i = c_i$ ($y_{n+1} = y_{n+2} = 0$), we find a five-diagonal symmetric matrix of order $n-1$

$$C_{n,2} = \begin{bmatrix} 1 & -2 & 1 & & & & & O \\ -2 & 5 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ O & & & & & 1 & -4 & 6 \end{bmatrix}.$$

Thus, using the minimal eigenvalue of this matrix we obtain the best constant $A_{n,2} = (\lambda_{\min}(C_{n,2}))^{-1/2}$. These constants, for $n = 4(1)10$ are presented in Table 4.1, with seven decimal digits (see G. V. Milovanović [40]). Numbers in parentheses indicate decimal exponents. For $n = 2$ and $n = 3$, we have exact values: $A_{2,2} = 1$ and $A_{3,2} = (3+2\sqrt{2})^{1/2}$, respectively.

Table 4.1

n	$\lambda_{\min}(C_{n,2})$	$A_{n,2}$
4	5.1590055(-2)	4.4026788
5	2.0635581(-2)	6.9613208
6	9.8237813(-3)	10.0892912
7	5.2614253(-3)	13.7863181
8	3.0685649(-3)	18.0522919
9	1.9090449(-3)	22.8871610
10	1.2494144(-3)	28.2908989

Remark 4.1. The last problem could be interpreted as an extremal problem of Wirtinger's type

$$\sum_{i=2}^{n} y_i^2 \le A_{n,2}^2 \sum_{i=2}^{n}(\Delta^2 y_i)^2, \quad y_{n+1} = y_{n+2} = 0\ .$$

Similar problems were considered in Fan, Taussky and Todd [17], G. V. Milovanović and I. Ž. Milovanović [42], and others.

Remark 4.2. In 1965, Shampine [60] proved that

$$\frac{1}{n^4}A_{n,2}^2 = \frac{1}{\nu_0^4} - R, \quad 0 < R \le \frac{1}{2n} - \frac{1}{6n^2}\ ,$$

where $\nu_0 = 1.8751041\ldots$ (ν_0 is the smallest root of the equation $1 + \cos\nu\cosh\nu = 0$).

Dörfler [11] gave upper and lower bounds for the constant $A_{n,k}$, in the case of the Laguerre measure $d\lambda(t) = \exp(-t)dt$, and investigated its behaviour as $n \to +\infty$. He proved that

$$\frac{1}{n-k+1}\sum_{\nu=0}^{n-k}\binom{n-\nu}{k}^2 \le A_{n,k}^2 \le \sum_{\nu=0}^{n-k}(\nu+1)\binom{n-1-\nu}{k-1}^2$$

and

$$\frac{1}{k!\sqrt{2k+1}} \le \liminf_{n\to+\infty}\frac{A_{n,k}}{n^k} \le \limsup_{n\to+\infty}\frac{A_{n,k}}{n^k} \le \frac{1}{(k-1)!\sqrt{2k(2k-1)}}\ .$$

A case with a special even weight function (including Gegenbauer weight) was considered by Milovanović [40].

5. Extremal Problems on a Circle and Some Opposite Inequalities

Another type of extremal problems goes back to S. N. Bernstein [8]. His first result in this field is the inequality

$$\|P'\| \le n\|P\| \quad (P \in \mathcal{P}_n)\ , \tag{5.1}$$

where $\|f\| = \max_{|z|=1} |f(z)|$. The equality in (5.1) is attained for $P(z) = cz^n$, $c =$ const.

In a restricted polynomial class, an improvement of (5.1) was conjectured by P. Erdős and later proved by Lax [29]:

Theorem 5.1. If $P(\in \mathcal{P}_n)$ does not vanish in $|z| < 1$, then

$$\|P'\| \le \frac{n}{2}\|P\|\ . \tag{5.2}$$

A simple proof of this theorem was given by Aziz and Mohammad [5].

Using Theorem 5.1, Ankeny and Rivlin [1] showed that if $P(\in \mathcal{P}_n)$ does not vanish in $|z| < 1$, then

$$\max_{|z|=R} |P(z)| \le \frac{R^n + 1}{2}\|P\|, \quad R > 1\ , \tag{5.3}$$

with equality case if $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. The case of polynomials of degree n having all their zeros in $|z| \ge K \ge 1$ was considered by Aziz and Mohammad [6].

