# Multiple Orthogonality and Applications in Numerical Integration

Gradimir V. Milovanović and Marija P. Stanić

*Dedicated to Professor Themistocles M. Rassias*
*on the occasion of his 60th birthday*

**Abstract** In this paper a brief survey of multiple orthogonal polynomials defined using orthogonality conditions spread out over $r$ different measures are given. We consider multiple orthogonal polynomials on the real line, as well as on the unit semicircle in the complex plane. Such polynomials satisfy a linear recurrence relation of order $r+1$, which is a generalization of the well known three-term recurrence relation for ordinary orthogonal polynomials (the case $r = 1$). Method for the numerical construction of multiple orthogonal polynomials by using the discretized Stieltjes-Gautschi procedure are presented. Also, some applications of such orthogonal systems to numerical integration are given. A numerical example is included.

## 1 Introduction

Multiple orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in Hermite-Padé approximation of a system of $r$ (Markov) functions. A good source for information on Hermite-Padé approximation is the book by Nikishin and Sorokin [23, Chapter 4], where the multiple orthogonal

Gradimir V. Milovanović
Mathematical Institute of the Serbian Academy of Sciences and Arts, Knez Mihailova 36, p.p. 367, 11001 Beograd, Serbia
e-mail: gvm@mi.sanu.ac.rs

Marija P. Stanić
Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac
Radoja Domanovića 12, 34000 Kragujevac, Serbia,
e-mail: stanicm@kg.ac.rs

polynomials are called polyorthogonal polynomials. Other good sources of information are the surveys by Aptekarev [1] and de Bruin [5], as well as the papers by Piñeiro [24], Sorokin [26–28] and Van Assche [30].

Historically, Hermite-Padé approximation was introduced by Hermite to prove the transcendence of e. Multiple orthogonal polynomials can be used to give a constructive proof of irrationality and transcendence of certain real numbers (see [30]).

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy $r \in \mathbb{N}$ orthogonality conditions. Let $r \geq 1$ be an integer and let $w_1, w_2, \ldots, w_r$ be $r$ weight functions on the real line such that the support of each $w_i$ is a subset of an interval $E_i$. Let $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ be a vector of $r$ nonnegative integers, which is called a *multi-index* with length

$$|\mathbf{n}| = n_1 + n_2 + \cdots + n_r.$$

There are two types of multiple orthogonal polynomials (see [32]).

1° *Type I multiple orthogonal polynomials.*

Here we want to find a vector of polynomials $(A_{\mathbf{n},1}, A_{\mathbf{n},2}, \ldots, A_{\mathbf{n},r})$ such that each $A_{\mathbf{n},i}$ is polynomial of degree $n_i - 1$ and the following orthogonality conditions hold:

$$\sum_{j=1}^{r} \int_{E_j} A_{\mathbf{n},j}\, x^k w_j(x)\,\mathrm{d}x = 0, \quad k = 0, 1, 2, \ldots, |\mathbf{n}| - 2.$$

2° *Type II multiple orthogonal polynomials.*

Type II multiple orthogonal polynomial is a monic polynomial $P_{\mathbf{n}}$ of degree $|\mathbf{n}|$ which satisfies the following orthogonality conditions:

$$\int_{E_1} P_{\mathbf{n}}(x)\, x^k w_1(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_1 - 1, \tag{1}$$

$$\int_{E_2} P_{\mathbf{n}}(x)\, x^k w_2(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_2 - 1, \tag{2}$$

$$\vdots$$

$$\int_{E_r} P_{\mathbf{n}}(x)\, x^k w_r(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_r - 1. \tag{3}$$

The conditions (1)–(3) give $|\mathbf{n}|$ linear equations for the $|\mathbf{n}|$ unknown coefficients $a_{k,\mathbf{n}}$ of the polynomial $P_{\mathbf{n}}(x) = \sum_{k=0}^{|\mathbf{n}|} a_{k,\mathbf{n}} x^k$, where $a_{|\mathbf{n}|,\mathbf{n}} = 1$. Since the matrix of coefficients of this system can be singular, we need some additional conditions on the $r$ weight functions to provide the uniqueness of the multiple orthogonal polynomial.

If the polynomial $P_{\mathbf{n}}(x)$ is unique, then $\mathbf{n}$ is *normal index*. If all indices are normal, then we have a *complete system*.

For $r = 1$ in the both cases we have the ordinary orthogonal polynomials. In the sequel we consider only the type II multiple orthogonal polynomials.

There are two distinct cases for which the type II multiple orthogonal polynomials are given (see [32]).

*1. Angelesco systems* – For this systems the intervals $E_i$, on which the weight functions are supported, are disjoint, i.e., $E_i \cap E_j = \emptyset$ for $1 \le i \ne j \le r$.

*2. AT systems* – AT systems are such that all weight functions are supported on the same interval $E$ and the following $|\mathbf{n}|$ functions: $w_1(x), xw_1(x), \ldots, x^{n_1-1}w_1(x), w_2(x), xw_2(x), \ldots, x^{n_2-1}w_2(x), \ldots, w_r(x), xw_r(x), \ldots, x^{n_r-1}w_r(x)$ form a Chebyshev system on $E$ for each multi-index $\mathbf{n}$.

The following two theorems have been proved in [32].

**Theorem 1.** *In an Angelesco system the type II multiple orthogonal polynomial $P_{\mathbf{n}}(x)$ factors into $r$ polynomials $\prod_{j=1}^{r} q_{n_j}(x)$, where each $q_{n_j}$ has exactly $n_j$ zeros on $E_j$.*

**Theorem 2.** *In an AT system the type II multiple orthogonal polynomial $P_{\mathbf{n}}(x)$ has exactly $|\mathbf{n}|$ zeros on $E$. For the type I vector of multiple orthogonal polynomials, the linear combination $\sum_{j=1}^{r} A_{\mathbf{n},j}(x)w_j(x)$ has exactly $|\mathbf{n}|-1$ zeros on $E$.*

For each of the weight functions $w_j$, $j = 1,2,\ldots,r$,

$$(f,g)_j = \int_{E_j} f(x)g(x)w_j(x)\,\mathrm{d}x \tag{4}$$

denotes the corresponding inner product of $f$ and $g$.

In the sequel by $\mathcal{P}_n$ we denote the set of algebraic polynomials of degree at most $n$, and by $\mathcal{P}$ the set of all algebraic polynomials.

The paper is organized as follows. Section 2 is devoted to recurrence relations for some cases of type II multiple orthogonal polynomials. In Section 3 a numerical procedure for construction of type II multiple orthogonal polynomials based on the discretized Stieltjes-Gautschi procedure [8] are presented. In Section 4 we transfer the concept of multiple orthogonality to the unit semicircle in the complex plane. Special attention is devoted to the case $r = 2$, for which the coefficients of the recurrence relation for multiple orthogonal polynomials on the semicircle are expressed in terms of the coefficients of recurrence relation for the corresponding type II multiple orthogonal (real) polynomials. Applications of multiple orthogonality to numerical integration are given in Section 5. Finally, in Section 6 a numerical example is included.

## 2 Recurrence Relations

It is well known that orthogonal algebraic polynomials satisfy the three-term recurrence relation (see [6], [9], [12]). Such a recurrence relation is one of the most important piece of information for the constructive and computational use of orthogonal polynomials. Knowledge of the recursion coefficients allows the zeros of orthogonal polynomials to be computed as eigenvalues of a symmetric tridiagonal matrix, and with them the Gaussian quadrature rule, and also allows an efficient evaluation of expansions in orthogonal polynomials.

The type II multiple orthogonal polynomials with nearly diagonal multi-index satisfy recurrence relation of order $r+1$. Let $n \in \mathbb{N}$ and write it as $n = \ell r + j$, with $\ell = [n/r]$ and $0 \le j < r$. The *nearly diagonal multi-index* $\mathbf{s}(n)$ corresponding to $n$ is given by

$$\mathbf{s}(n) = (\underbrace{\ell+1, \ell+1, \ldots, \ell+1}_{j \text{ times}}, \underbrace{\ell, \ell, \ldots, \ell}_{r-j \text{ times}}).$$

Let us denote the corresponding type II multiple (monic) orthogonal polynomials by $P_n(x) = P_{\mathbf{s}(n)}(x)$. Then, the following recurrence relation

$$xP_m(x) = P_{m+1}(x) + \sum_{i=0}^{r} a_{m,r-i} P_{m-i}(x)\,, \quad m \ge 0, \tag{5}$$

holds, with initial conditions $P_0(x) = 1$ and $P_i(x) = 0$ for $i = -1, -2, \ldots, -r$ (see [31]).

