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Orthogonal polynomials and Gauss quadratures associated with one integral of the Ramanujan type

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Abstract

Orthogonal polynomials related to the weight function

$$w^{(\alpha)}(x) = \frac{(1-x^2)^\alpha}{\pi^2 + 4\operatorname{arctanh}^2 x} \quad (\alpha \geq -1)$$

on $(-1,1)$, as well as the corresponding quadrature formulas of Gaussian type, are considered. For some particular values of the parameter α, the coefficients in the three-term recurrence relation for these orthogonal polynomials are obtained in an explicit form as fractions. Numerical examples are included. The case $\alpha = -1$ can be connected with one Ramanujan integral recently considered by Gautschi and Milovanović [Math. Comp. 93 (347), 1297–1308, 2024].

Keywords: Orthogonal polynomials, three-term recurrence relation, associated Legendre polynomials, weight functions, moments, modification of the weight function, Ramanujan integral, Gaussian quadrature

2020 MSC: 33C45, 33C47, 33D45, 33F10, 41A55, 65D32

1. Introduction

Let $\mathcal{P}$ be the space of all algebraic polynomials defined on $\mathbb{R}$ and $\mathcal{P}_n$ its subspace $\mathcal{P}_n \subset \mathcal{P}$ containing all polynomials of degree at most n ($n \in \mathbb{N}$). For a given weight function $w(x)$ on $[a,b]$ ($a < b$), for which all moments $\mu_k = \int_a^b x^k w(x)\,\mathrm{d}x$, $k \in \mathbb{N}_0$, exist and are finite, and $\mu_0 > 0$, there exists a unique sequence of monic polynomials $\{\pi_k(x)\}_{k=0}^{\infty}$ orthogonal on $[a,b]$, i.e.,

$$(\pi_k, \pi_n) = \int_a^b \pi_k(x)\pi_n(x)w(x)\,\mathrm{d}x = \|\pi_n\|^2\delta_{k,n}, \tag{1.1}$$

where $\delta_{k,n}$ is Kronecker's delta.

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1.1. Orthogonal polynomials and Gaussian quadratures

Orthogonal polynomials $\{\pi_k(x)\}_{k=0}^{\infty}$ satisfy the three-term recurrence relation

$$\pi_{k+1}(x) = (x - \alpha_k)\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, 2, \ldots, \tag{1.2}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$, where $\alpha_k = \alpha_k(w)$ and $\beta_k = \beta_k(w)$ are recursion coefficients. The coefficient β_0 may be arbitrary, but is conveniently defined by $\beta_0 = \mu_0 = \int_a^b w(x)\,\mathrm{d}x$.

Also, for a such kind of the weight functions, there exists the n-point Gauss-Christoffel quadrature rule for each $n \in \mathbb{N}$,

$$\int_a^b f(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{n} A_\nu^{(n)} f(x_\nu^{(n)}) + R_n(f), \tag{1.3}$$

which is exact for all polynomials of degree $\leq 2n - 1$ ($f \in \mathcal{P}_{2n-1}$). The quadrature nodes $x_\nu^{(n)}$, $\nu = 1, \ldots, n$, in (1.3) are eigenvalues of the Jacobi matrix

$$J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathrm{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathrm{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}, \tag{1.4}$$

and the first components of the corresponding normalized eigenvectors $\mathbf{v}_\nu = [v_{\nu,1} \ \ldots \ v_{\nu,n}]^{\mathrm{T}}$ (with $\mathbf{v}_\nu^{\mathrm{T}}\mathbf{v}_\nu = 1$) give the weight coefficients (Christoffel numbers) $A_\nu^{(n)} = \beta_0 v_{\nu,1}^2$, $\nu = 1, \ldots, n$. Such a construction of the Gauss-Christoffel quadrature rule (1.3) is done by the Golub-Welsch algorithm (*cf.* [6]).

Unfortunately, these recursion coefficients α_k and β_k in (1.2) are known explicitly only for some narrow classes of weight functions. Among them the most popular are the so-called classical weight functions:

- the Jacobi weight $(1 - x)^\alpha(1 + x)^\beta$, $\alpha, \beta > -1$, on $(-1, 1)$;
- the generalized Laguerre weight $x^\alpha \mathrm{e}^{-x}$, $\alpha > -1$, on $(0, +\infty)$;
- the Hermite weight function e^{-x^2} on $(-\infty, +\infty)$.

In general, for arbitrary weight functions, precisely for the strongly non-classical weights, the recursion coefficients in (1.2) must be constructed numerically, but such processes are usually ill-conditioned. Some of the available methods for such numerical construction are: method of (modified) moments, the discretized Stieltjes–Gautschi procedure, and the Lanczos algorithm (see [2, 4, 8, 11]). Fortunately, advances in recent decades in *computer architecture*, and especially in *variable precision arithmetic*, have made it possible direct generation of the recurrence coefficients α_k and β_k, using the original Chebyshev method of moments, but only with sufficiently high precision. Furthermore, the progress in *symbolic computation* additionally has increased this possibility. The corresponding software for such numerical and symbolic computation is now available: Gautschi's package `SOPQ` in Matlab, and our Mathematica package `OrthogonalPolynomials` (see [1] and [13]). This Mathematica package is downloadable from the Web Site (Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia):

`http://www.mi.sanu.ac.rs/~gvm/.`

1.2. Associated polynomials

The same recursion coefficients α_k and β_k as in (1.2) appear also in the case of the so-called associated (or numerator) polynomials, defined by the same weight function as follows

$$\sigma_k(x) = \int_a^b \frac{\pi_k(x) - \pi_k(t)}{x - t} w(t)\,\mathrm{d}t, \quad k \geq 0,$$

but with starting values $\sigma_0(x) = 0$ and $\sigma_{-1}(x) = -1$ (see [8, pp. 111–114]). Here for the monic associated polynomial $\widehat{\sigma}_{k+1}(x)$, we use the notation $\pi_k^{(0)}(x)$

$$\pi_k^{(0)}(x) = \frac{1}{\beta_0}\int_a^b \frac{\pi_{k+1}(x) - \pi_{k+1}(t)}{x-t} w(t)\,\mathrm{d}t, \quad k \geq 0, \tag{1.5}$$

where $\deg(\pi_k^{(0)}) = k$ and $\beta_0 = \mu_0 = \int_a^b w(x)\,\mathrm{d}x$, so that

$$\pi_{k+1}^{(0)}(x) = (x - \alpha_{k+1})\pi_k^{(0)}(x) - \beta_{k+1}\pi_{k-1}^{(0)}(x), \quad k = 0, 1, \ldots, \tag{1.6}$$

with $\pi_0^{(0)}(x) = 1$ and $\pi_{-1}^{(0)}(x) = 0$.

