An Error Expansion for Gauss–Turán Quadrature with Chebyshev Weight Function

G. V. Milovanović[a] and M. M. Spalević[b]
[a] University of Niš, Faculty of Electronic Engineering, P. O. Box 73
18000 Niš, Serbia, Yugoslavia
[b] University of Kragujevac, Faculty of Science, P. O. Box 60
34000 Kragujevac, Serbia, Yugoslavia

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Abstract

Our aim in this paper is to obtain an expansion for the error in the Gauss-Turán quadrature formula for approximating $\int_{-1}^{1} w(t)f(t)\,dt$ in the case when the function f is analytic in some region of the complex plane containing the interval $[-1,1]$ in its interior, and the remainder term is presented in the form of a contour integral over the confocal ellipses. In the case $w(t) = 1/\sqrt{1-t^2}$ we used such expansion to obtain very exact estimations of the error. Some numerical results and illustrations are included.

1 Introduction

Suppose a weight function w is positive and continuous in the open interval $(-1,1)$ and is integrable over $(-1,1)$. Our object is to obtain an expansion for the error

$$R_{n,s}(f) = I(f) - Q_{n,s}(f) \tag{1.1}$$

in the Gauss-Turán quadrature formula with multiple nodes

$$Q_{n,s}(f) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) \quad (n \in \mathbb{N};\ s \in \mathbb{N}_0), \tag{1.2}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,s)}, \tau_\nu = \tau_\nu^{(n,s)}$ $(i = 0, 1, \dots, 2s;\ \nu = 1, \dots, n)$ for approximating

$$I(f) = \int_{-1}^{1} f(t)w(t)\, dt \tag{1.3}$$

in the case when the function f is analytic in some region of the complex plane containing the interval $[-1, 1]$ in its interior. The quadrature formula (1.2) is exact for all algebraic polynomials of degree at most $2(s + 1)n - 1$.

The nodes τ_ν in (1.2) are the zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the integral $\int_{-1}^{1} \pi_n(t)^{2s+2}\, d\lambda(t)$. This gives

$$\int_{-1}^{1} \pi_n(t)^{2s+1} t^k\, d\lambda(t) = 0, \quad k = 0, 1, \ \dots\ , n - 1. \tag{1.4}$$

The polynomials $\pi_n(t) = \pi_{n,s}(t)$, which satisfy this type of orthogonality (1.4) are known as s-orthogonal polynomials with respect to the weight $w(t)$. For details and references about several classes of s-orthogonal polynomials, as well as their generalizations known as σ-orthogonal polynomials, and corresponding quadrature formulas with multiple nodes, see the survey paper [12], and some very recent papers [13], [14], [16], [17], [20], [21].

2 The Remainder Term for Analytic Functions

In this paper let $\pi_{n,s}(z)$ be the s-orthogonal polynomial of degree n with respect to the weight function $w(t)$ over $(-1, 1)$, scaled so that the coefficient of z^n in the expansion of $\pi_{n,s}(z)$ in powers of z is positive.

Let Γ be a simple closed cuvre in the complex surrounding the interval $[-1, 1]$ and D be its interior. If the integrand f is analytic in D and continuous on $\overline{D}$, then we take as our starting point the known expression (cf. [19], [18], [15]) of the remainder term (1.1) in the form of the contour integral

$$R_{n,s}(f) = \frac{1}{2\pi i}\oint_{\Gamma} K_{n,s}(z) f(z)\, dz. \tag{2.1}$$

The kernel is given by

$$K_{n,s}(z) = \frac{\varrho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}, \quad z \notin [-1,1], \tag{2.2}$$

where

$$\varrho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(z)]^{2s+1}}{z-t}\, w(t)\, dt, \quad n \in \mathbb{N},. \tag{2.3}$$

For $s = 0$ (2.1) and (2.2) reduce to the corresponding formulas for Gaussian quadratures.

Integral representation (2.1) leads to the error estimate

$$|R_{n,s}(f)| \le \frac{l(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{2.4}$$

where $l(\Gamma)$ is the length of the contour Γ. It means that is necessary to study the magnitude of $|K_{n,s}(z)|$ on Γ (cf. [15]).