Using a restricted class of polynomials, Turán [63] proved an opposite inequality:

Theorem 5.2. If P is a polynomial of degree n having all its zeros in $|z| \le 1$, then

$$\|P'\| \ge \frac{n}{2}\|P\|\ . \tag{5.4}$$

The result is best possible and equality in (5.4) holds for $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$.

It is interesting to see that in (5.2), as well as in (5.4), the equality holds for those polynomials of degree n which have all their zeros on $|z| = 1$.

Turán [63] also proved an opposite inequality, for polynomials $P \in \mathcal{P}_n$ having all their zeros in $[-1, 1]$,

$$\|P'\|_\infty > \frac{\sqrt{n}}{6}\|P\|_\infty\ , \tag{5.5}$$

where $\|f\|_\infty = \max_{-1\le t\le 1} |f(t)|$. The constant $\sqrt{n}/6$ is not best possible.

Turán's inequalities (5.4) and (5.5) have been generalized and extended by many mathematicians in several different ways. We will give a few of them.

Inequality (5.5) was sharpened by Erőd [16], who obtained

$$\|P'\|_\infty \ge B_n\|P\|_\infty\ , \tag{5.6}$$

where $B_2 = 1$, $B_3 = 3/2$, and

$$B_n = \begin{cases} \frac{n}{\sqrt{n-1}}\left(1 - \frac{1}{n-1}\right)^{(n-2)/2}, & n = 2k, \\ \frac{n^2}{(n-1)\sqrt{n+1}}\left(1 - \frac{\sqrt{n+1}}{n-1}\right)^{(n-3)/2}\left(1 + \frac{1}{\sqrt{n+1}}\right)^{(n-1)/2}, & n = 2k+1, \end{cases}$$

where $k = 2, 3, \ldots$.

Exactly, equality in (5.6) is attained for $P(t) = (1-t)^n$, if $n = 1, 2, 3$, and for $P(t) = (1-t)^{n-[n/2]}(1+t)^{[n/2]}$, if $n \ge 4$.

Malik [32] proved the following result (see also Govil and Rahman [21]):

Theorem 5.3. If P is a polynomial of degree n, with $\|P(z)\| \le 1$ on $|z| \le 1$ and P has no zero in the disk $|z| < K$, $K \ge 1$, then for $|z| \le 1$,

$$|P'(z)| \le \frac{n}{1+K}\ . \tag{5.7}$$

The result is best possible and equality holds for

$$P(z) = \left(\frac{z+K}{1+K}\right)^n .$$

If P is a polynomial of degree n with $\|P\|=1$ and P has all its zeros in the disk $|z|\le k\le 1$, then $Q(z)=z^nP(1/z)$ satisfies the hypothesis of Theorem 5.3, with $K=1/k$. Since

$$Q'(z)=nz^{n-1}P(1/z)-z^{n-2}P'(1/z)\ ,$$

Malik [32] concluded that

$$\max_{|z|=1}|P'(z)|\ge n-\max_{|z|=1}|Q'(z)|\ge n-\frac{n}{1+1/k}\ ,$$

i.e.

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{1+k}\ . \tag{5.8}$$

The equality in (5.8) is attained for $P(z)=(z+k)^n/(1+k)^n$.

A simple and direct proof of this result was given by Govil [20]. Namely, if $P(z)=a_n\prod_{\nu=1}^n(z-z_\nu)$ is a polynomial of degree n having all its zeros in $|z|\le k\le 1$, then

$$\left|\frac{P'(e^{i\theta})}{P(e^{i\theta})}\right|\ge \operatorname{Re}\left(e^{i\theta}\frac{P'(e^{i\theta})}{P(e^{i\theta})}\right)=\sum_{\nu=1}^n\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-z_\nu}\right)\ge\sum_{\nu=1}^n\frac{1}{1+k}\ ,$$

i.e.

$$\left|P'(e^{i\theta})\right|\ge\frac{n}{1+k}\left|P(e^{i\theta})\right|,$$

where θ is real. Choosing θ such that $|P(e^{i\theta})|=\|P\|=\max_{|z|=1}|P(e^{i\theta})|$, we get

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{1+k}\max_{|z|=1}|P(z)|\ .$$

The above argument does not hold for $k>1$ for then $\operatorname{Re}(e^{i\theta}/(e^{i\theta}-z_\nu))$ may not be greater than or equal to $1/(1+k)$.