1.3. One Ramanujan integral

In a joint paper with Walter Gautschi [5] we recently considered the Ramanujan integral

$$I_R(t) = \int_0^{+\infty} \frac{1}{x}\frac{\mathrm{e}^{-tx}}{\pi^2 + \log^2 x}\,\mathrm{d}x, \quad t > 0, \tag{1.7}$$

as well as its derivatives. Thanks to the representation of this integral in the form

$$I_R(t) = \mathrm{e}^t \int_0^1 \frac{\Gamma(a,t)}{\Gamma(a)}\,\mathrm{d}a, \quad t > 0,$$

where $\Gamma(a,t)$ is the upper incomplete gamma function we have shown that (1.7) can be efficiently calculated using the classical Gaussian quadrature on the interval $[0,1]$, as well as that $I_R(t)$ is a completely monotone function on $(0,+\infty)$, that is, $(-1)^k I_R^{(k)}(t) > 0$ on $(0,\infty)$ for all $k = 0, 1, 2, \ldots$. The inspiration for our research was the work of Van E. Wood [14], who considered integrals

$$I_n^k(t) = \int_{-\infty}^{(0+)} \mathrm{e}^{zt} z^{n-1} \log^k z\,\mathrm{d}z, \quad \operatorname{Re} t > 0,$$

occurring in the asymptotic expansions of the solutions of heat conduction problems in regions bounded internally by a circular cylinder, in problems on the flow of fluids through porous media, in electron slowing-down problems, etc. For nonnegative n and negative k, these integrals can be expressed, by means of change of variables and integrations by parts, in terms of derivatives of Ramanujan's integral (1.7).

Remark 1.1. For calculating $I_R(t)$, Wood [14] decomposed it into three integrals $I_R(t) = I_{R_1} + I_{R_2} + I_{R_3}$, where

$$\begin{aligned} I_{R_1} &= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}((\log 2t)/\pi), \\ I_{R_2} &= -\int_0^{1/2} \frac{1-\mathrm{e}^{-v}}{\pi^2 + \log^2(v/t)}\frac{\mathrm{d}v}{v}, \\ I_{R_3} &= -\int_{1/2}^{\infty} \frac{\mathrm{e}^{-v}}{\pi^2 + \log^2(v/t)}\frac{\mathrm{d}v}{v}, \end{aligned}$$

and then used some approximate calculations for each of them.

This paper is devoted to integrals of the form

$$I(f) = \int_0^1 \frac{1}{x}\frac{f(x)}{\pi^2 + \log^2 x}\,\mathrm{d}x, \tag{1.8}$$

which can be transformed to (see Remark 3.3)

$$I(F) = \int_0^1 \frac{2}{1-x^2}\frac{F(x)}{\pi^2 + \log^2 \dfrac{1-x}{1+x}}\,\mathrm{d}x, \tag{1.9}$$

taking $(1-x)/(1+x)$ instead of x, where $F(x) = f((1-x)/(1+x))$. The log-term in (1.9) can be written as the arctanh function, so that for even functions $x \mapsto F(x)$, Eq. (1.9) reduces to

$$I(F) = \int_{-1}^{1} \frac{1}{1-x^2} \frac{F(x)}{\pi^2 + 4\,\mathrm{arctanh}^2 x}\, \mathrm{d}x. \tag{1.10}$$

In this paper our primary task is to get orthogonal polynomials, as well as the corresponding quadratures of Gaussian type, for a more general weight function than one in the previous integral. Namely, we consider an even weight function with the parameter $\alpha \geq -1$,

$$w^{(\alpha)}(x) = \frac{(1-x^2)^\alpha}{\pi^2 + 4\,\mathrm{arctanh}^2 x} \quad \text{on } (-1,1). \tag{1.11}$$

This weight function $w^{(\alpha)}(x)$ for different values of α is displayed in Figure 1.

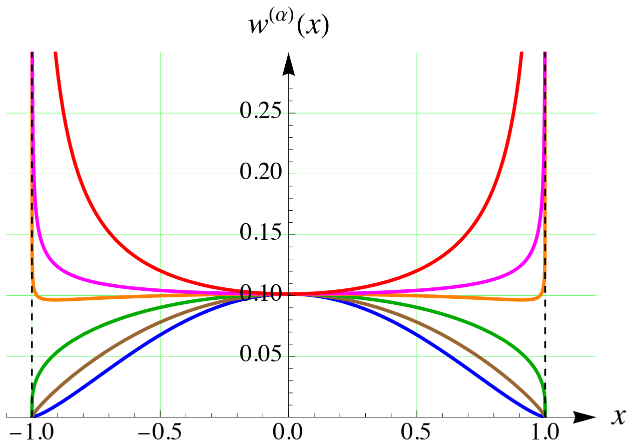

Figure 1. The weight functions $x \mapsto w^{(\alpha)}(x)$ for $\alpha = 1$ (blue), $\alpha = 1/2$ (brown), $\alpha = 0$ (green), $\alpha = -7/20$ (orange), $\alpha = -1/2$ (magenta) and $\alpha = -1$ (red)

The paper is organized as follows. In Section 2 we obtain the three-term recurrence relation for polynomials orthogonal with respect to the weight function (1.11), and the corresponding Gaussian rules in Section 3.