Two choices of the contour Γ have been widely used: a circle with center 0 and radius ϱ (> 0), and an ellipse with foci at ± 1. In this paper we take the contour Γ as an ellipse with foci at the points ± 1 and sum of semiaxes $\varrho > 1$

$$E_{\varrho} = \left\{ z \in \mathbb{C} :\ z = \frac{1}{2}(\varrho e^{i\theta} + \varrho^{-1}e^{-i\theta}),\ 0 \le \theta < 2\pi \right\}. \tag{2.5}$$

In Section 3 we adapt Hunter's approach [11] for Gaussian quadratures and obtain error expansions for Gauss-Turán quadrature formulae (1.2) based on elliptical contours.

We consider the following four weight functions $w(t)$:

(a) $w_1(t) = (1-t^2)^{-1/2}$, (b) $w_2(t) = (1-t^2)^{1/2+s}$,

(c) $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$, (d) $w_4(t) = (1-t)^{1/2+s}(1+t)^{-1/2}$.

S. Bernstein [1] shoved that the monic Chebyshev polynomial (with respect to the weight function (a)) $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\, dt, \quad k \geq 0.$$

Thus, the Chebyshev polynomials T_n are s-orthogonal on $[-1, 1]$ for each $s \geq 0$. Ossicini and Rosatti [19] found three other weights ((b), (c), (d)) for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind: U_n, V_n, and W_n, which are defined by

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, V_n(t) = \frac{\cos(n+1/2)\theta}{\cos\theta/2}, W_n(t) = \frac{\sin(n+1/2)\theta}{\sin\theta/2},$$

respectively (cf. [5]), where $t = \cos\theta$. However, such weights depend on s (see (b), (c), (d)). Notice that the weight function in (d) can be obtained by substitution $t := -t$ in the weight function in (c), and that $W_n(-t) = (-1)^n V_n(t)$. Because of that, the weights $w_3(t), w_4(t)$ can be treated in similar way.

Recently, Gori and Micchelli [9] have introduced for each n a class of weight functions defined on $[-1, 1]$ for which explicit Gauss-Turán quadrature formulas of all orders can be found. In the other words, these classes of weight functions have the peculiarity that the corresponding s-orthogonal polynomials, of the same degree, are independent of s. This class includes certain generalized Jacobi weight functions

$$w_{n,\mu}(t) = |U_{n-1}(t)/n|^{2\mu+1}(1-t^2)^{\mu}\,,$$

where $U_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$ (Chebyshev polynomial of the second kind) and $\mu > -1$. In this case, the Chebyshev polynomials T_n appear to be s-orthogonal polynomials.

In [15], following [6] (see also [7]) for $s = 0$, we studied the magnitude of $|K_{n,s}(z)|$ on the contour E_ϱ. Precisely, for the weight functions $w_k(t)$ ($k = 1, 2, 3$) we investigated the locations on the confocall ellipses (2.5) where the modulus of the corresponding kernels attain their maximum values. Basing on the calculation we conjectured in [15] for $w(t) = w_1(t)$ (also for $w(t) = w_3(t)$) that for each fixed $\varrho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\varrho, s) \in \mathbb{N}$ such that the maximum of the kernel is attained at $\theta = 0$ for each $n \geq n_0$.

In Section 4, following [11], we obtain a few new estimates of the remainder term (2.1). In particular, we concentrate our attention on the weight function $w_1(t)$ and obtain some very exact estimates of the remainder term. Some of them are the smallest, including and ones from [15].

3 An Error Expansion for Gauss–Turán Quadrature Formulae

If f is analytic in the interior of E_ϱ, then it has the expansion

$$f(z) = {\sum_{k=0}^{+\infty}}' \alpha_k T_k(z), \tag{3.1}$$

where

$$\alpha_k = \frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2} f(t) T_k(t)\, dt, \tag{3.2}$$

which converges for all z in the interior of E_ϱ. In terms of $\xi = \varrho e^{i\theta}$ ($\varrho > 1$), $T_k(z)$ is given by the equation

$$T_k(z) = \frac{1}{2}\left(\xi^k + \xi^{-k}\right). \tag{3.3}$$

We shall require the following two results (see [11]).