Setting $m = 0, 1, \ldots, n-1$ in (5), we get

$$x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{bmatrix} = H_n \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{bmatrix} + P_n(x) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

i.e.,

$$H_n \mathbf{P}_n(x) = x\mathbf{P}_n(x) - P_n(x)\mathbf{e}_n, \tag{6}$$

where $\mathbf{P}_n(x) = \begin{bmatrix} P_0(x) & P_1(x) & \ldots & P_{n-1}(x) \end{bmatrix}^T$, $\mathbf{e}_n = [0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1]^T$, and $H_n$ is the following lower (banded) Hessenberg matrix of order $n$

$$H_n = \begin{bmatrix} a_{0,r} & 1 & & & & & & & \\ a_{1,r-1} & a_{1,r} & 1 & & & & & & \\ \vdots & \ddots & \ddots & \ddots & & & & & \\ a_{r,0} & \cdots & a_{r,r-1} & a_{r,r} & 1 & & & & \\ & a_{r+1,0} & \cdots & a_{r+1,r-1} & a_{r+1,r} & 1 & & & \\ & & \ddots & & \ddots & \ddots & \ddots & & \\ & & & a_{n-2,0} & \cdots & a_{n-2,r-1} & a_{n-2,r} & 1 \\ & & & & a_{n-1,0} & \cdots & a_{n-1,r-1} & a_{n-1,r} \end{bmatrix}.$$

This kind of matrix has been obtained also in construction of orthogonal polynomials on the radial rays in the complex plane (see [15]).

Let $x_\nu \equiv x_\nu^{(n)}$, $\nu = 1, \ldots, n$, be the zeros of $P_n(x)$. Then (6) reduces to the following eigenvalue problem:

$$x_\nu \mathbf{P}_n(x_\nu) = H_n \mathbf{P}_n(x_\nu).$$

Thus, $x_\nu$ are eigenvalues of the matrix $H_n$ and $\mathbf{P}_n(x_\nu)$ are the corresponding eigenvectors. According to (6) it is easy to obtain the determinant representation $P_n(x) = \det(xI_n - H_n)$, where $I_n$ is the identity matrix of the order $n$.

For computing zeros of $P_n(x)$ as the eigenvalues of the matrix $H_n$ we use the EISPACK routine COMQR [25, pp. 277–284]. Notice that this routine needs an upper Hessenberg matrix, i.e., the matrix $H_n^T$. Also, the MATLAB or MATHEMATICA could be used.

Therefore, the main problem in the construction of the type II multiple orthogonal polynomials in this way is computation of the recurrence coefficients in (5), i.e., computation of entries of the Hessenberg matrix $H_n$. For the simplest case of multiple orthogonality, when $r = 2$, for some classical weight functions (Jacobi, Laguerre, Hermite) one can find explicit formulas for the recurrence coefficients (see [30], [32], [3]). An effective numerical method for constructing the Hessenberg matrix $H_n$ was given in [18].

## 3 Numerical Construction of Multiple Orthogonal Polynomials

In this section we describe the method for constructing the Hessenberg matrix $H_n$, presented in [18].

For the calculation of the recurrence coefficient we use some kind of the Stieltjes procedure (cf. [8]), called the *discretized Stieltjes-Gautschi procedure*. At first, we express the elements of $H_n$ in terms of the inner products[1] (4), and then we use the corresponding Gaussian rules to discretize these inner products. Of course, we suppose that the type II multiple orthogonal polynomials with respect to the inner products $(\cdot,\cdot)_k$, $k = 1, 2, \ldots, r$, given by (4), exist.

Taking $(\cdot,\cdot)_{j+\ell r} = (\cdot,\cdot)_j$, $\ell \in \mathbb{Z}$, for the inner products, the following result holds (see [18, Theorem 4.2]).

**Theorem 3.** *The type II multiple monic orthogonal polynomials* $\{P_n\}$*, with nearly diagonal multi-index, satisfy the recurrence relation*

$$P_{n+1}(x) = (x - a_{n,r})P_n(x) - \sum_{k=0}^{r-1} a_{n,k} P_{n-r+k}(x), \quad n \ge 0, \tag{7}$$

*where* $P_0(x) = 1$, $P_i(x) = 0$ *for* $i = -1, -2, \ldots, -r$,

$$a_{n,0} = \frac{\left(xP_n, P_{[(n-r)/r]}\right)_{\nu+1}}{\left(P_{n-r}, P_{[(n-r)/r]}\right)_{\nu+1}}$$

*and*

[1] Such formulas for coefficients of the three-term recurrence relation for standard orthogonal polynomials on the real line are known as Darboux formulas.

$$a_{n,k} = \frac{\left(xP_n - \sum_{i=0}^{k-1} a_{n,i}P_{n-r+i}, P_{[(n-r+k)/r]}\right)_{\nu+k+1}}{\left(P_{n-r+k}, P_{[(n-r+k)/r]}\right)_{\nu+k+1}}, \quad k = 1,2,\ldots,r.$$

*Here, we put $n = \ell r + \nu$, where $\ell = [n/r]$ and $\nu \in \{0,1,\ldots,r-1\}$ ($[t]$ denotes the integer part of $t$).*

We use alternatively recurrence relation (7) and given formulas for coefficients. Knowing $P_0$ we compute $a_{0,r}$, then knowing $a_{0,r}$ we compute $P_1$, and then again $a_{1,r}$ and $a_{1,r-1}$, etc. Continuing in this manner, we can generate as many polynomials, and therefore as many of the recurrence coefficients, as are desired.

All of the necessary inner products in the previous formulas can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature rule with respect to the corresponding weight function

$$\int_{E_j} g(t)w_j(t)\,\mathrm{d}t = \sum_{\nu=1}^{N} A_{j,\nu}^{(N)} g\big(\tau_{j,\nu}^{(N)}\big) + R_{j,N}(g), \quad j = 1,2,\ldots,r. \tag{8}$$

Thus, for all calculations we use only the recurrence relation (7) for the type II multiple orthogonal polynomials and the Gauss-Christoffel quadrature rules (8).

## 4 Multiple Orthogonal Polynomials on the Semicircle

Polynomials orthogonal on the semicircle have been introduced by Gautschi and Milovanović in [11]. Multiple orthogonal polynomials on the semicircle, investigated by Milovanović and Stanić in [19], are a generalization of orthogonal polynomials on the semicircle in the sense that they satisfy $r \in \mathbb{N}$ orthogonality conditions.

We repeat some basic facts about polynomials orthogonal on the semicircle, and then transfer the concept of multiple orthogonality to the semicircle.

Let $w$ be a weight function, which is positive and integrable on the open interval $(-1,1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$. Consider the following two inner products,

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,\mathrm{d}x, \tag{9}$$

$$[f,g] = \int_{\Gamma} f(z)g(z)w(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z = \int_{0}^{\pi} f\big(\mathrm{e}^{\mathrm{i}\theta}\big)g\big(\mathrm{e}^{\mathrm{i}\theta}\big)w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta, \tag{10}$$

where $\Gamma$ is the circular part of $\partial D_+$ and all integrals are assumed to exist, possibly as appropriately defined improper integrals.

The inner product (9) is positive definite and therefore generates a unique set of real orthogonal polynomials $\{p_k\}$ ($p_k$ is monic polynomial of degree $k$). The inner product (10) is not Hermitian and the existence of the corresponding orthogonal polynomials, therefore, is not guaranteed.

A system of complex polynomials $\{\pi_k\}$ ($\pi_k$ is monic of degree $k$) is called *orthogonal on the semicircle* if $[\pi_k, \pi_\ell] = 0$ for $k \neq \ell$ and $[\pi_k, \pi_\ell] \neq 0$ for $k = \ell$, $k, \ell = 0, 1, 2, \ldots$.

Gautschi, Landau and Milovanović in [10] have established the existence of orthogonal polynomials $\{\pi_k\}$ assuming only that

$$\mathrm{Re}\,[1,1] = \mathrm{Re}\int_0^\pi w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta \neq 0.$$

They have represented $\pi_n$ as a linear (complex) combination of $p_n$ and $p_{n-1}$, where $\{p_k\}$ is the sequence of the corresponding ordinary orthogonal (real) polynomials with respect to the inner product (9):

$$\pi_n(z) = p_n(z) - \mathrm{i}\theta_{n-1}p_{n-1}(z), \quad n \geq 0; \quad p_{-1}(x) = 0,\ p_0(x) = 1.$$

Under certain conditions zeros of polynomials orthogonal on the semicircle lie in $D_+$ (see [10, 11, 13, 14]).

Let $C_\varepsilon$, $\varepsilon > 0$, denotes the boundary of $D_+$ with small circular parts of radius $\varepsilon$ and centers at $\pm 1$ spared out. Let $c_{\varepsilon,\pm 1}$ are the circular parts of $C_\varepsilon$ with centers at $\pm 1$ and radii $\varepsilon$. We assume that $w$ is such that

$$\lim_{\varepsilon \downarrow 0} \int_{c_{\varepsilon,\pm 1}} g(z)w(z)\,\mathrm{d}z = 0, \quad \text{for all } g \in \mathcal{P},$$

the following equation holds

$$0 = \int_\Gamma g(z)w(z)\,\mathrm{d}z + \int_{-1}^1 g(x)w(x)\,\mathrm{d}x, \quad g \in \mathcal{P}.$$

It is well known that the real (monic) polynomials $\{p_k(z)\}$, orthogonal with respect to the inner product (9), as well as the associated polynomials of the second kind,

$$q_k(z) = \int_{-1}^1 \frac{p_k(z) - p_k(x)}{z - x} w(x)\,\mathrm{d}x, \quad k = 0, 1, 2, \ldots,$$

satisfy a three-term recurrence relation of the form

$$y_{k+1} = (z - a_k)y_k - b_k y_{k-1}, \quad k = 0, 1, 2, \ldots,$$

whit initial conditions $y_{-1} = 0$, $y_0 = 1$ for $\{p_k\}$, and $y_{-1} = -1$, $y_0 = 0$ for $\{q_k\}$.