2. A modification of the associated Legendre polynomials

Let π_n be monic Legendre polynomials orthogonal on $[-1,1]$ with respect to the inner product (1.1), with the constant weight function $w(x) = 1$. They satisfy the three-term recurrence relation (1.2), where the recurrence coefficients are $\alpha_k = 0$, $k \in \mathbb{N}_0$, and

$$\beta_0 = \mu_0 = 2, \quad \beta_k = \frac{k^2}{4k^2-1}, \quad k \in \mathbb{N}. \tag{2.1}$$

2.1. *Case $s = 0$*

The monic associated (or numerator) Legendre polynomials are ones whose recursion coefficients are simply those of the Legendre polynomials (2.1) shifted in their indices by 1 (see (1.5) and (1.6)), i.e.,

$$\pi_{k+1}^{(0)}(x) = x\pi_k^{(0)}(x) - \frac{(k+1)^2}{(2k+1)(2k+3)}\pi_{k-1}^{(0)}(x).$$

These polynomials are orthogonal on $[-1,1]$ with respect to the weight function (*cf.* [8, pp. 111–114], [12])

$$w^{(0)}(x) = \frac{1}{\pi^2 + 4\,\mathrm{arctanh}^2 x}.$$

Thus,

$$\beta_0^{(0)} = \mu_0^{(0)} = \int_{-1}^{1} w^{(0)}(x)\,\mathrm{d}x = \frac{1}{6}, \quad \beta_k^{(0)} = \frac{(k+1)^2}{(2k+1)(2k+3)}, \quad k \ge 1.$$

These coefficients for $1 \le k \le 10$ are presented in Figure 2.

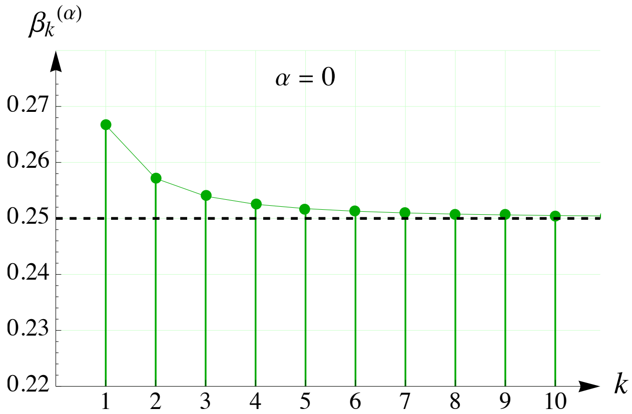

Figure 2. Recurrence coefficients $\beta_k^{(0)}$, $k = 1, 2, \ldots, 10$

Remark 2.1. For an interesting story about the weight function $w^{(0)}(x)$ see [3]. Otherwise, for the moments

$$\mu_k^{(0)} = \int_{-1}^{1} \frac{x^k}{\pi^2 + 4\,\mathrm{arctanh}^2(x)}\,\mathrm{d}x \quad (k \in \mathbb{N}_0),$$

we have $\mu_k^{(0)} = 0$ for odd k. For even k, after changing variables $x = \tanh t$, the moments can be expressed in the form

$$\mu_k^{(0)} = 2\int_0^{+\infty} \frac{\tanh^k t}{\pi^2 + 4t^2} \cdot \frac{\mathrm{d}t}{\cosh^2 t}, \tag{2.2}$$

and this kind of integrals can be calculated using quadrature formulas of Gaussian type with respect to the weight function $t \mapsto 1/\cosh^2 t$ on $(0, +\infty)$ developed in [9] for a fast summation of slowly convergent series (*cf.* [10]). The moment sequence $\{\mu_k^{(0)}\}$ is

$$\Big\{\frac{1}{6}, 0, \frac{2}{45}, 0, \frac{22}{945}, 0, \frac{214}{14175}, 0, \frac{5098}{467775}, 0, \frac{5359534}{638512875}, 0, \frac{12932534}{1915538625},$$
$$0, \frac{2736303958}{488462349375}, 0, \frac{37092982886}{7795859096025}, 0, \frac{132349236090514}{32157918771103125}, 0, \ldots\Big\}.$$

Now, in this paper we first consider two cases, one for $\alpha = 1$ and the other for $\alpha = -1$. In both cases the β-coefficients are fractions, for which we get explicit expressions in the case of $\alpha = -1$. In the last part of this section we give a numerical construction of β-coefficients for $\alpha > -1$ and different from an integer.

2.2. Case $\alpha = 1$

For calculating recurrence coefficients $\beta_k^{(1)}$ ($k \geq 0$), with respect to the weight function

$$w^{(1)}(x) = \frac{1 - x^2}{\pi^2 + 4\,\mathrm{arctanh}^2 x},$$

we can use one of modification algorithms (*cf.* [4, pp. 121–138, §2.4,]), the so-called *modification by a simplified symmetric quadratic factor* $x^2 + y^2$ (Algorithm 2.7 in [4, p. 127]), taking y to be the imaginary unit $y = \mathrm{i}$. Using the same notation as in this book, as the initialization we take:

$$r_0'' = y, \quad r_1'' = y + \beta_1^{(0)}/y, \quad r_2'' = y + \beta_2^{(0)}/r_1'',$$

$$\alpha_0^{(1)} = 0, \;\; \beta_0^{(1)} = \beta_0^{(0)}(\beta_1^{(0)} + y^2),$$

and then we continue to generate coefficients in the following way:

$$r_{k+2}'' = y + \beta_{k+2}^{(0)}/r_{k+1}'',$$

$$\alpha_k^{(1)} = 0, \;\; \beta_k^{(1)} = \beta_k^{(0)} r_{k+1}''/r_{k-1}'',$$

for $k = 1, 2, \ldots, n-1$. At the end we should change the negative sign of $\beta_0^{(1)} := -\beta_0^{(1)}$, because in our case, the factor is $1 - x^2$ (not $x^2 + \mathrm{i}^2 = x^2 - 1$!). Because, of symmetric case (even weight functions), the α-coefficients are equal to zero.

The following Mathematica code provides the first $n = 100$ recurrence coefficients:

```
beta[k_]:=If[k==0,1/6,(1+k)^2/((1+2k)(3+2k))]; n=100;
y=I; r0s=y; r1s=y+beta[1]/y; r2s=y+beta[2]/r1s;
beta1[0]=beta[0](beta[1]+y^2);
Do[p=y+beta[k+2]/r2s; beta1[k]=beta[k]r2s/r0s;
r0s=r1s; r1s=r2s; r2s=p,{k,1,n-1}];
beta1[0] = -beta1[0];
```

For $k \leq 10$ the recurrence β-coefficients are

$$\beta_0^{(1)} = \frac{11}{90},\; \beta_1^{(1)} = \frac{40}{231},\; \beta_2^{(1)} = \frac{411}{1925},\; \beta_3^{(1)} = \frac{784}{3425},\; \beta_4^{(1)} = \frac{20625}{87269},\; \beta_5^{(1)} = \frac{834056}{3468465},\; \beta_6^{(1)} = \frac{17116729}{70441965},$$

$$\beta_7^{(1)} = \frac{428688480}{1752329587},\; \beta_8^{(1)} = \frac{1720009917}{6998346817},\; \beta_9^{(1)} = \frac{24529748360}{99477213451},\; \beta_{10}^{(1)} = \frac{1023487494293}{4140519760325},$$

and they are presented in Figure 3.