Lemma 3.1 *If $z \notin [-1,1]$, $1/\pi_{n,s}(z)$ can be expanded in the form*

$$1/\pi_{n,s}(z) = \sum_{k=0}^{+\infty} \beta_{n,k}^{(s)} \xi^{-n-k}.$$

Furthermore, if w is an even function then $\beta_{n,2j+1} = 0$ ($j = 0, 1, 2, \ldots$).

Proof. The zeros of $\pi_{n,s}(z)$ are real, different, and all contained in the open interval $(-1,1)$ (cf. [8]). Then, the proof is the same as the proof of Lemma 3 in [11]. ■

Now, it is not difficult to see that (cf. [10, Eq. 0.314])

$$\frac{1}{[\pi_{n,s}(z)]^{2s+1}} = \sum_{k=0}^{+\infty} \overline{\beta}_{n,k}^{(s)}\, \xi^{-n(2s+1)-k}; \quad \xi = \varrho e^{i\theta},\ \varrho > 1, \tag{3.4}$$

where

$$\overline{\beta}_{n,0}^{(s)} = \left(\beta_{n,0}^{(s)}\right)^{2s+1}, \quad \overline{\beta}_{n,m}^{(s)} = \frac{1}{m\beta_{n,0}^{(s)}} \sum_{k=1}^{m} (2k(s+1)-m)\beta_{n,k}^{(s)}\, \overline{\beta}_{n,m-k}^{(s)},\ \ m \geq 1.$$

In particular, if $w(-t) = w(t)$ then

$$\frac{1}{[\pi_{n,s}(z)]^{2s+1}} = \sum_{k=0}^{+\infty} \overline{\beta}_{n,2k}^{(s)}\, \xi^{-n(2s+1)-2k}; \quad \xi = \varrho e^{i\theta},\ \varrho > 1. \tag{3.5}$$

Lemma 3.2 *If $z \notin [-1,1]$, $\varrho_{n,s}(z)$ can be expanded as*

$$\varrho_{n,s}(z) = \sum_{k=0}^{+\infty} \overline{\gamma}_{n,k}^{(s)}\, \xi^{-n-k-1}. \tag{3.6}$$

Furthermore, if w is an even function, then $\overline{\gamma}_{n,2j+1}^{(s)} = 0$ $(j = 0,1,\ldots)$.

Proof. It is well-known that if $w(t)$ is a weight function, then $W_{n,s}(t) = [\pi_{n,s}(t)]^{2s}\, w(t)$ is also a weight function (see [4, pp. 214–226]). Now, the proof can be given in an analogous way as one of Lemma 4 in [11].

From (2.3) we have

$$\varrho_{n,s}(z) = \int_{-1}^{1} W_{n,s}(z) \frac{\pi_{n,s}(t)}{z-t}\, dt = \sum_{k=0}^{+\infty} \overline{\gamma}_{n,k}^{(s)}\, \xi^{-n-k-1},$$

where

$$\overline{\gamma}_{n,k}^{(s)} = 2\int_{-1}^{1} w(t)[\pi_{n,s}(t)]^{2s+1}\, U_{n+k}(t)\, dt \quad (k = 0,1,\ldots). \tag{3.7}$$

If $w(-t) = w(t)$, then for k odd the integrand in (3.7) is odd, so $\overline{\gamma}_{n,k}^{(s)} = 0$.
■

Therefore, by the substitution (3.4), (3.6) in (2.2) we obtain

$$K_{n,s}(z) = \sum_{k=0}^{+\infty} \omega_{n,k}^{(s)} \, \xi^{-2n(s+1)-k-1}, \tag{3.8}$$

where

$$\omega_{n,k}^{(s)} = \sum_{j=0}^{k} \overline{\beta}_{n,j}^{(s)} \, \overline{\gamma}_{n,k-j}^{(s)} \, . \tag{3.9}$$

Theorem 3.3 *The remainder term $R_{n,s}(f)$ can be represented in the form*

$$R_{n,s}(f) = \sum_{k=0}^{+\infty} \alpha_{2n(s+1)+k} \, \varepsilon_{n,k}^{(s)}, \tag{3.10}$$

where the coefficients $\varepsilon_{n,k}^{(s)}$ are independent of f. Furthermore, if f is an even function then $\varepsilon_{n,2j+1}^{(s)} = 0$ $(j = 0, 1, \dots)$.