Govil [20] also proved the following result:

Theorem 5.4. If $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n with $\|P\|=\max_{|z|=1}|P(e^{i\theta})|=1$ and P has all its zeros in the disk $|z|\le K$, $K\ge 1$, then

$$\|P'\|\ge\frac{n}{1+K^n}\ . \tag{5.9}$$

The result is best possible with equality for the polynomial

$$P(z)=\frac{z^n+K^n}{1+K^n}\ . \tag{5.10}$$

For $K>1$, the extremal polynomial turns out to be of the form (5.10), whereas for $K<1$ it has the form $(z+K)^n/(1+K)^n$. Thus 1 is a critical value of this parameter for the problem under consideration, and one should not expect the same kind of reasoning to work both for $K<1$ and for $K>1$.

The following results were given by Govil, Rahman and Schmeisser [22]:

Theorem 5.5. If $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\ge K\ge 1$, then

$$\|P'\|\le n\frac{n|a_0|+K^2|a_1|}{(1+K^2)n|a_0|+2K^2|a_1|}\|P\|\ ;$$

furthermore

$$\|P'\|\le\frac{n}{1+K}\cdot\frac{(1-|\lambda|)(1+K^2|\lambda|)+K(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-K+K^2+K|\lambda|)+K(n-1)|\mu-\lambda^2|}\|P\|\ ,$$

where

$$\lambda=\frac{K}{n}\frac{a_1}{a_0}\quad\text{and}\quad\mu=\frac{2K^2}{n(n-1)}\frac{a_2}{a_0}\ .$$

Theorem 5.6. If $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\le k\le 1$, then

$$\|P'\|\ge n\frac{n|a_n|+|a_{n-1}|}{(1+k^2)n|a_n|+2|a_{n-1}|}|P\|\ ;$$

furthermore

$$\|P'\|\ge\frac{n(1-|\omega|)(1+k^2|\omega|)+k(n-1)|\Omega-\omega^2|}{1+k(1-|\omega|)(1-k+k^2+k|\omega|)+k(n-1)|\Omega-\omega^2|}\|P\|\ ,$$

where

$$\omega=\frac{1}{nk}\frac{a_{n-1}}{a_n}\quad\text{and}\quad\Omega=\frac{2}{n(n-1)k^2}\frac{a_{n-2}}{a_n}\ .$$

In this connection, E. B. Saff formulated the following problem: Let

$$P(z) = \prod_{\nu=1}^{n} (z - z_\nu) \tag{5.11}$$

be a polynomial having all its zeros in $\operatorname{Re} z \geq 1$. Is it true that

$$\|P'\| \leq \sum_{\nu=1}^{n} \frac{1}{1 + \operatorname{Re} z_\nu} \|P\| \ ?$$

Here equality must hold if in addition the zeros are all real.

Giroux, Rahman and Schmeisser [18] solved completely this problem only for polynomials of degree $n \leq 2$. Namely, for $n = 1$ and $n = 2$, they proved that the answer in the above problem is affirmative. Also, they considered the inequality

$$\|P'\| \leq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \|P\| \tag{5.12}$$

and its opposite inequality.

Theorem 5.7. If the polynomial P in (5.11) is real for real z, then (5.12) holds, provided all the zeros lie in $D = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0, \ |z| \geq 1\}$.

Theorem 5.8. Under conditions $|z_\nu| \leq 1$, $\nu = 1, \ldots, n$, the opposite inequality of (5.12) is valid. There is equality if the zeros are all positive.

Aziz [2] obtained a generalization of Theorem 5.8.