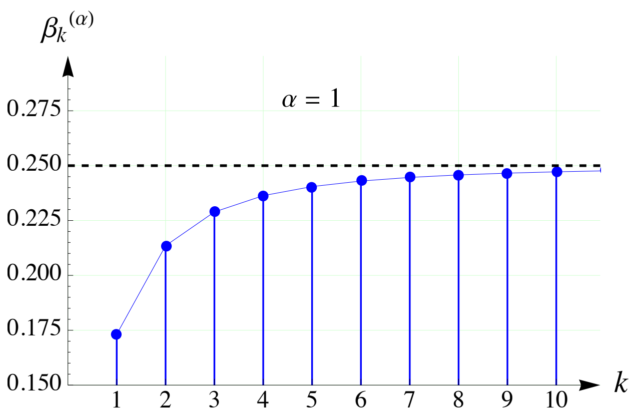

Figure 3. Recurrence coefficients $\beta_k^{(1)}$, $k = 1, 2, \ldots, 10$

Remark 2.2. As we can see from Figure 3 this sequence $\left\{\beta_k^{(1)}\right\}_{k\in\mathbb{N}}$ is (most likely) increasing, with the limit $\lim\limits_{k\to\infty}\beta_k^{(1)} = 1/4$. It could be interesting to find the analytic expression for the coefficients $\beta_k^{(1)}$, $k \ge 1$. Our attempts to do so have been unsuccessful.

2.3. Case $\alpha = -1$

This weight function

$$w^{(-1)}(x) = \frac{1}{1-x^2}\cdot\frac{1}{\pi^2+4\,\mathrm{arctanh}^2 x}, \quad x\in(-1,1), \tag{2.3}$$

appears in the integral (1.10).

We need the moments

$$\begin{aligned}\mu_k^{(-1)} &= \int_{-1}^{1} x^k w^{(-1)}(x)\,\mathrm{d}x \qquad (2.4)\\ &= \begin{cases} 0, & k \text{ is odd,}\\[2mm] 2\displaystyle\int_0^1 \frac{x^k}{1-x^2}\cdot\frac{1}{\pi^2+4\,\mathrm{arctanh}^2 x}\,\mathrm{d}x, & k \text{ is even.}\end{cases}\end{aligned}$$

Similarly as in (2.2) we obtain

$$\mu_k^{(-1)} = 2\int_0^{+\infty}\frac{\tanh^k t}{\pi^2+4t^2}\,\mathrm{d}t \quad \text{for even } k.$$

In particular, for $k=0$, we have

$$\mu_0^{(-1)} = 2\int_0^{+\infty}\frac{\mathrm{d}t}{\pi^2+4t^2} = \frac{1}{2}.$$

Since it is obvious

$$\mu_k^{(-1)} - \mu_{k+2}^{(-1)} = \mu_k^{(0)},$$

in this case, the moment sequence $\{\mu_k^{(-1)}\}$ is

$$\Big\{\frac{1}{2},\ 0,\ \frac{1}{3},\ 0,\ \frac{13}{45},\ 0,\ \frac{251}{945},\ 0,\ \frac{3551}{14175},\ 0,\ \frac{22417}{93555},\ 0,\ \frac{147636491}{638512875},\ 0,$$
$$\frac{61425277}{273648375},\ 0,\ \frac{9718892317}{44405668125},\ 0,\ \frac{41728893807163}{194896477400625},\ 0,\ \ldots\Big\}.$$

Using our Mathematica package `OrthogonalPolynomials` (see [1] and [13]), with these moments (the sequence `momm1`), we can obtain recurrence coefficients $\beta_k^{(-1)}$ (and also $\alpha_k^{(-1)} = 0$ for each k), by only one command:

```
{alm1,bem1} = aChebyshevAlgorithm[momm1,Algorithm->Symbolic];
```

Thus, the coefficients $\beta_k^{(-1)}$ for $0 \le k \le 25$ are

$$\Big\{\frac{1}{2},\frac{2}{3},\frac{1}{5},\frac{8}{35},\frac{5}{21},\frac{8}{33},\frac{35}{143},\frac{16}{65},\frac{21}{85},\frac{80}{323},\frac{33}{133},\frac{40}{161},\frac{143}{575},\frac{56}{225},\frac{65}{261},$$
$$\frac{224}{899},\frac{85}{341},\frac{96}{385},\frac{323}{1295},\frac{120}{481},\frac{133}{533},\frac{440}{1763},\frac{161}{645},\frac{176}{705},\frac{575}{2303},\frac{208}{833}\Big\}$$

(see also Figure 4). Precisely, the following statement holds:

Proposition 2.3. *The polynomials $\pi_k^{(-1)}(x)$, orthogonal with respect to the weight function* (2.3) *on* $(-1,1)$ *satisfy the recurrence relation*

$$\pi_{k+1}^{(-1)}(x) = x\pi_k^{(-1)}(x) - \beta_k^{(-1)}\pi_{k-1}^{(-1)}(x), \quad k=0,1,\ldots, \tag{2.5}$$

with $\pi_0^{(-1)}(x) = 1$ and $\pi_{-1}^{(-1)}(x) = 0$, where the β-coefficients are given by

$$\beta_0^{(-1)} = \frac{1}{2}, \quad \beta_1^{(-1)} = \frac{2}{3}, \quad \beta_k^{(-1)} = \frac{k^2-1}{4k^2-1}, \quad k=2,3,\ldots. \tag{2.6}$$