Proof. By substitution (3.1), (3.8) in (2.1) we obtain

$$\begin{aligned} R_{n,s}(f) &= \frac{1}{2\pi i} \int_{E_\varrho} \left({\sum_{j=0}^{+\infty}}' \alpha_j T_j(z) \sum_{k=0}^{+\infty} \omega_{n,k}^{(s)} \, \xi^{-2n(s+1)-k-1} \right) dz \\ &= \sum_{k=0}^{+\infty} \left[\frac{1}{2\pi i} {\sum_{j=0}^{+\infty}}' \alpha_j \int_{E_\varrho} T_j(z) \, \xi^{-2n(s+1)-k-1} \, dz \right] \omega_{n,k}^{(s)}. \end{aligned}$$

On applying Lemma 5 from [11], this reduces to (3.10), with

$$\begin{aligned} \varepsilon_{n,0}^{(s)} &= \frac{1}{4} \, \omega_{n,0}^{(s)} \, , \\ \varepsilon_{n,1}^{(s)} &= \frac{1}{4} \, \omega_{n,1}^{(s)} \, , \\ \varepsilon_{n,k}^{(s)} &= \frac{1}{4} \left(\omega_{n,k}^{(s)} - \omega_{n,k-2}^{(s)} \right) \quad (k = 2, 3, 4, \dots). \end{aligned} \tag{3.11}$$

If $w(-t) = w(t)$ and k is odd it follows from (3.9) and Lemmas 3.1 and 3.2 that $\omega_{n,k}^{(s)} = 0$ and hence $\varepsilon_{n,k}^{(s)} = 0$. ■

Remark 3.1 One follows from (3.10) on setting $f(z) = T_{2n(s+1)+k}(z)$ that

$$\varepsilon_{n,k}^{(s)} = \sigma_{2n(s+1)+k} - \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} T_{2n(s+1)+k}^{(i)}(\tau_\nu) \quad (k = 0, 1, 2, \ldots),$$

where

$$\sigma_k = \int_{-1}^{1} w(t) T_k(t)\, dt \quad (k = 0, 1, 2, \ldots).$$

Therefore, we conclude that

$$\left|\varepsilon_{n,k}^{(s)}\right| \leq \int_{-1}^{1} w(t)\, dt + \sum_{\nu=1}^{n} \sum_{i=0}^{2s} |A_{i,\nu}| \left|T_{2n(s+1)+k}^{(i)}(\tau_\nu)\right| \quad (k = 0, 1, 2, \ldots).$$

If $s = 0$ then $\left|\varepsilon_{n,k}^{(0)}\right| \leq 2\int_{-1}^{1} w(t)\, dt$, and this fact can be used to obtain some global upper bounds of the remainder term (see Hunter [11]). Unfortunately, for now, such conclusion cannot be made in the general case for $s > 0$, because of the difficulties with finding upper bounds in $\left|T_{2n(s+1)+k}^{(i)}(\tau_\nu)\right|$.

4 Error Estimates for Gauss–Turán Quadratures with Chebyshev Weight Function of First Kind

If $u \in \mathbb{C}$, $|u| < 1$, then by differentiating the well-known identity

$$\frac{1}{1-u} = \sum_{k=0}^{+\infty} u^k$$

we obtain

$$\frac{1}{(1-u)^{l+1}} = \sum_{k=l}^{+\infty} \binom{k}{l} u^{k-l} \quad (l = 0, 1, 2, \ldots). \tag{4.1}$$

In this section we consider the weight function $w(t) = w_1(t)$. Then $\pi_{n,s}(t) = T_n(t)$. By using (4.1), and by the representation $u = 1/\xi$ ($\xi = \varrho e^{i\theta}$, $\varrho > 1$), we obtain

$$\begin{aligned}\frac{1}{[T_n(z)]^{2s+1}} &= \frac{1}{\left[\frac{1}{2}(\xi^n + \xi^{-n})\right]^{2s+1}} = 2^{2s+1}\,\xi^{-n(2s+1)}\left(\frac{1}{1+\xi^{-2n}}\right)^{2s+1} \\ &= 2^{2s+1}\,\xi^{-n(2s+1)}\sum_{k=2s}^{+\infty}\binom{k}{2s}\left(-\xi^{-2n}\right)^{k-2s}.\end{aligned}$$