**Definition 1.** For a positive integer $r$, a set $W = \{w_1, \ldots, w_r\}$ is *an admissible set of weight functions* if for the set $W$ there exist a unique system of the (real) type II multiple orthogonal polynomials and for each $w_j$, $j = 1, \ldots, r$, there exists a unique system of (monic, complex) orthogonal polynomials relative to the inner product (10).

Let $r \geq 1$ be an integer and let $W = \{w_1, w_2, \dots, w_r\}$ be an admissible set of weight functions. Let $\mathbf{n} = (n_1, n_2, \dots, n_r)$ be the multi-index with length $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$. Multiple orthogonal polynomial on the semicircle is the monic polynomial $\Pi_{\mathbf{n}}(z)$ of degree $|\mathbf{n}|$ such that it satisfies the following orthogonality conditions:

$$\int_\Gamma \Pi_{\mathbf{n}}(z) z^k w_j(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z = 0, \quad k = 0, 1, \dots, n_j - 1,\ j = 1, 2, \dots, r. \tag{11}$$

For $r = 1$ we have the ordinary orthogonal polynomials on the semicircle.

Let use denote by

$$[f,g]_j = \int_\Gamma f(z)g(z)w_j(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z = \int_0^\pi f\big(\mathrm{e}^{\mathrm{i}\theta}\big)g\big(\mathrm{e}^{\mathrm{i}\theta}\big)w_j\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta, \quad j = 1, 2, \dots, r, \tag{12}$$

the corresponding complex inner products.

The equations

$$0 = \int_\Gamma g(z)w_j(z)\,\mathrm{d}z + \int_{-1}^1 g(x)w_j(x)\,\mathrm{d}x \tag{13}$$

and

$$\int_\Gamma \frac{g(z)w_j(z)}{\mathrm{i}z}\,\mathrm{d}z = \pi g(0)w_j(0) + \mathrm{i}\,⨍_{-1}^1 \frac{g(x)w_j(x)}{x}\,\mathrm{d}x \tag{14}$$

hold for any polynomial $g$ for all $j = 1, 2, \dots, r$.

We consider only the nearly diagonal multi-indices $\mathbf{s}(n)$ and denote the corresponding multiple orthogonal polynomial on the semicircle by $\Pi_n(z) = \Pi_{\mathbf{s}(n)}(z)$. The corresponding type II multiple orthogonal polynomials (real) $\{P_n\}$ satisfy recurrence relation (7). Also, it is easy to see that for $j = 1, 2, \dots, r$ associated polynomials of the second kind

$$Q_n^{(j)}(z) = \int_{-1}^1 \frac{P_n(z) - P_n(x)}{z - x} w_j(x)\,\mathrm{d}x, \quad n = 0, 1, \dots,$$

satisfy the same recurrence relation (but with different initial conditions).

Let us denote by $\mu_k^{(j)}$, $k \in \mathbb{N}_0$, $j = 1, 2, \dots, r$, the moments for the inner products (12) , i.e.,

$$\mu_k^{(j)} = [z^k, 1]_j = \int_\Gamma z^k w_j(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z, \quad j = 1, 2, \dots, r, \quad k \in \mathbb{N}_0.$$

For zero moments we have

$$\mu_0^{(j)} = \int_\Gamma \frac{w_j(z)}{\mathrm{i}z}\,\mathrm{d}z = \pi w_j(0) + \mathrm{i}\,⨍_{-1}^1 \frac{w_j(x)}{x}\,\mathrm{d}x, \quad j = 1, 2, \dots, r. \tag{15}$$

Let us also denote

$$D_n = \begin{bmatrix} Q_{n-1}^{(1)}(0) - \mathrm{i}\mu_0^{(1)} P_{n-1}(0) & \cdots & Q_{n-r}^{(1)}(0) - \mathrm{i}\mu_0^{(1)} P_{n-r}(0) \\ Q_{n-1}^{(2)}(0) - \mathrm{i}\mu_0^{(2)} P_{n-1}(0) & \cdots & Q_{n-r}^{(2)}(0) - \mathrm{i}\mu_0^{(2)} P_{n-r}(0) \\ \vdots & & \vdots \\ Q_{n-1}^{(r)}(0) - \mathrm{i}\mu_0^{(r)} P_{n-1}(0) & \cdots & Q_{n-r}^{(r)}(0) - \mathrm{i}\mu_0^{(r)} P_{n-r}(0) \end{bmatrix}. \tag{16}$$

By using equations (13)–(14) for appropriately chosen polynomials $g$ and orthogonality conditions (11), one can prove existence and uniqueness of multiple orthogonal polynomials on the semicircle with additional conditions that all matrices $D_n$ are regular. The following theorem was proved in [21].

**Theorem 4.** *Let $r$ be a positive integer and $W = \{w_1, \dots, w_r\}$ be an admissible set of weight functions. Assume in addition that all matrices $D_n$, given by* (16)*, are regular. Denoting by $\{P_k\}$ the (real) type II multiple orthogonal polynomials, relative to the set $W$, we have the following representation*

$$\Pi_k(z) = P_k(z) + \theta_{k,1} P_{k-1}(z) + \theta_{k,2} P_{k-2}(z) + \cdots + \theta_{k,r} P_{k-r}(z).$$

*The coefficients $\theta_{k,j}$, $j = 1, 2, \dots, r$, are the solution of the following system of linear equations*

$$\sum_{j=1}^{r} \theta_{k,j} \left( Q_{k-j}^{(m)}(0) - \mathrm{i}\mu_0^{(m)} P_{k-j}(0) \right) = \mathrm{i}\mu_0^{(m)} P_k(0) - Q_k^{(m)}(0), \quad m = 1, 2, \dots, r.$$

The multiple orthogonal polynomials on the semicircle with nearly diagonal multi-index satisfy the recurrence relation of order $r+1$, too. In a similar way as in the real case, the recurrence coefficients and the multiple orthogonal polynomials on the semicircle could be obtained by using some kind of the discretized Stieltjes-Gautschi procedure. Taking $[f,g]_{j+\ell r} = [f,g]_j$ for each $\ell \in \mathbb{Z}$, the following theorem could be proved (see [19]).

**Theorem 5.** *The multiple orthogonal polynomials on the semicircle $\{\Pi_n\}$, with nearly diagonal multi-index, satisfy the recurrence relation*

$$\Pi_{n+1}(z) = (z - \alpha_{n,r}) \Pi_n(z) - \sum_{k=0}^{r-1} \alpha_{n,k} \Pi_{n-r+k}(x), \quad n \ge 0,$$

*where $\Pi_0(z) = 1$, $\Pi_{-1}(z) = \Pi_{-2}(z) = \cdots = \Pi_{-r}(z) = 0$,*

$$\alpha_{n,0} = \frac{\left[ z\Pi_n, \Pi_{[(n-r)/r]} \right]_{\nu+1}}{\left[ \Pi_{n-r}, \Pi_{[(n-r)/r]} \right]_{\nu+1}} \tag{17}$$

*and*

$$\alpha_{n,k} = \frac{\left[z\Pi_n - \sum_{i=0}^{k-1} \alpha_{n,i}\Pi_{n-r+i}, \Pi_{[(n-r+k)/r]}\right]_{\nu+k+1}}{\left[\Pi_{n-r+k}, \Pi_{[(n-r+k)/r]}\right]_{\nu+k+1}}, \quad k = 1,2,\ldots,r. \tag{18}$$

*Here, we put $n = \ell r + \nu$, where $\ell = [n/r]$ and $\nu \in \{0,1,\ldots,r-1\}$ ($[t]$ denotes the integer part of $t$).*

In order to apply the previous theorem, one has to calculate all of the inner products (17)–(18), i.e., the integrals of the following type $\int_\Gamma z^j \Pi_l(z) w_k(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z$. For $j \geq 1$, because of (13), these integrals could be calculated exactly, except for rounding errors, by using the corresponding Gaussian quadratures. For $j = 0$ one has

$$\int_\Gamma \frac{\Pi_l(z)w_k(z)\,\mathrm{d}z}{\mathrm{i}z} = \mu_0^{(k)}\Pi_l(0) + \mathrm{i}\int_{-1}^{1} \frac{\Pi_l(x) - \Pi_l(0)}{x} w_k(x)\,\mathrm{d}x,$$

and the corresponding Gaussian quadratures and (15) could be used.

Knowing the recurrence coefficients we form a complex lower banded Hessenberg matrix $H_n$ as in the real case. The zeros of the multiple orthogonal polynomials on the semicircle are the eigenvalues of the complex Hessenberg matrix $H_n$.