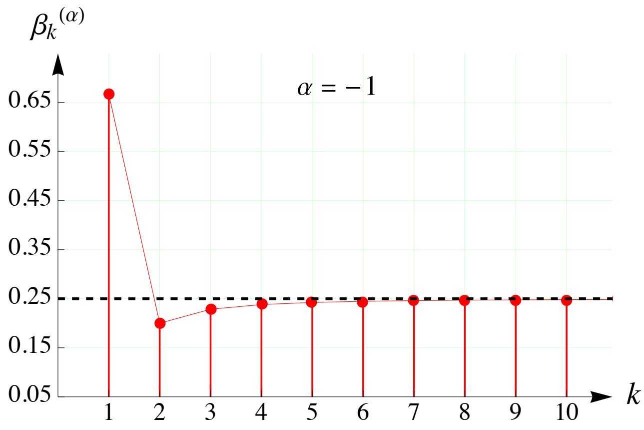

Figure 4. Recurrence coefficients $\beta_k^{(-1)}$, $k = 1, 2, \ldots, 10$

2.4. Numerical approach for an arbitrary $\alpha > -1$

For numerical calculation of the recurrence coefficients $\beta_k^{(\alpha)}$ for $\alpha > -1$ we use the method of modified moments (see [2], [8, pp. 160–162]). In order to construct the first N coefficients $\{\beta_k^{(\alpha)}\}_{k=0}^{N-1}$, we need the first $2N$ modified moments

$$\mu_k^{(\alpha)} = \int_{-1}^{1} p_k(x) w^{(\alpha)}(x)\,\mathrm{d}x, \qquad j = 0, 1, \ldots, 2N-1, \tag{2.7}$$

where the sequence of polynomials $\{p_k\}$ is chosen as in [7],

$$p_k(x) = \begin{cases} (x^2-1)^{k/2}, & \text{if } k \text{ is even}, \\ x(x^2-1)^{(k-1)/2}, & \text{if } k \text{ is odd}, \end{cases} \tag{2.8}$$

so that the moment of odd order are equal to zero, i.e., $\mu_k = 0$ for odd k.

It is obvious the polynomials $p_k(x)$ satisfy the three-term recurrence relation

$$p_{k+1}(x) = (x - a_k)p_k(x) - b_k p_{k-1}(x), \qquad p_0(x) = 1,\ p_{-1}(x) = 0,$$

with $a_k = 0$ and $b_k = 1$ for odd k and $b_k = 0$ for even k.

Thus, for even k we consider the integrals (2.7) on $(0, 1)$ and introduce a new variable by $x = \tanh t$, so that

$$\begin{aligned} \mu_k^{(\alpha)} &= 2\int_0^1 (x^2-1)^{k/2} w^{(\alpha)}(x)\,\mathrm{d}x \\ &= 2(-1)^{k/2} \int_0^1 \frac{(1-x^2)^{k/2+\alpha}}{\pi^2 + 4\,\mathrm{arctanh}^2 x}\,\mathrm{d}x, \qquad k = 0, 2, \ldots, 2N-2, \end{aligned}$$

i.e.,

$$\mu_k^{(\alpha)} = 2(-1)^{k/2} \int_0^{+\infty} \frac{1}{\pi^2 + 4t^2} \cdot \frac{\mathrm{d}t}{\cosh^{k+2\alpha+2} t}, \qquad k = 0, 2, \ldots, 2N-2. \tag{2.9}$$

For calculating the first $N = 30$ recurrence coefficients $\alpha_k^{(\alpha)}$ $(= 0)$ and $\beta_k^{(\alpha)}$, we need the first $2N = 60$ modified moments $\mu_k^{(\alpha)}$, $k = 0, 1, \ldots, 59$ (the sequence `ModMom`). Using the Mathematica package `OrthogonalPolynomials` and executing the following commands

```
<<orthogonalPolynomials`
Fk[k_,t_,alpha_]:=2(-1)^(k/2)/((Pi^2+4t^2)Cosh[t]^(k+2alpha+2));
ak=Table[0,{k,0,59}]; bk=Table[If[OddQ[k],1,0],{k,0,59}];
alpha=-1/2;
ModMom=Table[If[OddQ[k],0, NIntegrate[Fk[k,t,alpha],
 {t,0,Infinity},WorkingPrecision->55]],{k,0,59}];
{alf,bet}=aChebyshevAlgorithmModified[ModMom,ak,bk, WorkingPrecision->35];
```

we obtain the recurrence coefficients (the sequences `alf` and `bet`). The sequences `ak` and `bk` describe the polynomials (2.8).

The obtained coefficients $\beta_k^{(\alpha)}$ for $\alpha = \pm 1/2$ are given in Table 1 with 33 decimal digits.

k	$\beta_k^{(-1/2)}$	$\beta_k^{(1/2)}$
0	0.220635600152651593396456432117998	0.139393934778513395904068133798040
1	0.368216485997406421221417667713285	0.209828461140439353963403673070145
2	0.271761169652707040816871520639319	0.234829748126929328474574136722756
3	0.258754277050327977963237092006281	0.242288795626735744126709016733210
4	0.254625958525324078380024449482067	0.245415884137259973700512908231515
5	0.252825418106109020908059461153271	0.246995914174621288351123287453131
6	0.251889572895927930109395001989429	0.247895617405908436827210373375403
7	0.251345050448647184421522645138922	0.248452552589670815345978912768160
8	0.251002117369457262485560522405465	0.248819285464220218213230545036414
9	0.250773069179688060659612922031017	0.249072555873738002973633705834681
10	0.250612976189327397429364673612923	0.249254218732305870884873517134723
11	0.250496955305702550532410934559111	0.249388600820137797967477629633768
12	0.250410353869893912948274513960787	0.249490586076166670474732914991518
13	0.250344104650692462771693366126065	0.249569675053899175520770187761686
14	0.250292361926185020117908592348747	0.249632151677671968165204313785406
15	0.250251224716591125303421899816806	0.249682300239331309853634485232483
16	0.250218012117391582029834389209147	0.249723119101132285427692962008941
17	0.250190834323508349399659101627634	0.249756755047129793733202662122012
18	0.250168329345612301455275025241506	0.249784776269947633337021535872672
19	0.250149496263814323219801054780205	0.249808348851579335444107432116092
20	0.250133586670511840370744616345382	0.249828353737761803326034773729162
21	0.250120032251390135037243275892083	0.249845466013372636256425272325500
22	0.250108395412601667359596603769675	0.249860209710309807566224559592982
23	0.250098334957086950064709177498842	0.249872996386862886568213772928204
24	0.250089581799297780937175862734641	0.249884152730033774278047124813188
25	0.250081921505972956625421368821888	0.249893940599489896649435587938430
26	0.250075181560396326934570185490500	0.249902571781869963953321618134050
27	0.250069221947735721351778151344357	0.249910218987674205547974282365617
28	0.250063928109801353465761730636937	0.249917024142358415116318481823169
29	0.250059205613134026704822526493082	0.249923104704166239874147682928246