Therefore,

$$\frac{1}{[T_n(z)]^{2s+1}} = 2^{2s+1}\sum_{j=0}^{+\infty}(-1)^j\binom{j+2s}{2s}\xi^{-n(2s+1)-2nj}. \tag{4.2}$$

On the other hand, one holds (3.4) with $\pi_{n,s}(t) = T_n(t)$. By comparing the right sides of these equalities we obtain

$$\overline{\beta}_{n,k}^{(s)} = \begin{cases} 2^{2s+1}(-1)^j\binom{j+2s}{2s}; & k = 2jn\ (j = 0,1,2,\dots), \\ 0; & \text{otherwise.}\end{cases} \tag{4.3}$$

By using (3.7) and the substitution $t = \cos\theta$, we obtain

$$\begin{aligned}\overline{\gamma}_{n,k}^{(s)} &= 2\int_{-1}^{1}(1-t^2)^{-1/2}\,[T_n(t)]^{2s+1}\,U_{n+k}(t)\,dt \\ &= 2\int_0^{\pi}\frac{1}{\sin\theta}[\cos n\theta]^{2s+1}\sin{(n+k+1)\theta}\,d\theta.\end{aligned} \tag{4.4}$$

If k is odd, then $\overline{\gamma}_{n,k}^{(s)} = 0$. For $[\cos n\theta]^{2s+1}$ in the last integral we use the known representation

$$[\cos n\theta]^{2s+1} = \sum_{m=0}^{2s+1} a_m^{(s)}\,\cos mn\theta,$$

with $a_m^{(s)} = \overline{a}_m^{(s)} \,/ \int_0^\pi \cos^2 mn\theta \, d\theta$, where

$$\overline{a}_m^{(s)} = \int_0^\pi [\cos n\theta]^{2s+1} \, \cos mn\theta \, d\theta. \tag{4.5}$$

Because of $\overline{a}_m^{(s)} = 0$ if m is even (cf. [10, Eq. 3.631.17]), and $\int_0^\pi \cos^2 mn\theta \, d\theta = \pi/2$, (4.4) becomes

$$\overline{\gamma}_{n,k}^{(s)} = \frac{4}{\pi} \sum_{\substack{m=0 \\ (m \text{ is odd})}}^{2s+1} \overline{a}_m^{(s)} I_{n,k,m}^{(s)}, \tag{4.6}$$

where k is even, and the integrals

$$I_{n,k,m}^{(s)} = \int_0^\pi \frac{\sin{(n+k+1)\theta} \cos mn\theta}{\sin\theta} \, d\theta$$

can be found by [10, Eq. 3.612.1]. Therefore, (4.6) reduces to

$$\overline{\gamma}_{n,k}^{(s)} = \begin{cases} 4 \displaystyle\sum_{l=0}^{j} \overline{a}_{2l+1}^{(s)}; & k = 2nj, 2nj+2, \ldots, 2n(j+1)-2 \\ & \quad (j = 0, 1, \ldots, s-1), \\ 2\pi; & k = 2sn, 2sn+2, \ldots, \\ 0; & \text{otherwise.} \end{cases} \tag{4.7}$$

Now, consider the integrals in (4.5). By substitution $n\theta = t$ we obtain

$$\overline{a}_m^{(s)} = \frac{1}{n} \int_0^{n\pi} \cos^{2s+1} t \, \cos mt \, dt.$$

If m is odd (by substitution $t := t - \pi$) one holds

$$\int_0^\pi \cos^{2s+1} t \, \cos mt \, dt = \int_\pi^{2\pi} \cos^{2s+1} t \, \cos mt \, dt, \tag{4.7.1}$$

then (cf. [10, Eq. 3.631.17])

$$\overline{a}_m^{(s)} = \int_0^\pi \cos^{2s+1} t \, \cos mt \, dt = \frac{\pi}{2^{2s+1}} \binom{2s+1}{(2s+1-m)/2} \; (> 0). \tag{4.8}$$