Theorem 5.9. Under conditions $|z_\nu| \leq K$, $\nu = 1, \ldots, n$, we have

$$\|P'\| \geq \frac{2}{1 + K^n} \sum_{\nu=1}^{n} \frac{K}{K + |z_\nu|} \|P\| \, . \tag{5.13}$$

The result is best possible and equality in (5.13) holds for $P(z) = z^n + K^n$.

Inequality (5.13) is also a refinement of (5.9).

Aziz [2] also mentioned that the inequality (5.12) holds, for $|z_\nu| \geq 1$ ($\nu = 1, \ldots, n$), provided $|P(z)|$ and $|P'(z)|$ become maximum at the same point on $|z| = 1$.

Referring to Turán inequality (5.4), Aziz [2, Thm. 5] proved:

Theorem 5.10. If P is a polynomial of degree n satisfying $P(z) \equiv z^n P(1/z)$, then (5.4) holds. The result is sharp with equality for $P(z) = z^n + 1$.

Aziz and Dawood [4] proved the following results:

Theorem 5.11. If P is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)|$$

and

$$\min_{|z|=R>1} |P(z)| \geq R^n \min_{|z|=1} |P(z)| \, .$$

Both the estimates are sharp with equality for $P(z) = m e^{i\alpha} z^n$, $m > 0$.

Theorem 5.12. If P is a polynomial of degree n which does not vanish in the disk $|z| < 1$, then

$$\|P'\| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\} \, . \tag{5.14}$$

The result is best possible and equality in (5.14) holds for the polynomial $P(z) = \alpha z^n + \beta$, where $|\beta| \geq |\alpha|$.

Theorem 5.13. If P is a polynomial of degree n which does not vanish in the disk $|z| < 1$, then

$$\max_{|z|=R>1} |P| \leq \left(\frac{R^n + 1}{2} \right) \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{2} \right) \min_{|z|=1} |P(z)| \, . \tag{5.15}$$

The result is best possible and equality in (5.15) holds for $P(z) = \alpha z^n + \beta$, where $|\beta| \geq |\alpha|$.

This result is a generalization of the inequality (5.3) and it can be obtained as an application of Theorem 5.12.

Theorem 5.14. If P is a polynomial of degree n which has all its zeros in $|z| \leq 1$, then

$$\|P\| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\} \, . \tag{5.16}$$

The result is best possible and equality in (5.16) holds for $P(z) = \alpha z^n + \beta$, where $|\beta| \le |\alpha|$.

A refinement of the Bernstein's inequality (5.1) was given by Aziz [3].

There are generalizations of the above results in some mixed norms.

Using the uniform norm on the unit circle, $||f|| = \max_{|z|=1} |P(z)|$, and the integral L^q-norm,

$$||f||_q = \left(\int_0^{2\pi} |f(e^{i\theta})|^q \, d\theta \right)^{1/q},$$

Saff and Sheil-Small [56] proved the following result:

Theorem 5.15. Let P be a polynomial of degree n having all its zeros on the unit circle. Then for each $q > 0$, we have

$$||P||_q \le \frac{(A_q)^{1/q}}{2} ||P||, \tag{5.17}$$

where

$$A_q = 2^{q+1}\sqrt{\pi}\,\Gamma(\tfrac{1}{2}q + \tfrac{1}{2}) \big/ \Gamma(\tfrac{1}{2}q + 1). \tag{5.18}$$

The result is best possible and equality in (5.17) holds if and only if $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$.

In order to obtain (5.17), Saff and Sheil-Small established the inequality

$$n||P||_q \le (A_q)^{1/q} ||P'|| \tag{5.19}$$

for polynomials having all their zeros on $|z| = 1$, and then in (5.19) used Theorem 5.1.

An analogous result of the Turán inequality (5.4) was proved by Malik [33]:

Theorem 5.16. If P is a polynomial of degree n having all its zeros in $|z| \le 1$, then for each $q > 0$

$$n||P||_q \le (A_q)^{1/q} ||P'||, \tag{5.20}$$

where A_q is given by (5.18). The result is best possible and equality in (5.20) holds for $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$.

Making $q \to +\infty$ in (5.20), it gets the Turán inequality (5.2).

Analogous inequalities of (5.5), i.e. (5.6), in L^2-norm had been considered by Varma [67–72].