### 4.1 *Case* $r = 2$

Let $W = \{w_1, w_2\}$ be an admissible set of weight functions. The type II (real) multiple orthogonal polynomials satisfy the following recurrence relation

$$P_{k+1}(x) = (x - b_k)P_k(x) - c_k P_{k-1}(x) - d_k P_{k-2}(x), \quad k \geq 0, \tag{19}$$

with initial conditions $P_0(x) = 1$, $P_{-1}(x) = P_{-2} = 0$. The multiple orthogonal polynomials on the semicircle satisfy the following recurrence relation

$$\Pi_{k+1}(z) = (z - \beta_k)\Pi_k(z) - \gamma_k \Pi_{k-1}(z) - \delta_k \Pi_{k-2}(z), \quad k \geq 0, \tag{20}$$

with initial conditions $\Pi_0(z) = 1$, $\Pi_{-1}(z) = \Pi_{-2}(z) = 0$.

Using Theorem 4 for $k \geq 2$ we have the following equation

$$\Pi_k(z) = P_k(z) + \theta_{k,1}P_{k-1}(z) + \theta_{k,2}P_{k-2}(z), \tag{21}$$

where $\theta_{k,1}$ and $\theta_{k,2}$ are solution of the following system of linear equations

$$\begin{aligned}
\theta_{k,1}\big(Q_{k-1}^{(1)}(0) - \mathrm{i}\mu_0^{(1)}P_{k-1}(0)\big) + \theta_{k,2}\big(Q_{k-2}^{(1)}(0) - \mathrm{i}\mu_0^{(1)}P_{k-2}(0)\big)& \\
= \mathrm{i}\mu_0^{(1)}P_k(0) - Q_k^{(1)}(0)&, \\
\theta_{k,1}\big(Q_{k-1}^{(2)}(0) - \mathrm{i}\mu_0^{(2)}P_{k-1}(0)\big) + \theta_{k,2}\big(Q_{k-2}^{(2)}(0) - \mathrm{i}\mu_0^{(2)}P_{k-2}(0)\big)& \\
= \mathrm{i}\mu_0^{(2)}P_k(0) - Q_k^{(2)}(0)&.
\end{aligned}$$

Relations between $\theta_{k,1}$, $\theta_{k,2}$ and recurrence coefficients $b_k$, $c_k$, $d_k$ were derived in [21]:

$$\theta_{1,1} = b_0 - \frac{\mu_1^{(1)}}{\mu_0^{(1)}}, \quad \theta_{2,1} = b_0 + b_1 - \frac{\mu_0^{(1)}\mu_2^{(2)} - \mu_2^{(1)}\mu_0^{(2)}}{\mu_0^{(1)}\mu_1^{(2)} - \mu_1^{(1)}\mu_0^{(2)}},$$
$$\theta_{k,1} = b_{k-1} - \frac{d_{k-1}}{\theta_{k-1,2}}, \quad k \geq 3,$$
$$\theta_{2,2} = c_1 + b_0^2 - b_0\frac{\mu_0^{(1)}\mu_2^{(2)} - \mu_2^{(1)}\mu_0^{(2)}}{\mu_0^{(1)}\mu_1^{(2)} - \mu_1^{(1)}\mu_0^{(2)}} + \frac{\mu_1^{(1)}\mu_2^{(2)} - \mu_2^{(1)}\mu_1^{(2)}}{\mu_0^{(1)}\mu_1^{(2)} - \mu_1^{(1)}\mu_0^{(2)}},$$
$$\theta_{k,2} = c_{k-1} - d_{k-1}\frac{\theta_{k-1,1}}{\theta_{k-1,2}}, \quad k \geq 3.$$

Also, in [21], the recurrence coefficients $\beta_k$, $\gamma_k$ and $\delta_k$ were given as functions of $b_k$, $c_k$, $d_k$, $\theta_{k,1}$ and $\theta_{k,2}$:

$$\beta_0 = b_0 - \theta_{1,1},$$
$$\beta_1 = b_1 + \theta_{1,1} - \theta_{2,1}, \quad \gamma_1 = c_1 + \theta_{1,1}b_0 - \theta_{2,2} - \beta_1\theta_{1,1},$$
$$\gamma_2 = \theta_{2,2} + \theta_{2,1}(b_1 - \theta_{2,1}), \quad \delta_2 = d_2 - \gamma_2\theta_{1,1} - \beta_2\theta_{2,2} + c_1\theta_{2,1} + b_0\theta_{2,2},$$
$$\delta_3 = \theta_{3,2}(b_1 - \theta_{2,1}),$$
$$\beta_k = \theta_{k,1} + \frac{d_k}{\theta_{k,2}}, \quad \gamma_k = \theta_{k,2} + d_{k-1}\frac{\theta_{k,1}}{\theta_{k-1,2}}, \quad \delta_k = d_{k-2}\frac{\theta_{k,2}}{\theta_{k-2,2}}, \quad k \geq 4.$$

# 5 Applications of Multiple Orthogonality to Numerical Integration

## 5.1 An Optimal Set of Quadrature Rules

Starting with a problem that arise in the evaluation of computer graphics illumination models, Borges [4] has examined the problem of numerically evaluating a set of $r$ definite integrals taken with respect to distinct weight functions, but related to a common integrand and interval of integration. For such a problem it is not efficient to use a set of $r$ Gauss-Christoffel quadrature rules, because valuable information is wasted.

Borges has introduced a performance ratio, defined as:

$$R = \frac{\text{Overall degree of precision} + 1}{\text{Number of integrand evaluation}}.$$

Taking the set of $r$ Gauss-Christoffel quadrature rules, one has $R = 2/r$ and, hence, $R < 1$ for all $r > 2$.

If we select a set of $n$ distinct nodes, common for all quadrature rules, weight coefficients for each of $r$ quadrature rules can be chosen in such a way that $R=1$. Since the selection of nodes is arbitrary, the quadrature rules may not be the best possible. The aim is to find an optimal set of nodes, by simulating the development of the Gauss-Christoffel quadrature rules.

Let us denote by $W=\{w_1,w_2,\ldots,w_r\}$ an AT system. Following [4, Definition 3], we introduce the following definition.

**Definition 2.** Let $W$ be an AT system (the weight functions $w_j$, $j=1,2,\ldots,r$, are supported on the interval $E$), $\mathbf{n}=(n_1,n_2,\ldots,n_r)$ be a multi-index, and $n=|\mathbf{n}|$. A set of quadrature rules of the form

$$\int_E f(x)\,w_j(x)\,\mathrm{d}x \approx \sum_{\nu=1}^{n} A_{j,\nu} f(x_\nu), \quad j=1,2,\ldots,r, \tag{22}$$

is *an optimal set with respect to* $(W,\mathbf{n})$ if and only if the weight coefficients, $A_{j,\nu}$, and the nodes, $x_\nu$, satisfy the following equations:

$$\sum_{\nu=1}^{n} A_{j,\nu} x_\nu^{m+n_j-1} = \int_E x^{m+n_j-1} w_j(x)\,\mathrm{d}x, \quad m=0,1,\ldots,n; \quad j=1,2,\ldots,r.$$

The next generalization of fundamental theorem of Gauss-Christoffel quadrature rules holds (see [18] for the proof).

**Theorem 6.** *Let $W$ be an AT system, $\mathbf{n}=(n_1,n_2,\ldots,n_r)$, $n=|\mathbf{n}|$. The quadrature rules* (22) *form an optimal set with respect to $(W,\mathbf{n})$ if and only if*

$1^\circ$ *they are exact for all polynomials of degree less than or equal to $n-1$;*

$2^\circ$ *the polynomial $q(x)=\prod_{\nu=1}^{n}(x-x_\nu)$ is the type II multiple orthogonal polynomial $P_\mathbf{n}$ with respect to $W$.*

*Remark 1.* All zeros of the type II multiple orthogonal polynomial $P_\mathbf{n}$ are distinct and located in the interval $E$ (Theorem 2).

For $r=1$ in Definition 2 we have the Gauss-Christoffel quadrature rule.

According to Theorem 6, the nodes of the optimal set of quadrature rules (of Gaussian type) with respect to $(W,\mathbf{n})$ are the zeros of the type II multiple orthogonal polynomial $P_\mathbf{n}$ with respect to the given AT system $W$. When the nodes are known, the weight coefficients $A_{j,\nu}$, $j=1,2,\ldots,r$, $\nu=1,2,\ldots,n$, can be obtained as the solutions of the following Vandermonde systems of equations

$$V(x_1,x_2,\ldots,x_n)\begin{bmatrix} A_{j,1} \\ A_{j,2} \\ \vdots \\ A_{j,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{(j)} \\ \mu_1^{(j)} \\ \vdots \\ \mu_{n-1}^{(j)} \end{bmatrix}, \quad j=1,2,\ldots,r,$$

where

$$\mu_\nu^{(j)} = \int_E x^\nu w_j(x)\,\mathrm{d}x, \quad j = 1,2,\ldots,r, \quad \nu = 0,1,\ldots,n-1.$$

Each of these Vandermonde systems always has the unique solution, because the zeros of the type II multiple orthogonal polynomial $P_{\mathbf{n}}$ are distinct.

For the case of the nearly diagonal multi-indices $\mathbf{s}(n)$ we can compute the nodes $x_\nu$, $\nu = 1,2,\ldots,n$, of the Gaussian type quadrature rules as eigenvalues of the corresponding banded Hessenberg matrix $H_n$. Then, from the corresponding recurrence relation, it follows that the eigenvector associated with $x_\nu$ is given by $\mathbf{P}_n(x_\nu)$. We can use this fact to compute the weight coefficients $A_{j,\nu}$ by requiring that each rule correctly generate the first $n$ modified moments.