Now, (4.7) can be expressed in an explicit form

$$\overline{\gamma}_{n,k}^{(s)} = \begin{cases} \dfrac{\pi}{2^{2s-1}} \displaystyle\sum_{l=0}^{j} \binom{2s+1}{s-l}; & k = 2nj, 2nj+2, \dots, 2n(j+1)-2 \\ & \quad (j = 0, 1, \dots, s-1), \\ 2\pi; & k = 2sn, 2sn+2, \dots, \\ 0; & \text{otherwise.} \end{cases} \tag{4.9}$$

Remark 4.1 From (4.9) we conclude that $\overline{\gamma}_{n,k}^{(s)} > 0$ for each even k, and, since $\sum_{l=0}^{s} \binom{2s+1}{s-l} = 2^{2s}$ (cf. [15]), then

$$\frac{\pi}{2^{2s-1}} \binom{2s+1}{s} \leq \overline{\gamma}_{n,k}^{(s)} \leq 2\pi.$$

4.1 First type of error estimates

In general, the Chebyshev coefficients α_k in (3.1) are unknown. However, Elliot [3] describes a number of ways of estimating or bounding them. In particular, under our assumptions,

$$|\alpha_k| \leq \frac{2\left(\max_{z \in E_\varrho} |f(z)|\right)}{\varrho^k}. \tag{4.10}$$

Let

$$h_k(t) := \sum_{n=1}^{+\infty} n^k t^{n-1} \quad (|t| < 1).$$

To derive its recurrence relation we have

$$\begin{aligned} h_k(t) - t h_k(t) &= \sum_{n=0}^{+\infty} (n+1)^k t^n - \sum_{n=1}^{+\infty} n^k t^n \\ &= 1 + \sum_{n=1}^{+\infty} [(n+1)^k - n^k] t^n = 1 + \sum_{n=1}^{+\infty} \left[\sum_{i=0}^{k-1} \binom{k}{i} n^i \right] t^n \\ &= 1 + t \sum_{i=0}^{k-1} \binom{k}{i} h_i(t). \end{aligned}$$

Hence

$$h_k(t) = \frac{1}{1-t}\left[1 + t\sum_{i=0}^{k-1}\binom{k}{i}h_i(t)\right], \quad k \geq 1.$$

Here, we consider the case $s = 1$.

By using (4.3), we find

$$\overline{\beta}_{n,k}^{(1)} = \begin{cases} 8(-1)^j\binom{j+2}{2}; & k = 2jn \ (j = 0, 1, 2, \dots), \\ 0; & \text{otherwise.} \end{cases}$$

From (4.9) we obtain

$$\overline{\gamma}_{n,k}^{(1)} = \begin{cases} \dfrac{3\pi}{2}; & k = 0, 2, \dots, 2n-2, \\ 2\pi; & k = 2n, 2n+2, \dots\ , \\ 0; & \text{otherwise.} \end{cases} \tag{4.11}$$

Using (3.9) and (3.11) we get

$$\begin{aligned} \varepsilon_{n,k}^{(1)} &= \frac{1}{4}\left[\frac{3\pi}{2}\overline{\beta}_{n,2jn}^{(1)} + 2\pi\sum_{\substack{\overline{l}=2ln \\ l<j}}\overline{\beta}_{n,\overline{l}}^{(1)} - \left(\frac{3\pi}{2}\overline{\beta}_{n,2(j-1)n}^{(1)} + 2\pi\sum_{\substack{\overline{l}=2ln \\ l<j-1}}\overline{\beta}_{n,\overline{l}}^{(1)}\right)\right] \\ &= (-1)^j\pi\left(j^2 + 4j + 3\right)\,, \end{aligned}$$

for $k = 2jn$ $(j = 0, 1, 2, \dots)$ and $\varepsilon_{n,k}^{(1)} = 0$, otherwise. Now, by using just obtained results and (3.10), (4.10), we have

$$\begin{aligned} |R_{n,1}(f)| &= \left|\sum_{k=0}^{+\infty}\alpha_{4n+k}\,\varepsilon_{n,k}^{(1)}\right| = \left|\sum_{j=0}^{+\infty}\alpha_{4n+2jn}\,\varepsilon_{n,2jn}^{(1)}\right| \\ &\leq \frac{2\pi\left(\max_{z\in E_\varrho}|f(z)|\right)}{\varrho^{4n}}\sum_{j=0}^{+\infty}\frac{j^2+4j+3}{\varrho^{2jn}}\,. \end{aligned}$$