Let W_n be the set of all algebraic polynomials of degree n whose zeros are all real and lie inside $[-1, 1]$, and let $||f||^2 = ||f||_2^2 = \int_{-1}^{1} f(t)^2 \, dt$.

Theorem 5.17. Let $P \in W_n$ and $n = 2m$; then

$$||P'||^2 \ge \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)} \right) ||P||^2,$$

where equality holds iff $P(t) = (1-t^2)^m$. Moreover, if $n = 2m-1$, then

$$||P'||^2 \ge \left(\frac{n}{2} + \frac{3}{4} + \frac{5}{4(n-2)} \right) ||P||^2, \quad n \ge 3,$$

where equality holds iff $P(t) = (1-t)^{m-1}(1+t)^m$ or $P(t) = (1-t)^m (1+t)^{m-1}$.

This theorem has been proved by Varma [72] and it gives an improvement of one of his earlier result (see Varma [67–69]. A similar problem in L^p-norm ($p \ge 1$) on $(-1, 1)$ was considered by Zhou [74].

Varma [72] also proved:

Theorem 5.18. Let $P \in W_n$, subject to the condition $P(1) = 1$; then

$$||P'||^2 \ge \frac{n}{2} + \frac{1}{8} + \frac{1}{8(2n-1)}, \quad n \ge 1, \tag{5.21}$$

where equality holds for $P(t) = ((1+t)/2)^n$.

Inequality (5.21) is an improvement over $||P'||^2 > n/4$, given by Szabados and Varma [62].

In 1979, Varma [70] proved the three following results:

Theorem 5.19. Let $||f||^2 = \int_{-1}^{1} (1-t^2) f(t)^2 \, dt$ and $P \in W_n$. Then for $n \ge 2$, we have

$$||P'||^2 \ge \left(\frac{n}{2} + \frac{1}{4} - \frac{1}{4(n+1)} \right) ||P||^2.$$

The equality holds for $P(t) = (1-t^2)^m$, $n = 2m$.

Theorem 5.20. Let P be an algebraic polynomial of degree at most n whose roots are all real and lie in the complement of the interval $[-1,1]$. Then we have

$$||P'||^2 \le \frac{n(n+1)(2n+3)}{4(2n+1)} ||P||^2 .$$

The equality holds for $P(t) = (1+t)^n$ or $P(t) = (1-t)^n$. The norm is the same as in the above theorem.

Theorem 5.21. Let P be an algebraic polynomial of degree n whose zeros τ_ν $(\nu = 1, \dots, n)$ all lie in the interval $[0, \infty)$. Let $P(0) = 0$ or

$$\sum_{\nu=1}^{n} \tau_\nu^{-1} \ge \frac{1}{2} ;$$

then

$$||P'||^2 \ge \frac{n}{2(2n-1)} ||P||^2 .$$

The equality holds for $P(t) = t^n$. Here $||f||^2 = \int_0^\infty e^{-t} f(t)^2 \, dt$.

The corresponding extremal problems of Markov's type in L^2-norm had been considered by Varma [71], Milovanović [39], Milovanović and Đorđević [41], G. V. Milovanović and I. Ž. Milovanović [43], Milovanović and Petković [45], etc. Similar problems in L^r-norm, where $r \in \mathbb{N}$, have been recently investigated by Guessab and Milovanović [24], and Guessab, Milovanović and Arino [25].

Recently, Babenko and Pichugov [7] proved the following inequality:

$$||T||_\infty \ge \sqrt{\frac{n}{2}} \left(1 - \frac{1}{2n}\right)^{n-1/2} ||T||_\infty , \tag{5.22}$$

for trigonometric polynomials of degree n, with all real zeros. Here $||f||_\infty = \max_{0 \le t \le 2\pi} |f(t)|$. The equality in (5.22) holds for

$$T(t) = C\left(\sin \frac{t-\gamma}{2}\right)^{2n} \quad (\forall \gamma \in \mathbb{R}, C \ne 0) .$$

Some generalizations of these results in integral metrics were obtained by Tyrygin [65–66].