Let us denote by

$$V_n = \big[\mathbf{P}_n(x_1)\ \mathbf{P}_n(x_2)\ \ldots\ \mathbf{P}_n(x_n)\big]$$

the matrix of the eigenvectors of $H_n$, each normalized so that the first component is equal to 1. Then, the weight coefficients $A_{j,\nu}$ can be obtained by solving systems of linear equations

$$V_n \begin{bmatrix} A_{j,1} \\ A_{j,2} \\ \vdots \\ A_{j,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{*(j)} \\ \mu_1^{*(j)} \\ \vdots \\ \mu_{n-1}^{*(j)} \end{bmatrix}, \quad j = 1,2,\ldots,r,$$

where

$$\mu_\nu^{*(j)} = \int_E P_\nu(x)\, w_j(x)\,\mathrm{d}x, \quad j = 1,2,\ldots,r; \quad \nu = 0,1,\ldots,n-1,$$

are modified moments, $P_\nu = P_{\mathbf{s}(\nu)}$. All modified moments can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature rules with respect to the corresponding weight function $w_j$, $j = 1,2,\ldots r$.

In the same way as in the real case, we can generate the optimal set of quadrature rules

$$\int_0^\pi f\big(\mathrm{e}^{\mathrm{i}\theta}\big) w_j\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta \approx \sum_{\nu=1}^{n} \sigma_{j,\nu} f(\zeta_\nu), \quad j = 1,2,\ldots,r,$$

where for each $w_j$, $j = 1,2,\ldots,r$, the corresponding quadrature is exact for all polynomials of degree less than or equal to $n + n_j - 1$. The nodes of such optimal set of quadratures are zeros of the multiple orthogonal polynomial on the semicircle $\Pi_{\mathbf{n}}(z)$, i.e., in the case of the nearly diagonal multi-index, nodes are the eigenvalues of the Hessenberg matrix $H_n$. Using the corresponding eigenvectors we obtain the weight coefficients in a similar way as in the real case.

### 5.2 An Optimal Set of Quadrature Rules with Preassigned Nodes

Let $W = \{w_1, w_2, \ldots, w_r\}$ be an AT system. Following Definition 2 and ordinary quadrature rules of Gaussian type with preassigned abscissas (see, e.g., [7, Subsection 2.2.1]) we introduce the following definition (see [20]).

**Definition 3.** Let $W$ be an AT system (the weight functions $w_j$, $j = 1,2,\ldots,r$, are supported on the interval $E$), $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index, $n = |\mathbf{n}|$. A set of quadrature rules of the form:

$$\int_E f(x) w_j(x)\,\mathrm{d}x \approx \sum_{i=1}^{k} a_{j,i} f(y_i) + \sum_{\nu=1}^{n} A_{j,\nu} f(x_\nu), \quad j = 1,2,\ldots,r, \tag{23}$$

where the nodes $y_i \in E$, $i = 1,2,\ldots,k$, are fixed and prescribed in advance, is called *an optimal set of quadrature rules with preassigned nodes* $\{y_i\}_{i=1}^k$ with respect to $(W, \mathbf{n})$ if and only if the weight coefficients, $a_{j,i}, A_{j,\nu}$, and the nodes, $x_\nu$, satisfy the following equations:

$$\sum_{i=1}^{k} a_{j,i} y_i^{m+n_j+k-1} + \sum_{\nu=1}^{n} A_{j,\nu} x_\nu^{m+n_j+k-1} = \int_E x^{m+n_j+k-1} w_j(x)\,\mathrm{d}x, \quad m = 0,1,\ldots,n;$$

for $j = 1,2,\ldots,r$.

For the set of preassigned nodes $\{y_i\}_{i=1}^k$ we introduce $s(x)$ as a polynomial of degree $k$, with zeros at $y_i$, $i = 1,2,\ldots,k$. Let us denote

$$\widetilde{W} = \{\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_r\}, \;\; \widetilde{w}_j(x) = s(x) w_j(x), \quad j = 1,2,\ldots,r.$$

**Theorem 7.** *Let $W$ be an AT system, $\mathbf{n} = (n_1, n_2, \ldots, n_r)$, $n = |\mathbf{n}|$. Suppose that for preassigned nodes, $\{y_i\}_{i=1}^k$, $\widetilde{W}$ is also AT system. The set of quadrature rules* (23) *form the optimal set with preassigned nodes $\{y_i\}_{i=1}^k$ with respect to $(W, \mathbf{n})$ if and only if:*

$1^\circ$ *they are exact for all polynomials of degree less than or equal to $n+k-1$;*
$2^\circ$ *the polynomial $q(x) = \prod_{\nu=1}^{n}(x - x_\nu)$ is the type II multiple orthogonal polynomial $P_{\mathbf{n}}$ with respect to $\widetilde{W}$.*

*Proof.* Let us suppose first that the quadrature rules (23) form the optimal set with preassigned nodes $\{y_i\}_{i=1}^k$ with respect to $(W, \mathbf{n})$. In order to prove $1^\circ$ we note that for each $j = 1,2,\ldots,r$, the corresponding quadrature rule (23) is exact for all polynomials from $\mathcal{P}_{n+n_j+k-1}$ and then it is exact for those from $\mathcal{P}_{n+k-1}$. To prove $2^\circ$, for $j = 1,2,\ldots r$, we assume that $p_j(x) \in \mathcal{P}_{n_j-1}$. Then, $q(x) p_j(x) s(x) \in \mathcal{P}_{n+n_j+k-1}$. Since the corresponding quadrature rule is exact for all such polynomials, it follows that

$$\int_E q(x) p_j(x) s(x) w_j(x)\,\mathrm{d}x = \sum_{i=1}^{k} a_{j,i} q(y_i) p_j(y_i) s(y_i) + \sum_{\nu=1}^{n} A_{j,\nu} q(x_\nu) p_j(x_\nu) s(x_\nu).$$

Since $s(y_i)=0$ for $i=1,2,\ldots,k$ and $q(x_\nu)=0$ for $\nu=1,2,\ldots,n$, the both sums on the right hand side in the previous equation are identically zero. Thus, we have

$$\int_E q(x)p_j(x)\,s(x)w_j(x)\,\mathrm{d}x=0,\quad j=0,1,\ldots,r,$$

and 2° follows.

Let us now suppose that for quadrature rules (23) 1° and 2° hold.

For $j=1,2,\ldots,r$, let $t_j(x)$ be a polynomial from $\mathcal{P}_{n+n_j+k-1}$. We can write $t_j(x)=u_j(x)\cdot q(x)s(x)+v(x)$, where $u_j(x)\in\mathcal{P}_{n_j-1}$ and $v(x)\in\mathcal{P}_{n+k-1}$. It is easy to see that

$$t_j(y_i)=v(y_i),\ \ i=1,2,\ldots,k,\quad t_j(x_\nu)=v(x_\nu),\ \ \nu=1,2,\ldots,n. \tag{24}$$

Then, we obtain

$$\begin{aligned}\int_E t_j(x)\,w_j(x)\,\mathrm{d}x&=\int_E[u_j(x)q(x)s(x)+v(x)]\,w_j(x)\,\mathrm{d}x\\&=\int_E q(x)u_j(x)\,s(x)w_j(x)\,\mathrm{d}x+\int_E v(x)\,w_j(x)\,\mathrm{d}x.\end{aligned}$$

According to 2° we have $\int_E q(x)u_j(x)\,s(x)w_j(x)\,\mathrm{d}x=0$ and, therefore,

$$\int_E t_j(x)\,w_j(x)\,\mathrm{d}x=\int_E v(x)\,w_j(x)\,\mathrm{d}x.$$

Since $v(x)\in\mathcal{P}_{n+k-1}$, it follows from 1° that

$$\int_E v(x)\,w_j(x)\,\mathrm{d}x=\sum_{i=1}^{k}a_{j,i}v(y_i)+\sum_{\nu=1}^{n}A_{j,\nu}v(x_\nu)$$

and hence, using (24), we obtain

$$\begin{aligned}\int_E t_j(x)\,w_j(x)\,\mathrm{d}x&=\sum_{i=1}^{k}a_{j,i}v(y_i)+\sum_{\nu=1}^{n}A_{j,\nu}v(x_\nu)\\&=\sum_{i=1}^{k}a_{j,i}t_j(y_i)+\sum_{\nu=1}^{n}A_{j,\nu}t_j(x_\nu).\end{aligned}$$

This proves that for each $j=1,2,\ldots,r$, the corresponding quadrature rule is exact for all polynomials of degree $\le n+n_j+k-1$. □

According to Theorem 7, the nodes $x_\nu$, $\nu=1,2,\ldots,n$, of the optimal set of quadrature rules with preassigned nodes (23) are the zeros of the type II multiple orthogonal polynomial $P_{\mathbf{n}}$ with respect to the AT system $\widetilde{W}$. In the case of nearly diagonal multi-index we use the discretized Stieltjes–Gautschi procedure to compute those zeros. When the nodes are known, then for $j=1,2,\ldots,r$ we can choose the weight coefficients $a_{j,i}$, $i=1,2,\ldots,k$ and $A_{j,\nu}$, $\nu=1,2,\ldots,n$, such that they satisfy the following Vandermonde system of equations

$$V(y_1,\dots,y_k,x_1,\dots,x_n)\begin{bmatrix} a_{j,1}\\ \vdots \\ a_{j,k}\\ A_{j,1}\\ \vdots \\ A_{j,n}\end{bmatrix} = \begin{bmatrix} \mu_0^{(j)}\\ \mu_1^{(j)}\\ \vdots \\ \mu_{n+k-1}^{(j)}\end{bmatrix}, \quad j=1,2,\dots,r, \tag{25}$$

where

$$\mu_i^{(j)} = \int_E x^i w_j(x)\,\mathrm{d}x, \quad j=1,2,\dots,r; \quad i=0,1,\dots,n+k-1,$$

are moments which can be computed exactly, except for rounding errors, by using the Gauss–Christoffel quadrature rules with respect to the corresponding weight function $w_j$, $j=1,2,\dots r$.