The sums $\sum\limits_{j=0}^{+\infty} \dfrac{j^l}{\varrho^{2jn}}$ $(l=0,1,2)$ can be calculated by using the method for $h_l(t)$ and putting $t = 1/\varrho^{2n}$.

Therefore, we obtain the error estimate

$$|R_{n,1}(f)| \le 2\pi \left(\max_{z\in E_\varrho} |f(z)| \right) \frac{3\varrho^{2n}-1}{(\varrho^{2n}-1)^3} \,. \tag{4.12}$$

For $s = 0$ the error estimate has been obtained by Hunter [11] (see also Chawla and Jain [2]).

4.2 Second type of error estimates

In this subsection, we use (2.1) in order to derive some error estimates, which are different from the previous, as well as ones derived in [15]. It follows immediately from (2.1) that

$$|R_{n,s}(f)| \le \overline{K}_{n,s}(E_\varrho) \left(\max_{z\in E_\varrho} |f(z)| \right), \tag{4.14}$$

what is, in fact, (2.4) with $\Gamma \equiv E_\varrho$ and

$$\overline{K}_{n,s}(E_\varrho) = \frac{l(E_\varrho)}{2\pi} \left(\max_{z\in E_\varrho} |K_{n,s}(z)| \right). \tag{4.15}$$

It is known that the ellipse has length $l(E_\varrho) = 4\varepsilon^{-1}E(\varepsilon)$, where ε is the eccentricity of E_ϱ, i. e., $\varepsilon = 2/(\varrho+\varrho^{-1})$, and

$$E(\varepsilon) = \int_0^{\pi/2} \sqrt{1-\varepsilon^2 \sin^2\theta}\, d\theta$$

is the complete elliptic integral of the second kind. This approach is used by, e. g., Gautschi and Varga [6] (see also [7]) in the case $s = 0$, and recently extended to the case $s \in \mathbb{N}_0$ by the authors of this paper (cf. [15]).

Here, we follow a different approach (cf. [11]). Directly from (2.1) ($\Gamma \equiv E_\varrho$) we obtain the error estimate

$$|R_{n,s}(f)| \le L_{n,s}(E_\varrho) \left(\max_{z\in E_\varrho} |f(z)| \right), \tag{4.16}$$

where

$$L_{n,s}(E_\varrho) = \frac{1}{2\pi} \oint_{E_\varrho} |K_{n,s}(z)|\, |dz|\,, \tag{4.17}$$

and $K_{n,s}(z)$ is given by (2.2). Obviously,

$$L_{n,s}(E_\varrho) \le \overline{K}_{n,s}(E_\varrho).$$

For $z = \frac{1}{2}(\varrho e^{i\theta} + \varrho^{-1}e^{-i\theta})$ (4.17) can be represented in the form

$$L_{n,s}(E_\varrho) = \frac{1}{4\pi} \int_0^{2\pi} \frac{|\varrho_{n,s}(z)|\, \left(\varrho^2 + \varrho^{-2} - 2\cos 2\theta\right)^{1/2}}{|\pi_{n,s}(z)|^{2s+1}}\, d\theta\,. \tag{4.18}$$

Here, we concentrate to the error estimates based on (4.18). This integral can be evaluated numerically by using a quadrature formula. However if $w(t) = w_1(t)$ we can obtain explicit expressions for $L_{n,s}(E_\varrho)$ or for their bounds. In this case (4.18) becomes

$$L_{n,s}(E_\varrho) = \frac{2^{2s-1}}{\pi} \int_0^{2\pi} \frac{|\varrho_{n,s}(z)|\, \left(\varrho^2 + \varrho^{-2} - 2\cos 2\theta\right)^{1/2}}{\left[(\varrho^n + \varrho^{-n})^2 - 4\sin^2 n\theta\right]^{(2s+1)/2}}\, d\theta. \tag{4.19}$$