Table 1. Recurrence coefficients $\beta_k^{(\alpha)}$, $0 \leq k \leq 29$, for $\alpha = \mp 1/2$

3. Gaussian quadratures related to the weight function $w^{(\alpha)}(x)$

In this section we consider quadrature formulas of Gaussian type

$$\int_{-1}^{1} \frac{(1-x^2)^\alpha}{\pi^2 + 4\,\mathrm{arctanh}^2 x}\, f(x)\, \mathrm{d}x = \sum_{\nu=1}^{n} A_\nu^{(n)} f(x_\nu^{(n)}) + R_n(f), \tag{3.1}$$

where the remainder term $R_n(\cdot)$ vanishes for all algebraic polynomials of degree at most $2n-1$.

As we mentioned in §1.1, the nodes $x_\nu^{(n)}$ and the weight coefficients (Christoffel numbers) $A_\nu^{(n)}$ are connected to the symmetric tridiagonal Jacobi matrix $J_n(w^{(\alpha)})$, given by (1.4).

The obtained recurrence coefficients $\beta_k^{(\alpha)}$, $0 \le k \le N-1$, in the previous section, enable us to construct the quadrature parameters $x_\nu^{(n)}$ and $A_\nu^{(n)}$ in (3.1) for each $n \le N$. For example, using the following simple commands

```
pq[n_]:=aGaussianNodesWeights[n,alf,bet,WorkingPrecision->35, Precision->30];
{n20, w20} = N[pq[20],30];
```

we can get quadrature parameters (`n20` and `w20`) in (3.1) for $n = 20$ nodes, with 30 decimal digits. Table 2 shows these parameters for $\alpha = -1$. Numbers in parenthesis indicate the decimal exponents.

ν	$x_\nu^{(-1)}$	$A_\nu^{(-1)}$
1, 20	∓0.999178968436867050558373617149	8.46564767317383397145781444122(−2)
2, 19	∓0.976394515885833819536949321654	2.57799096570423383911940711872(−2)
3, 18	∓0.928771911184512946281106156836	2.08898104551302067686741194825(−2)
4, 17	∓0.857837685992294196815523062277	1.88444734647755411076015882747(−2)
5, 16	∓0.765478770296464356225485330371	1.77227434631692315686739169213(−2)
6, 15	∓0.654052515874030526792879676751	1.70335326888289388059260992763(−2)
7, 14	∓0.526360656827159643522158127562	1.65893017267934731216185869748(−2)
8, 13	∓0.385592221505761512649722344354	1.63031890646593114165879006862(−2)
9, 12	∓0.235250682564827616763480604645	1.61309024793346092522696165784(−2)
10, 11	∓0.079070878744265441541283732725	1.60496602685280098528759562064(−2)

Table 2. Nodes $x_\nu^{(-1)}$ and weight coefficients $A_\nu^{(-1)}$ for 20-point Gaussian rule (3.1), with 30 decimal digits

In the sequel we give some numerical computations.

Example 3.1. Consider the integrals

$$I_j(\alpha) = \int_{-1}^{1} \frac{(1-x^2)^\alpha}{\pi^2 + 4\,\mathrm{arctanh}^2 x} f_j(x)\,\mathrm{d}x \quad (j = 1,2,3,4), \tag{3.2}$$

where the function $x \mapsto f_j(x)$ are given by (see Figure 5)

$$f_j(x) = \cos \pi x, \quad f_2(x) = \exp\left(\frac{x-1}{x+1}\right), \quad f_3(x) = \left|x - \frac{1}{2}\right|^{7/2}, \quad f_4(x) = \left|\cos\frac{\pi x}{2}\right|^{5/4}.$$

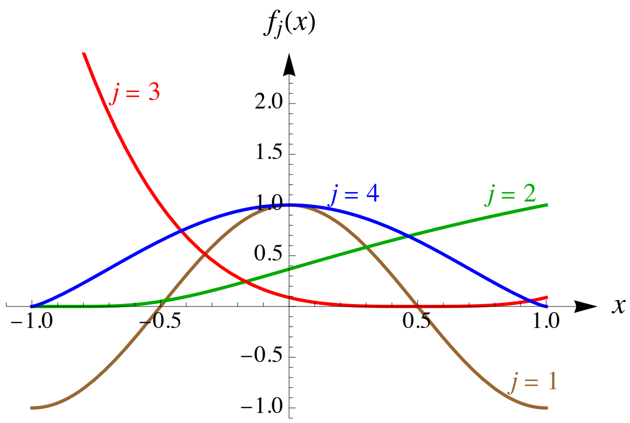

Figure 5. The functions $x \mapsto f_j(x)$ for $j = 1$ (brown), $j = 2$ (green), $j = 3$ (red) and $j = 4$ (blue)

We will apply the quadrature formula (3.1), with $n = 5(5)50$ nodes, to (3.2), in notation

$$Q_n(f;\alpha) = \sum_{\nu=1}^{n} A_\nu^{(n)} f(x_\nu^{(n)}),$$

and calculate the corresponding relative errors

$$\mathrm{Err}_j^{(n)}(\alpha) = \left| \frac{Q_n(f_j;\alpha) - I_j(\alpha)}{I_j(\alpha)} \right| \tag{3.3}$$

in two cases, when $\alpha = -1$ and $\alpha = 1/2$.