Each of Vandermonde systems (25) has a unique solution if all of the preassigned nodes are distinct from the zeros of type II multiple orthogonal polynomial $P_{\mathbf{n}}$ with respect to $\widetilde{W}$. This is always satisfied in cases when the preassigned nodes are at the end points of the interval $E$, i.e., in the case of quadrature rules of Gauss-Radau or Gauss-Lobatto type.

### 5.3 Connections with Generalized Birkhoff-Young Quadrature Rules

In 1950 Birkhoff and Young [2] proposed a quadrature formula of the form

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z \approx \frac{h}{15}\Big\{24f(z_0)+4\big[f(z_0+h)+f(z_0-h)\big]-\big[f(z_0+\mathrm{i}h)+f(z_0-\mathrm{i}h)\big]\Big\}$$

for numerical integration over a line segment in the complex plane, where $f(z)$ is a complex analytic function in $\{z : |z-z_0|\le r\}$ and $|h|\le r$. This five point quadrature formula is exact for all algebraic polynomials of degree at most five and for its error $R_5^{BY}(f)$ can be proved the following estimate [33] (see also Davis and Rabinowitz [7, p. 136])

$$|R_5^{BY}(f)| \le \frac{|h|^7}{1890}\max_{z\in S}|f^{(6)}(z)|,$$

where $S$ denotes the square with vertices $z_0+\mathrm{i}^k h$, $k=0,1,2,3$.

Without loss of generality the previous quadrature rule can be considered over $[-1,1]$ for analytic functions in the unit disk $\{z : |z|\le 1\}$, so that it becomes

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = \frac{16}{15}f(0)+\frac{4}{15}\big[f(1)+f(-1)\big]-\frac{1}{15}\big[f(\mathrm{i})+f(-\mathrm{i})\big]+R_5(f). \tag{26}$$

In 1978 Tošić [29] obtained a significant improvement of (26) in the form

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = Af(0)+\frac{1}{6}\left(\frac{7}{5}+\sqrt{\frac{7}{3}}\right)\big[f(r)+f(-r)\big] + \frac{1}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right)\big[f(\mathrm{i}r)+f(-\mathrm{i}r)\big]+R_5^T(f),$$

where $r=\sqrt[4]{3/7}$ and

$$R_5^T(f)=\frac{1}{793800}f^{(8)}(0)+\frac{1}{61122600}f^{(10)}(0)+\cdots\,.$$

This formula was extended by Milovanović and Đorđević [17] to the following quadrature formula of interpolatory type

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = Af(0)+C_{11}\big[f(r_1)+f(-r_1)\big]+C_{12}\big[f(\mathrm{i}r_1)+f(-\mathrm{i}r_1)\big] + C_{21}\big[f(r_2)+f(-r_2)\big]+C_{22}\big[f(\mathrm{i}r_2)+f(-\mathrm{i}r_2)\big]+R_9(f;r_1,r_2),$$

where $0<r_1<r_2<1$. They proved that for

$$r_1=r_1^*=\sqrt[4]{\frac{63-4\sqrt{114}}{143}} \quad \text{and} \quad r_2=r_2^*=\sqrt[4]{\frac{63+4\sqrt{114}}{143}},$$

this formula has the algebraic degree of precision $p=13$, with the error-term

$$R_9(f;r_1^*,r_2^*)=\frac{1}{28122661066500}f^{(14)}(0)+\cdots \approx 3.56\cdot 10^{-14}f^{(14)}(0).$$

In this subsection we consider a kind of generalized Birkhoff-Young quadrature formulas and give a connection with multiple orthogonal polynomials (cf. [16]). We introduce $N$-point quadrature formula for weighted integrals of analytic functions in the unit disc $\{z : |z|\le 1\}$,

$$I(f):=\int_{-1}^{1} f(z)w(z)\,\mathrm{d}z = Q_N(f)+R_N(f),$$

where $w:(-1,1)\to\mathbb{R}^+$ is an even positive weight function, for which all moments $\mu_k=\int_{-1}^{1}z^k w(z)\,\mathrm{d}z$, $k=0,1,\ldots$, exist. For a given fixed integer $m\ge 1$ and for each $N\in\mathbb{N}$, we put $N=2mn+\nu$ and define the node polynomial

$$\Omega_N(z)=z^\nu\omega_{n,\nu}(z^{2m})=z^\nu\prod_{k=1}^{n}(z^{2m}-r_k), \quad 0<r_1<\cdots<r_n<1, \tag{27}$$

where $n=[N/2m]$ and $\nu\in\{0,1,\ldots,2m-1\}$.

Now we consider the interpolatory quadrature rule $Q_N$ of the form

$$Q_N(f)=\sum_{j=0}^{\nu-1}C_j f^{(j)}(0)+\sum_{k=1}^{n}\sum_{j=1}^{m}A_{k,j}\big[f\big(x_k\mathrm{e}^{\mathrm{i}\theta_j}\big)+f\big(-x_k\mathrm{e}^{\mathrm{i}\theta_j}\big)\big],$$

where

$$x_k=\sqrt[2m]{r_k},\quad k=1,\ldots,n;\quad \theta_j=\frac{(j-1)\pi}{m},\quad j=1,\ldots,m.$$

If $\nu=0$, the first sum in $Q_N(f)$ is empty.

Following [16] we can prove the next result:

**Theorem 8.** *Let $m$ be a fixed positive integer and $w$ be an even positive weight function $w$ on $(-1,1)$, for which all moments $\mu_k=\int_{-1}^{1}z^k w(z)\,\mathrm{d}z$, $k\geq 0$, exist. For any $N\in\mathbb{N}$ there exists a unique interpolatory quadrature rule $Q_N(f)$ with a maximal degree of exactness $d_{\max}=2(m+1)n+s$, where*

$$n=\left[\frac{N}{2m}\right],\quad \nu=N-2mn,\quad s=\begin{cases}\nu-1, & \nu\ \text{even},\\ \nu, & \nu\ \text{odd}.\end{cases}\tag{28}$$

*The node polynomial $(27)$ is characterized by the following orthogonality relations*

$$\int_{-1}^{1}z^{2k+s+1}\omega_{n,\nu}(z^{2m})w(z)\,\mathrm{d}z=0,\quad k=0,1,\ldots,n-1.\tag{29}$$

The conditions (29) can be expressed in the form

$$\int_{-1}^{1}p_{2k}(z)z^{s+1}\omega_{n,\nu}(z^{2m})w(z)\,\mathrm{d}z=0,\quad k=0,1,\ldots,n-1,$$

where $\{p_k\}_{k\in\mathbb{N}_0}$ is a system of polynomials orthogonal with respect to the weight $w$ on $(-1,1)$.

The case with the Chebyshev weight of the first kind $w(z)=1/\sqrt{1-z^2}$ and $m=2$ was recently considered by Milovanović, Cvetković and Stanić [22]. In that case the previous conditions reduce to

$$\big(T_{2k},z^{s+1}p_{n,\nu}(z^4)\big)=\int_{-1}^{1}\frac{T_{2k}(z)z^{s+1}p_{n,\nu}(z^4)}{\sqrt{1-z^2}}\,\mathrm{d}z=0,\quad k=0,1\ldots,n-1,$$

where $T_k$ is the Chebyshev polynomial of the first kind of degree $k$. The corresponding quadrature rules are

$$Q_{4n+\nu}(f)=\sum_{j=0}^{\nu-1}C_j f^{(j)}(0)+\sum_{k=1}^{n}\Big\{A_k\big[f(x_k)+f(-x_k)\big]+B_k\big[f(\mathrm{i}x_k)+f(-\mathrm{i}x_k)\big]\Big\},$$

where $\nu=0,1,2,3$. For $\nu=0$, the first sum on the right-hand side is empty. Also, in order to have $Q_{4n+\nu}(f)=I(f)=0$ for $f(z)=z$, it must be $C_1=0$, so that $Q_{4n+1}(f)\equiv Q_{4n+2}(f)$.