References

1. N. C. Ankeny and T. J. Rivlin, *On a theorem of S. Bernstein*, Pacific J. Math. 5(1955) 849–852.
2. A. Aziz, *Inequalities for the derivative of a polynomial*, Proc. Amer. Math. Soc. 89(1983) 259–266.
3. A. Aziz, *A refinement of an inequality of S. Bernstein*, J. Approx. Theory 144(1989) 226–235.
4. A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory 54(1988) 306–313.
5. A. Aziz and Q. G. Mohammad, *Simple proof of a theorem of Erdős and Lax*, Proc. Amer. Math. Soc. 80(1980) 119–122.
6. A. Aziz and Q. G. Mohammad, *Growth of polynomials with zeros outside a circle*, Proc. Amer. Math. Soc. 81(1981) 549–553.
7. V. F. Babenko and S. A. Pichugov, *Exact inequality for the derivative of a trigonometric polynomial having only real zeros*, Mat. Zametki 39(1986) 330–336.
8. S. N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mémoires de l'Académie Royale de Belgique 4(2) (1912) 1–103.
9. R. P. Boas, *Inequalities for the derivatives of polynomials*, Math. Mag. 42 (1969) 165–174.
10. P. Dörfler, *New inequalities of Markov type*, SIAM J. Math. Anal. 18 (1987) 490–494.
11. P. Dörfler, *A Markov type inequality for higher derivatives of polynomials*, Monatsh. Math 109(1990) 113-122.
12. R. J. Duffin and A. C. Schaeffer, *A refinement of an inequality of the brothers Markoff*, Trans. Amer. Math. Soc. 50(1941) 517–528.
13. A. Durand, *Quelques Aspects de la Theorie Analytique des Polynomes. I et II*, Université de Limoges (1984).
14. V. K. Dzyadyk, *On a constructive characteristic of functions satisfying the Lipschitz condition* α $(0 < \alpha < 1)$, *on a finite segment of the real axis*, Izv. Akad. Nauk SSSR Ser. Mat. 20(1956) 623–642 (Russian).
15. V. K. Dzyadyk, *Introduction to Uniform Approximation Theory by Polynomials*, Nauka, Moscow, 1977 (Russian).
16. J. Erőd, *Bizonyos polinomok maximumának*, Mat. Fiz. Lapok 46(1939) 58–82.
17. K. Fan, O. Taussky and J. Todd, *Discrete analogs of inequalities of Wirtinger*, Monatsh. Math. 59(1955) 73–79.
18. A. Giroux, Q. I. Rahman and G. Schmeisser, *On Bernstein's inequality*, Canad. J. Math. 31(1979) 347–353.
19. P. Goetgheluck, *Généralisation de l'inégalité de Markov*, J. Approx. Theory 38(1983) 167–171.
20. N. K. Govil, *On the derivative of a polynomial*, Proc. Amer. Math. Soc. 41(1973) 543–546.