In the case $\alpha = -1$ the exact values of the integrals (3.2) are

$$\begin{aligned}
I_1(-1) &= -0.27388139339203145933884902274686299119752 35\ldots, \\
I_2(-1) &= 0.225873660379598200144161806837\ldots, \\
I_3(-1) &= 0.695185585471566248859540912251\ldots, \\
I_4(-1) &= 0.141398114354298142664612153942\ldots,
\end{aligned}$$

while for $\alpha = 1/2$ these values are

$$\begin{aligned}
I_1(1/2) &= 0.0389595945520207999063218785983317349786289\ldots, \\
I_2(1/2) &= 0.0539023893638563664422286283\ldots, \\
I_3(1/2) &= 0.0632119429443514939090728652\ldots, \\
I_4(1/2) &= 0.1012377374899969338041610614\ldots.
\end{aligned}$$

Applying quadrature formulas (3.1) to (3.2), with $\alpha = -1$ and $\alpha = 1/2$, we get quadrature approximations $Q_n(f_j;\alpha)$, with relative errors presented in Tables 3 and 4, respectively.

n	$\mathrm{Err}_1^{(n)}(-1)$	$\mathrm{Err}_2^{(n)}(-1)$	$\mathrm{Err}_3^{(n)}(-1)$	$\mathrm{Err}_4^{(n)}(-1)$
5	6.53(−5)	7.20(−4)	1.93(−5)	4.30(−3)
10	9.39(−15)	3.05(−8)	3.69(−6)	5.12(−4)
15	8.01(−27)	8.11(−8)	6.61(−7)	1.52(−4)
20	2.40(−40)	9.46(−9)	4.08(−8)	6.46(−5)
25		2.19(−10)	4.77(−8)	3.35(−5)
30		3.16(−11)	2.87(−8)	1.97(−5)
35		2.62(−13)	3.97(−9)	1.26(−5)
40		1.63(−13)	5.43(−9)	8.54(−6)
45		2.04(−14)	4.60(−9)	6.08(−6)
50		1.57(−15)	8.52(−10)	4.49(−6)

Table 3. Relative errors in quadrature sums $Q_n(f_j;-1)$ $(k = 1,2,3,4)$ for $n = 5(5)50$ nodes

n	$\mathrm{Err}_1^{(n)}(1/2)$	$\mathrm{Err}_2^{(n)}(1/2)$	$\mathrm{Err}_3^{(n)}(1/2)$	$\mathrm{Err}_4^{(n)}(1/2)$
5	5.60(−5)	5.28(−4)	2.73(−4)	7.08(−5)
10	8.00(−15)	4.43(−6)	6.44(−6)	1.82(−6)
15	6.86(−27)	8.45(−8)	1.82(−6)	1.90(−7)
20	2.07(−40)	3.15(−10)	1.05(−6)	3.72(−8)
25		1.34(−10)	2.38(−7)	1.03(−8)
30		2.81(−13)	6.55(−8)	3.60(−9)

Table 4. Relative errors in quadrature sums $Q_n(f_j;-1)$ $(k = 1,2,3,4)$ for $n = 5(5)30$ nodes

As we can see from Tables 3 and 4 the speed of convergence of quadrature formulas depends on the properties of functions f_j. In the case of the holomorphic function f_1, we have a very fast convergence; already with n=10 nodes an accuracy of 15 decimal digits is achieved. In the case of the function f_2, such an accuracy requires about $n = 50$ nodes. Convergence is significantly slower for functions f_3 and f_4 due to reduced smoothness.

The integrands in (3.2) are displayed in Figure 6 for $\alpha = -1$ (left) and $\alpha = 1/2$ (right).

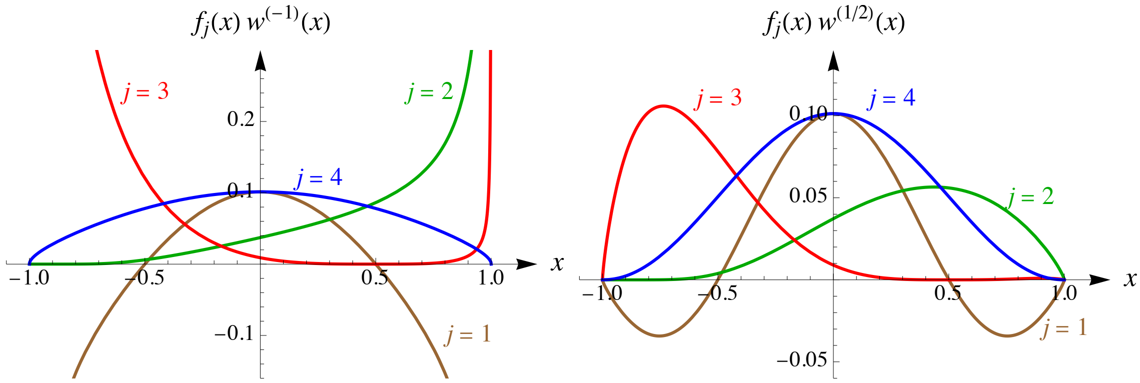

Figure 6. The integrands $x \mapsto f_j(x)w^{(\alpha)}(x)$, $j = 1, 2, 3, 4$, for $\alpha = -1$ (left) and $\alpha = 1/2$ (right)

Remark 3.2. The error estimate in the quadrature formulas (3.1), as well as their convergence in different classes of functions can be analysed using the usual techniques.

Remark 3.3. Taking

$$g(x;t) = \exp\left(-t\,\frac{1-x}{1+x}\right) + \exp\left(-t\,\frac{1+x}{1-x}\right),$$

we can see that the Ramanujan integral (1.7) reduces to (3.1). Indeed,

$$\begin{aligned} I_R(t) &= \int_0^\infty \frac{1}{x}\,\frac{\mathrm{e}^{-tx}}{\pi^2 + \log^2 x}\,\mathrm{d}x \\ &= \left(\int_0^1 + \int_1^\infty\right)\frac{1}{x}\,\frac{\mathrm{e}^{-tx}}{\pi^2 + \log^2 x}\,\mathrm{d}x \\ &= \int_0^1 \frac{1}{x}\,\frac{\mathrm{e}^{-tx} + \mathrm{e}^{-t/x}}{\pi^2 + \log^2 x}\,\mathrm{d}x. \end{aligned}$$

Then, taking $(1-x)/(1+x)$ instead of x, the last integral reduces to

$$I_R(t) = 2\int_0^1 \frac{1}{1-x^2}\,\frac{\mathrm{e}^{-t(1-x)/(1+x)} + \mathrm{e}^{-t(1+x)/(1-x)}}{\pi^2 + \log^2 \dfrac{1-x}{1+x}}\,\mathrm{d}x,$$

i.e.,

$$I_R(t) = \int_{-1}^1 \frac{1}{1-x^2}\cdot\frac{g(x;t)}{\pi^2 + \log^2 \dfrac{1-x}{1+x}}\,\mathrm{d}x = \int_{-1}^1 g(x;t)w^{(-1)}(x)\,\mathrm{d}x. \tag{3.4}$$

Note that $g(-x;t) = g(x,t)$, $g(x;0) = 2$, $g(0;t) = 2\mathrm{e}^{-t}$, and

$$\lim_{x\to 1-} g(x;t) = 1.$$

Applying the quadrature formula (3.1) to (3.4), we get approximations $Q_n(g(\,\cdot\,;t))$ of $I_R(t)$, with relative errors

$$\mathrm{Err}_n(t) = \left|\frac{Q_n(g(\,\cdot\,;t)) - I_R(t)}{I_R(t)}\right|,$$

where the "exact" value of $I_R(t)$ is obtained by a sufficiently high precision in Wolfram's Mathematica 14.0. Graphics of relative errors in log-scale for $n = 10(10)50$ are presented in Figure 7. As we can see, a lower accuracy is appeared for large values of t, and especially for small values close to zero.