By using (3.6) and (4.9) we obtain

$$\begin{aligned}\varrho_{n,s}(z) = \; & \frac{\pi}{2^{2s-1}}\,\frac{1}{\xi^{n+1}}\,\Bigg[\binom{2s+1}{s}\left(1+\frac{1}{\xi^2}+\cdots+\frac{1}{\xi^{2sn-2}}\right)\\ & +\binom{2s+1}{s-1}\left(\frac{1}{\xi^{2n}}+\frac{1}{\xi^{2n+2}}+\cdots+\frac{1}{\xi^{2sn-2}}\right)\\ & +\;\cdots\\ & +\binom{2s+1}{1}\left(\frac{1}{\xi^{2(s-1)n}}+\frac{1}{\xi^{2(s-1)n+2}}+\cdots+\frac{1}{\xi^{2sn-2}}\right)\Bigg]\\ & +\frac{2\pi}{\xi^{n+1}}\left(\frac{1}{\xi^{2sn}}+\frac{1}{\xi^{2sn+2}}+\ldots\right).\end{aligned}$$

After a little computation we obtain (cf. [15, p. 6])

$$\varrho_{n,s}(z) = \frac{\pi}{2^{2s-1}}\,\frac{1}{\xi^n\,(\xi-\xi^{-1})}\sum_{l=0}^{s}\binom{2s+1}{s-l}\frac{1}{\xi^{2ln}}\,. \tag{4.20}$$

Now, let $s=1$. From (4.20) we obtain

$$\varrho_{n,1}(z) = \frac{\pi}{2\xi^{2n}(\xi-\xi^{-1})}\,\left(3\xi^n+\xi^{-n}\right).$$

This gives

$$|\varrho_{n,1}(z)| = \frac{\pi\,\left(9\varrho^{2n}+\varrho^{-2n}+6\cos 2n\theta\right)^{1/2}}{2\varrho^{2n}\,\left(\varrho^2+\varrho^{-2}-2\cos 2\theta\right)^{1/2}}\,.$$

By substitution $|\varrho_{n,1}(z)|$ in (4.19) we get

$$L_{n,1}(E_\varrho) = \varrho^{-2n}\int_0^{2\pi}\frac{\left[(3\varrho^n+\varrho^{-n})^2-12\sin^2 n\theta\right]^{1/2}}{\left[(\varrho^n+\varrho^{-n})^2-4\sin^2 n\theta\right]^{3/2}}\,d\theta\,,$$

i.e.,

$$L_{n,1}(E_\varrho) = \frac{4\,(3\varrho^n+\varrho^{-n})}{\varrho^{2n}\,\left(\varrho^n+\varrho^{-n}\right)^3}\int_0^{\pi/2}\frac{\left[1-\left(\sqrt{12}/(3\varrho^n+\varrho^{-n})\right)^2\,\sin^2\theta\right]^{1/2}}{\left[1-(2/(\varrho^n+\varrho^{-n}))^2\,\sin^2\theta\right]^{3/2}}\,d\theta\,.$$

The last expression enables us to obtain the following upper bound of $L_{n,1}(E_\varrho)$

$$L_{n,1}(E_\varrho) \le \frac{4\,(3\varrho^n + \varrho^{-n})}{\varrho^{2n}\,\left(\varrho^n - \varrho^{-n}\right)^3} \int_0^{\pi/2} \left[1 - \left(\frac{\sqrt{12}}{3\varrho^n + \varrho^{-n}}\right)^2 \sin^2\theta\right]^{1/2} d\theta\,,$$

i.e.,

$$L_{n,1}(E_\varrho) \le \frac{4\,(3\varrho^n + \varrho^{-n})}{\varrho^{2n}\,\left(\varrho^n - \varrho^{-n}\right)^3}\; E\left(\frac{\sqrt{12}}{3\varrho^n + \varrho^{-n}}\right).$$

In a similar way we can obtain the error estimates for $s > 1$, but they are very heavy.

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