The parameters of the quadrature formula $Q_{4n+\nu}(f)$ as well as the corresponding maximal degree of exactness $d = 6n+s$, where $s$ is defined by (28), are presented in Table 1 for $n = 1$ and $\nu = 0,1,2,3$.

**Table 1** Parameters and the maximal degree of exactness of the generalized Birkhoff-Young-Chebyshev quadrature formula $Q_{4+\nu}(f)$ for $\nu = 0,1,2,3$

| $\nu$ | $x_1$ | $A_1$ | $B_1$ | $C_0$ | $C_2$ | $d$ |
|---|---|---|---|---|---|---|
| 0 | $\sqrt[4]{\frac{3}{8}}$ | $\frac{\pi}{2}\left(\frac{1}{2}+\frac{1}{\sqrt{6}}\right)$ | $\frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{\sqrt{6}}\right)$ | | | 5 |
| $1,2$ | $\sqrt[4]{\frac{5}{8}}$ | $\frac{3+\sqrt{10}}{20}\pi$ | $\frac{3-\sqrt{10}}{20}\pi$ | $\frac{2\pi}{5}$ | | 7 |
| 3 | $\frac{1}{2}\sqrt[4]{\frac{35}{3}}$ | $\frac{3(21+2\sqrt{105})}{490}\pi$ | $\frac{3(21-2\sqrt{105})}{490}\pi$ | $\frac{17\pi}{35}$ | $\frac{\pi}{28}$ | 9 |

By substitution $z^2 = t$, the orthogonality conditions (29) can be expressed in the form
$$\int_0^1 t^k \omega_{n,\nu}(t^m) t^{s/2} w(\sqrt{t})\,\mathrm{d}t = 0, \quad k = 0,1\ldots,n-1.$$
This means that the polynomial $t \mapsto \omega_{n,\nu}(t^m)$ of degree $mn$ is orthogonal to $\mathcal{P}_{n-1}$ with respect to the weight function $t^{s/2}w(\sqrt{t})$ on $(0,1)$, and it can be interpreted in terms of multiple orthogonal polynomials (see Milovanović [16]). Namely, these conditions are equivalent to
$$\int_0^1 t^{k/m} p_{n,\nu}(t) t^{(s+2)/(2m)-1} w(t^{1/(2m)})\,\mathrm{d}t = 0, \quad k = 0,1,\ldots,n-1.$$
Putting $k = m\ell + j - 1,\ \ell = [k/m]$, we get for each $j = 1,\ldots,m$,
$$\int_0^1 t^\ell p_{n,\nu}(t) w_j(t)\,\mathrm{d}t = 0, \quad \ell = 0,1\ldots,n_j - 1,$$
where
$$w_j(t) = t^{(s+2j)/(2m)-1} w(t^{1/(2m)}) \quad \text{and} \quad n_j = 1 + \left[\frac{n-j}{m}\right].$$
Notice that these weight functions, defined on the same interval $E_1 = E_2 = \cdots = E_m = E = (0,1)$, can be expressed in the form $w_j(t) = t^{(j-1)/m} w_1(t),\ j = 1,\ldots,m$, where $w_1(t) = t^{(s+2)/(2m)-1} w(t^{1/(2m)})$. Since the Müntz system
$$\left\{t^{k+(j-1)/m}\right\}, \quad k = 0,1,\ldots,n_j - 1;\ j = 1,\ldots,m,$$
is a Chebyshev system on $[0,\infty)$, and also on $E = (0,1)$, and $w_1(t) > 0$ on $E$, we conclude that $\{w_j,\ j = 1,\ldots,m\}$ is an AT system on $E$.

Therefore, according to Theorem 2, the unique type II multiple orthogonal polynomial $\omega_{n,\nu}(t) = P_{\mathbf{n}}(t)$ has exactly

$$|\mathbf{n}| := \sum_{j=1}^{m} n_j = \sum_{j=1}^{m} \left(1 + \left[\frac{n-j}{m}\right]\right) = n$$

zeros in $(0,1)$. Thus, we have the following result [16]:

**Theorem 9.** *Under conditions of Theorem* 8*, for any* $N \in \mathbb{N}$ *there exists a unique interpolatory quadrature rule* $Q_N(f)$*, with a maximal degree of exactness*

$$d_{\max} = 2(m+1)n + s,$$

*if and only if the polynomial* $\omega_{n,\nu}(t)$ *is the type II multiple orthogonal polynomial* $P_{\mathbf{n}}(t)$*, with respect to the weights* $w_j(t) = t^{(s+2j)/(2m)-1} w(t^{1/(2m)})$*, with*

$$n_j = 1 + \left[\frac{n-j}{m}\right], \quad j = 1, \ldots, m.$$

## 6 Numerical Example

As an example we consider the type II multiple orthogonal Jacobi polynomials, i.e., the type II multiple orthogonal polynomials with respect to an AT system consisting of Jacobi weight functions on $[-1,1]$ with different singularities at $-1$ and the same singularity at 1. Weight functions are

$$w_j(x) = (1-x)^{\alpha}(1+x)^{\beta_j}, \quad j = 1, 2, \ldots, r,$$

where $\alpha, \beta_j > -1$, $j = 1, 2, \ldots, r$, and $\beta_i - \beta_l \notin \mathbb{Z}$ whenever $i \neq l$.

In Table 2 the coefficients of recurrence relation (7) for multiple orthogonal Jacobi polynomials in the case $r = 3$, $\alpha = 1/2$, $\beta_1 = -1/4$, $\beta_2 = 1/4$, $\beta_3 = 1$ for $n \leq 16$ are given (numbers in parentheses denote decimal exponents). The nodes $x_\nu$ and the weights $A_{j,\nu}$, $\nu = 1, \ldots, 16$, $j = 1, 2, 3$, of the corresponding optimal set of quadrature rules (22) are given in Table 3.

**Table 2** Recursion coefficients $a_{n,k}$, $k = 0, 1, \ldots, r$, for the type II multiple orthogonal Jacobi polynomials with $r = 3$, $\alpha = 1/2$, $\beta_1 = -1/4$, $\beta_2 = 1/4$, $\beta_3 = 1$; $n \le 16$

| $n$ | $a_{n,3}$ | $a_{n,2}$ |
|---|---|---|
| 0 | −3.333333333333333(−1) | |
| 1 | −1.282051282051282(−1) | 2.735042735042735(−1) |
| 2 | −8.082010868388577(−2) | 2.661439536886072(−1) |
| 3 | −1.797818980050774(−1) | 2.623762626705582(−1) |
| 4 | −1.559462948426531(−1) | 2.653111297708491(−1) |
| 5 | −1.239638179278716(−1) | 2.659979011724685(−1) |
| 6 | −1.709146651380284(−1) | 2.654960405557197(−1) |
| 7 | −1.579012355168128(−1) | 2.662346749483896(−1) |
| 8 | −1.359869263770880(−1) | 2.664756863684053(−1) |
| 9 | −1.669363956328833(−1) | 2.662496940945655(−1) |
| 10 | −1.580814662624477(−1) | 2.665641681228860(−1) |
| 11 | −1.415386037831715(−1) | 2.666771188543775(−1) |
| 12 | −1.646602203100053(−1) | 2.665436153306013(−1) |
| 13 | −1.579776557002493(−1) | 2.667136879056348(−1) |
| 14 | −1.447181043954951(−1) | 2.667770009974078(−1) |
| 15 | −1.631843805063865(−1) | 2.666880187224645(−1) |
| 16 | −1.578284743344368(−1) | 2.667934513861474(−1) |

| $n$ | $a_{n,1}$ | $a_{n,0}$ |
|---|---|---|
| 2 | 2.970182155702518(−2) | |
| 3 | 1.746702080553980(−2) | −1.086753955083950(−3) |
| 4 | 4.394216071462117(−2) | 7.836954608420134(−4) |
| 5 | 3.763075042610465(−2) | 3.283125040895112(−3) |
| 6 | 2.909135291014223(−2) | 9.936110019727833(−4) |
| 7 | 4.156697542465302(−2) | 1.563768261907128(−3) |
| 8 | 3.808465719477277(−2) | 2.779000545083734(−3) |
| 9 | 3.222685629651563(−2) | 1.386312233788444(−3) |
| 10 | 4.046385429052276(−2) | 1.739912682414390(−3) |
| 11 | 3.809384444106908(−2) | 2.551616250383902(−3) |
| 12 | 3.367302798492897(−2) | 1.555345369944956(−3) |
| 13 | 3.983305940039197(−2) | 1.813811205210830(−3) |
| 14 | 3.804538508869144(−2) | 2.424240420958617(−3) |
| 15 | 3.450289136608302(−2) | 1.649497148723416(−3) |
| 16 | 3.942565410008006(−2) | 1.853693596062819(−3) |

**Acknowledgements** The authors were supported in part by the Serbian Ministry of Education and Science (Project: Approximation of Integral and Differential Operators and Applications, grant number #174015).