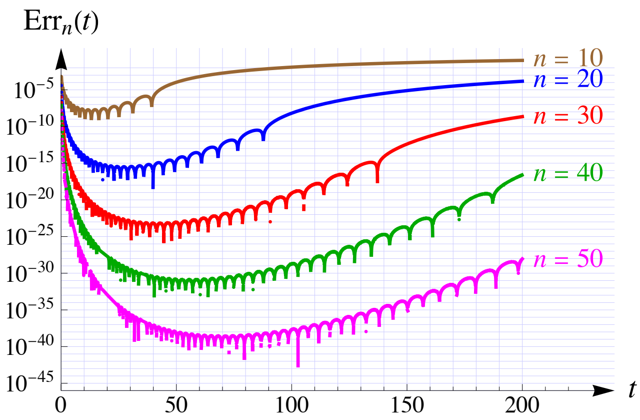

Figure 7. Relative errors $\mathrm{Err}_n(t)$ for $t \le 200$ and $n = 10(10)50$ nodes in log-scale

Finally, we repeat a diagram from [14], obtained by this kind of integration (Figure 8).

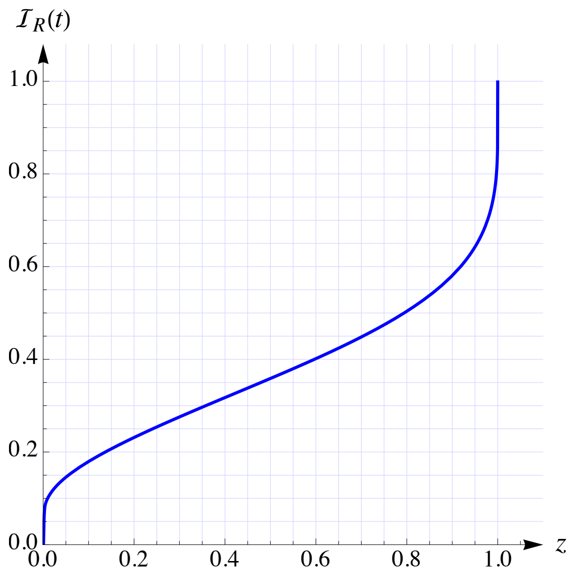

Figure 8. Wood's diagram of $z \mapsto I_R(t)$, when $t = 1/z^2 - 1$, $0 < z < 1$

Remark 3.4. It could be interesting to consider the corresponding orthogonal polynomials and quadrature formulas with respect to the two-parametric weight function

$$w^{(\alpha,\beta)}(x) = \frac{(1-x)^\alpha(1+x)^\beta}{\pi^2 + 4\,\mathrm{arctanh}^2 x} \quad (\alpha,\beta \geq -1)$$

on $(-1,1)$.

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References

[1] A. S. Cvetković and G. V. Milovanović, *The mathematica package "OrthogonalPolynomials"*, Facta Univ. Ser. Math. Inform. **19**, 17–36, 2004.
[2] W. Gautschi, *On generating orthogonal polynomials*, SIAM J. Sci. Statist. Comput. **3**, 289–317, 1982.
[3] W. Gautschi, *Reflections and recollections,* In: Approximation and Computation: A Festschrift in Honor of Walter Gautschi, Proceedings of the Purdue Conference (Ed. by R. V. M. Zahar), December 2-5, 1993, pp. xvii–xxxv, Birkhäuser, Boston, Basel, Berlin, 1994.
[4] W. Gautschi, *Orthogonal polynomials: Computation and approximation,* Clarendon Press, Oxford, 2004.
[5] W. Gautschi and G. V. Milovanović, *A Ramanujan integral and its derivatives: Computation and analysis*, Math. Comp. **93 (347)**, 1297–1308, 2024.
[6] G. Golub and J. H. Welsch, *Calculation of Gauss quadrature rules*, Math. Comp. **23**, 221–230, 1969.
[7] P. Laxmi, S. Jain, P. Agarwal and G. V. Milovanović, *Numerical calculation of the extension of k-beta function and some new extensions by using two parameter k-Mittag-Leffler function*, Appl. Math. Comput. **479**, 2024; Article ID: 128857, https://doi.org/10.1016/j.amc.2024.128857.
[8] G. Mastroianni and G. V. Milovanović, *Interpolation processes-basic theory and applications*, Springer Monographs in Mathematics, Springer Verlag, Berlin, Heidelberg, New York, 2008.
[9] G. V. Milovanović, *Summation of series and Gaussian quadratures*, In: Approximation and Computation (Ed. by R. V. M. Zahar), 459–475, ISNM (Volume 119), Birkhäuser, Basel-Boston-Berlin, 1994.
[10] G.V. Milovanović, *Quadrature processes and new applications*, Bull. Cl. Sci. Math. Nat. Sci. Math. **38**, 83–120, 2013.
[11] G. V. Milovanović, *Orthogonal polynomials on the real line*, In: Walter Gautschi: Selected Works and Commentaries (Ed. by C. Brezinski and A. Sameh) (Volume 2) Birkhäuser, Basel, 2014.
[12] G. V. Milovanović, *A note on extraction of orthogonal polynomials from generating function for reciprocal of odd numbers*, Indian J. Pure Appl. Math. **50 (1)**, 15–22, 2019.
[13] G. V. Milovanović and A. S. Cvetković, *Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type*, Math. Balkanica **26**, 169–184, 2012.
[14] V. E. Wood, *Some integrals of Ramanujan and related contour integrals*, Math. Comp. **20 (95)**, 424–429, 1966.

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