21. N. K. Govil and Q. I. Rahman, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math. Soc. **137**(1969) 501–517.
22. N. K. Govil, Q. I. Rahman and Schmeisser, *On the derivative of a polynomial*, Illinois J. Math. **23**(1979) 319–329.
23. N. K. Govil, Q. I. Rahman and Schmeisser, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math. Soc. **137**(1969) 501–517.
24. A. Guessab and G. V. Milovanović, *An extremal problem for polynomials with nonnegative coefficients. IV*, Math. Balkanica **3**(1989) 142–148.
25. A. Guessab, G. V. Milovanović and O. Arino, *Extremal problems for nonnegative polynomials in L^p norm with generalized Laguerre weight*, Facta. Univ. Ser. Math. Inform. **3**(1988) 1–8.
26. V. I. Ivanov, *Some extremal properties of polynomials and inverse inequalities in approximation theory* Trudy Mat. Inst. Steklov **145**(1979) 79–110 (Russian).
27. M. A. Lachance, *Bernstein and Markov inequalities for constrained polynomials*, Rational Approximation and Interpolation - Proceedings, Tampa, Florida, 1983 (P. R. Graves-Morris, E. B. Saff and R. S. Varga, eds.), Lect. Notes Math. No. 1105, Springer Verlag, Berlin-Heidelberg-New York (1984) 125–135.
28. M. A. Lachance, E. B. Saff and R. S. Varga, *Bounds for incomplete polynomials vanishing at both endpoints of an interval*, Constructive Approaches to Mathematical Models (C. V. Coffman and G. J. Fix, eds.), Academic Press, New York (1979) 421–437.
29. P. D. Lax, *Proof of a conjecture of P. Erdős on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50**(1944) 509–513.
30. G. G. Lorentz, *Approximation of functions*, Holt, Rinehart and Winston, New York (1966).
31. G. G. Lorentz, *Approximation by incomplete polynomials (problems and results)*, Padé and Rational Approximation: Theory and Applications, Proceedings of an international symposium (Tampa, 1976) (E. B. Saff and R. S. Varga, eds.), Academic Press, London (1977) 289–302.
32. M. A. Malik, *On the derivative of a polynomial*, J. London. Math. Soc. **1**(1969) 57–60.
33. M. A. Malik, *An integral mean estimate for polynomials*, Proc. Amer. Math. Soc. **91**(1984) 281–284.
34. Dzh. M. Mamedhanov, *On some subclasses of entire and rational functions*, Special Problems in Function Theory, Baku (1977) 111–154 (Russian).
35. A. A. Markov, *On a problem of D. I. Mendeleev* Zap. Imp. Akad. Nauk, St. Petersburg **62**(1889) 1–24 (Russian).
36. V. A. Markov, *On functions deviating least from zero in a given interval* Izdat. Imp. Akad. Nauk, St. Petersburg, 1892 (Russian). [German transl. Math. Ann. **77**(1916) 218–258.]

37. G. Meinardus, *Approximation of Functions: Theory and Numerical Methods*, Springer–Verlag, New York (1967).
38. D. Mendeleev, *Investigation of aqueous solutions based on specific gravity*, St. Petersburg (1887) (Russian).
39. G. V. Milovanović, *An extremal problem for polynomials with nonnegative coefficients*, Proc. Amer. Math. Soc. **94**(1985) 423–426.
40. G. V. Milovanović, *Various extremal problems of Markov's type for algebraic polynomials* Facta. Univ. Ser. Math. Inform. **2**(1987) 7–28.
41. G. V. Milovanović and R. Ž. Đorđević, *An extremal problem for polynomials with nonnegative coefficients. II*, Facta. Univ. Ser. Math. Inform. **1**(1986) 7–11.
42. G. V. Milovanović and I. Ž. Milovanović, *On discrete inequalities of Wirtinger's type*, J. Math. Annal. Appl. **88**(1982) 378–387.
43. G. V. Milovanović and I. Ž. Milovanović, *An extremal problem for polynomials with nonnegative coefficients. III*, Constructive Function Theory '87 (Varna, 1987), Bulgar. Acad. Sci., Sofia (1988) 315–321.
44. G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *On some extremal problems for algebraic polynomials in L^r norm*, Generalized Functions and Convergence (P. Antosik and A. Kamiński, eds.), World Scientific, Singapore (1990) 343–354.
45. G. V. Milovanović and M. S. Petković, *Extremal problems for Lorentz classes of nonnegative polynomials in L^2 metric with Jacobi weight*, Proc. Amer. Math. Soc. **102**(1988) 283–289.
46. L. Mirsky, *An inequality of the Markov-Bernstein type for polynomials*, SIAM J. Math. Anal. **14**(1983) 1004–1008.
47. R. Pierre and Q. I. Rahman, *On a problem of Turán about polynomials*, Proc. Amer. Math. Soc. **56**(1976) 231–238.
48. R. Pierre and Q. I. Rahman, *On a problem of Turán about polynomials II*, Canad. J. Math. **33**(1981) 701–733.
49. R. Pierre, Q. I. Rahman and G. Schmeisser, *On polynomials with curved majorants*, J. Approx. Theory **57**(1989) 211–222.
50. Q. I. Rahman, *On a problem on Turán about polynomials with curved majorants*, Trans. Amer. Math. Soc. **163**(1972) 447-455.
51. Q. I. Rahman, *Addendum to "On a problem on Turán about polynomials with curved majorants"*, Trans. Amer. Math. Soc. **168**(1972) 517–518.
52. Q. I. Rahman and Q. G. Mohammad, *Remarks on Schwarz's lemma*, Pacific J. Math. **23**(1967) 139–142.
53. Q. I. Rahman and G. Schmeisser, *Les inégalités de Markoff et de Bernstein*, Presses Univ. Montréal, Montréal, Québec (1983).
54. Q. I. Rahman and G. Schmeisser, *Markov-Duffin-Schaeffer inequality for polynomials with a circular majorant*, Trans. Amer. Math. Soc. **310**(1988) 693–702.