**Table 3** The parameters of the optimal set of quadrature rules in the case of AT system of Jacobi weights for $r = 3$, $\alpha = 1/2$, $\beta_1 = -1/4$, $\beta_2 = 1/4$, $\beta_3 = 1$; $n = 16$

| $\nu$ | $x_\nu$ | $A_{1,\nu}$ |
|---|---|---|
| 1 | −9.991207278514688(−1) | 2.593845860971087(−2) |
| 2 | −9.903618344136677(−1) | 7.241932746868121(−2) |
| 3 | −9.638312475017886(−1) | 1.232021800649184(−1) |
| 4 | −9.114418918738332(−1) | 1.705746075613651(−1) |
| 5 | −8.280210844814640(−1) | 2.096867833494312(−1) |
| 6 | −7.115498578342734(−1) | 2.372180134962362(−1) |
| 7 | −5.631228057882331(−1) | 2.511227240543505(−1) |
| 8 | −3.867448648543452(−1) | 2.506499390258389(−1) |
| 9 | −1.889812469123731(−1) | 2.363793559115719(−1) |
| 10 | 2.346960814946570(−1) | 1.750234363552845(−1) |
| 11 | 2.153170148057211(−2) | 2.101750909061207(−1) |
| 12 | 4.396510791687633(−1) | 1.347552229954631(−1) |
| 13 | 6.255191622587539(−1) | 9.367498172796770(−2) |
| 14 | 7.821521528902294(−1) | 5.613339333990859(−2) |
| 15 | 9.008275402581413(−1) | 2.608777198579168(−2) |
| 16 | 9.748476093398128(−1) | 6.697740217113879(−3) |
| $\nu$ | $A_{2,\nu}$ | $A_{3,\nu}$ |
| 1 | 7.595267817320088(−4) | 3.896535630665977(−6) |
| 2 | 7.109430002756949(−3) | 2.187023588317833(−4) |
| 3 | 2.343071830873815(−2) | 1.943281045595175(−3) |
| 4 | 5.076081012916438(−2) | 8.240428332930785(−3) |
| 5 | 8.695782331878246(−2) | 2.322282583304115(−2) |
| 6 | 1.274039941410160(−1) | 5.014597864881742(−2) |
| 7 | 1.659837991885244(−1) | 8.919391597730677(−2) |
| 8 | 1.962854916209727(−1) | 1.360251129413198(−1) |
| 9 | 2.128751628357952(−1) | 1.819274220554127(−1) |
| 10 | 1.944805801244912(−1) | 2.277961048571476(−1) |
| 11 | 2.124257539351905(−1) | 2.158470188824615(−1) |
| 12 | 1.616866751974687(−1) | 2.125040277473365(−1) |
| 13 | 1.194317136600792(−1) | 1.719347786751889(−1) |
| 14 | 7.493654857422641(−2) | 1.155852112758036(−1) |
| 15 | 3.596734227391774(−2) | 5.822578930636818(−2) |
| 16 | 9.412285520984374(−3) | 1.567997205810892(−2) |

## References


1. Aptekarev, A.I.: Multiple orthogonal polynomials. J. Comput. Appl. Math. **99**, 423–447 (1998)
2. Birkhoff, G., Young, D.M.: Numerical quadrature of analytic and harmonic functions. J. Math. Phys. **29**, 217–221 (1950)
3. Beckermann, B., Coussement, J., Van Assche, W.: Multiple Wilson and Jacobi-Piñeiro polynomials. J. Approx. Theory **132**(2), 155–181 (2005)
4. Borges, C.F.: On a class of Gauss-like quadrature rules. Numer. Math. **67**, 271–288 (1994)
5. De Bruin, M.G.: Simultaneous Padé approximation and orthogonality. In C. Brezinski, A. Draux, A.P. Magnus, P. Maroni, and A. Ronveaux, editors, Proc. Polynômes Orthogoneaux et

Applications, Bar-le-Duc, 1984, volume 1171 of Lecture Notes in Math., pp. 74–83, Springer (1985)
6. Chihara, T.S.: An Introduction to Orthogonal Polynomials. Gordon and Breach, New York (1978)
7. Davis, P.J., Rabinowitz, P.: Methods of Numerical Integration. Academic Press, New York, San Francisco, London (1975)
8. Gautschi, W.: Orthogonal polynomials: applications and computation. Acta Numerica **5**, 45–119 (1996)
9. Gautschi, W.: Orthogonal Polynomials: Computation and Approximation. Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford (2004)
10. Gautschi, W., Landau, H.J., Milovanović, G.V.: Polynomials orthogonal on the semicircle, II. Constr. Approx. **3**, 389–404 (1987)
11. Gautschi, W., Milovanović, G.V.: Polynomials orthogonal on the semicircle. J. Approx. Theory **46**, 230–250 (1986)
12. Mastroianni, G., Milovanović, G.V.: Interpolation Processes - Basic Theory and Applications. Springer Monographs in Mathematics, Springer - Verlag, Berlin - Heidelberg (2008)
13. Milovanović, G.V.: Complex orthogonality on the semicircle with respect to Gegenbauer weight: Theory and aplications. In: Topics in Mathematical Analysis (T. M. Rassias, ed.), pp. 695–722, Ser. Pure Math., 11, World Sci. Publishing, Teaneck, NJ (1989)
14. Milovanović, G.V.: On polynomials orthogonal on the semicircle and applications. J. Comput. Appl. Math. **49**, 193–199 (1993)
15. Milovanović, G.V.: Orthogonal polynomials on the radial rays in the complex plane and applications. Rend. Circ. Mat. Palermo, Serie II, Suppl. **68**, 65–94 (2002)
16. Milovanović, G.V.: Numerical quadratures and orthogonal polynomials. Stud. Univ. Babeş-Bolyai Math. **56** (2011) (to appear)
17. Milovanović, G.V., Đorđević, R.Ž.: On a generalization of modified Birkhoff-Young quadrature formula. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fis., No. 735 – No. 762, 130–134 (1982)
18. Milovanović, G.V., Stanić, M.: Construction of multiple orthogonal polynomials by discretized Stieltjes-Gautschi procedure and corresponding Gaussian quadratures. Facta Univ. Ser. Math. Inform. **18**, 9–29 (2003)
19. Milovanović, G.V., Stanić, M.: Multiple orthogonal polynomials on the semicircle and corresponding quadratures of Gaussian type. Math. Balkanica (N. S.) **18**, 373–387 (2004)
20. Milovanović, G.V., Stanić, M.: Multiple orthogonality and quadratures of Gaussian type, Rend. Circ. Mat. Palermo, Serie II, Suppl. **76**, 75–90 (2005)
21. Milovanović, G.V., Cvetković A.S., Stanić, M.P.: Multiple orthogonal polynomials on the semicircle, Facta Univ. Ser. Math. Inform. **20**, 41–55 (2005)
22. Milovanović, G.V., Cvetković A.S., Stanić, M.P.: A generalized Birkhoff-Young-Chebyshev quadrature formula for analytic functions. Appl. Math. Comput. doi:10.1016/j.amc.2011.02.007 (to appear)
23. Nikishin, E.M., Sorokin, V.N.: Rational Approximations and Orthogonality, vol. 92, Transl. Amer. Math. Soc., Providence, Rhode Island (1991)
24. Piñeiro, L.R.: On simultaneous approximations for a collection of Markov functions. Vestnik Mosk. Univ., Ser. I, no. 2, 67–70 (1987) [English translation in Moscow Univ. Math. Bull. **42**(2), 52–55 (1987)]
25. Smith, B.T., Boyle, J.M., Dongarra, J.J, Garbow, B.S., Ikebe, Y., Klema, V.C., Moler, C.B.: Matrix Eigensystem Routines – EISPACK Guide. Lect. Notes Comp. Science Vol. 6, Springer Verlag, Berlin – Heidelberg – New York (1976)
26. Sorokin, V.N.: Generalization of classical polynomials and convergence of simultaneous Padé approximants. Trudy Sem. Petrovsk. **11**, 125–165 (1986) [English translation in J. Soviet Math. **45**, 1461–1499 (1986)]
27. Sorokin, V.N.: Simultaneous Padé approximation for functions of Stieltjes type. Sib. Mat. Zh. **31**(5), 128–137 (1990) [English translation in Sib. Math. J. **31**(5), 809–817 (1990)]

28. Sorokin, V.N.: Hermite-Padé approximations for Nikishin systems and the irrationality of $\zeta(3)$. Uspekhi Mat. Nauk **49**(2), 167–168 (1994) [English translation in Russian Math. Surveys **49**(2), 176–177 (1994)]
29. Tošić, D.Đ.: A modification of the Birkhoff-Young quadrature formula for analytic functions. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fis. No. 602 – No. 633, 73–77 (1978)
30. Van Assche, W.: Multiple orthogonal polynomials, irratinality and transcendence. In: Continued Fractions: From Analytic Number Theory to Constructive Approximation (B.C. Berndt and F. Gesztesy, eds.), Contemporary Mathematics **236**, 325–342 (1999)
31. Van Assche, W.: Non-symetric linear difference equations for multiple orthogonal polynomials. CRM Proceedings and Lecture Notes, Vol. **25**, Amer. Math. Soc., Providence, RI, pp. 391–405 (2000)
32. Van Assche, W., Coussement, E.: Some classical multiple orthogonal polynomials. J. Comput. Appl. Math. **127**, 317–347 (2001)
33. Young, D.M.: An error bound for the numerical quadrature of analytic functions. J. Math. Phys. **31**, 42–44 (1952)