55. Th. M. Rassias, *On certain properties of polynomials and their derivative*, Topics in Mathematical Analysis (Th. M. Rassias, ed.), World Scientific, Singapore (1989) 758–802.
56. E. B. Saff and T. Sheil-Small, *Coefficient and integral mean estimates for algebraic and trigonometric polynomials with restricted zeros*, J. London. Math. Soc. 9(1974) 16–22.
57. A. C. Schaeffer and R. J. Duffin *On some inequalities of S. Bernstein and W. Markoff for derivatives of polynomials*, Bull. Amer. Math. Soc. **44**(1938) 289–297.
58. E. Schmidt, *Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum* Math. Ann. **119**(1944) 165–204.
59. L. F. Shampine, *Asymptotic L_2 inequalities of Markoff type*, Ph. D. Thesis, Calif. Inst. of Tech., Pasadena (1964).
60. L. F. Shampine, *Some L_2 Markoff inequalities*, J. Res. Nat. Bur. Standards, Sect. B **69**(1965) 155–158.
61. L. F. Shampine, *An inequality of E. Schmidt*, Duke. Math. J. **33**(1966) 145–150.
62. J. Szabados and A. K. Varma, *Inequalities for derivatives of polynomial having real zeros*, Approximation Theory III (E. W. Cheney, ed.) Academic Press, New York (1980) 881–887.
63. P. Turán, *Über die Ableitung von Polynomen*, Compositio Math. **7**(1939) 85–95.
64. P. Turán, *Remark on a theorem of Erhard Schmidt*, Mathematica (Cluj) **2**(25) (1960) 373–378.
65. I. Ya. Tyrygin, *Inequalities of P. Turán type in mixed integral metrics*, Dokl. Akad. Nauk Ukrain. SSR Ser. A (9)**1988** 14–17 (Russian).
66. I. Ya. Tyrygin, *On inequalities of Turán type in some integral metrics*, Ukrain. Mat. Zh. **40**(1988) 256–260 (Russian).
67. A. K. Varma, *An analogue of some inequalities of P. Turán concerning algebraic polynomials having all zeros inside* $[-1, 1]$, Proc. Amer. Math. Soc. **55**(1976) 305–309.
68. A. K. Varma, *An analogue of some inequalities of P. Erdős and P. Turán concerning algebraic polynomials satisfying certain conditions*, Fourier Analysis and Approximation Theory, Vol. II, Budapest 1976, Colloq. Math. Soc. János Bolyai **19**(1978) 877–890.
69. A. K. Varma, *An analogue of some inequalities of P. Turán concerning algebraic polynomials having all zeros inside* $[-1, 1]$, *II*, Proc. Amer. Math. Soc. **69**(1978) 25–33.
70. A. K. Varma, *Some inequalities of algebraic polynomials having real zeros*, Proc. Amer. Math. Soc. **75**(1979) 243–250.
71. A. K. Varma, *Derivatives of polynomials with positive coefficients*, Proc. Amer. Math. Soc. **83**(1981) 107–112.
72. A. K. Varma, *Some inequalities of algebraic polynomials having all zeros inside* $[-1, 1]$, Proc. Amer. Math. Soc. **88**(1983) 227–233.

73. E. V. Voronovskaja, *The functional method and its applications*, Trans. Math. Monographs, **28**, Amer. Math. Soc., Providence (1970).
74. S. P. Zhou *On the Turán inequality in L_p-norm* J. Hangzhou Univ. **11**(1984) 28–33 (Chinese, English summary).

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