Numerical integration of highly–oscillating functions

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Abstract

1 Introduction

By a highly–oscillating function we mean one with large number of local maxima and minima over some interval. The computation of integrals of highly–oscillating functions is one of the most important issues in numerical analysis since such integrals abound in applications in many branches of mathematics as well as in other sciences, e.g., quantum physics, fluid mechanics, electromagnetics, etc. The principal examples of highly–oscillating integrands occur in various transforms, e.g., Fourier transform, Fourier–Bessel transform, etc. The standard methods of numerical integration often require too much computation work and cannot be successfully applied. Because of that, for integrals of highly–oscillating functions there are a large number of special approaches, which are effective. In this paper we give survey of some special quadrature methods for different types of highly-oscillating integrands.

The earliest formulas for numerical integration of highly–oscillating functions were given by Filon [12] in 1928. Filon's approach for the Fourier integral on the finite interval,

$$I[f;\omega] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x,$$

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is based on the piecewise approximation of $f(x)$ by parabolic arcs on the integration interval. The resulting integrals over subintervals are then integrated exactly. One can divide interval $[a,b]$ into $2N$ subintervals of equal length $h = (b-a)/(2N)$. Let $x_k = a + kh$, $k = 0,1,\ldots,2N$. The Filon's formula is based on a quadratic fit of function $f(x)$ on every subinterval $[x_{2k-2}, x_{2k}]$, $k = 1,\ldots,N$, by interpolation at the mesh points. The error estimate was given by Håvie [19] and Ehrenmark [7].

It can be said that Filon's idea is one of the most fruitful in topic of integration of highly–oscillating functions, because of a wide range of improvements of the previous technique. Luke [36] in 1954 approximated the function $f(x)$ in a certain interval by a polynomial of at most 10th degree. Flinn [13] used 5th degree polynomials in order to approximate $f(x)$ taking values of function and values of its derivative at the points x_{2k-2}, x_{2k-1}, and x_{2k}. Stetter [60] used the idea of approximating the transformed function by polynomials in $1/t$. Tuck [61] suggested so called Filon–trapezoidal rule, where polygonal arches were used instead of parabolic arches. The Filon modification in such rule is nothing more than a simple multiplicative factor applied to the results of the crude trapezoidal rule. Einarsson [8] derived so called Filon–spline rule by passing cubic splines through functional values. Shampine [56] proposed method based on a smooth cubic spline, implemented in a Matlab program. An adaptive implementation of his method deal with function f that have peaks. His basic method can be adjusted to deal effectively with function f that have a moderate singularity at one or both ends of $[a,b]$. Miklosko [38] used an interpolatory quadrature formula with the Chebyshev nodes. Van de Vooren and van Linde [63] obtained the Fourier integral quadrature rules which for the real part are exact if f is of at most 7th degree, and for the imaginary part if f is of at most 8th degree.

Ixary and Paternoster [25, 26] derived exponential fitting approach for the Fourier integral on $[-1,1]$, designed to be exact when the integrand is some suitably chosen combination of exponential functions, e.g., with polynomial terms, or products of polynomials and exponentials.

Recently, Ledoux and Van Daele [29] made connection between Filon–type and exponential fitting methods. By introducing some S-shaped functions, they constructed Gauss–type rules for the Fourier integral $I[f;\omega]$ on $[-1,1]$ interpolating f in frequency dependent nodes along with Chebyshev nodes. In such a way they derived rules with an optimal asymptotic rate of decay of the error with increasing frequency, which are effective for small or moderate frequencies, too.

Very simple methods can be obtained by integration between the zeros. If the zeros of the oscillatory part of the integrand are located in the points $a \le x_1 < x_2 < \cdots < x_m \le b$, then the integral on each subinterval $[x_k, x_{k+1}]$ can be calculated by an appropriate rule. A Lobatto rule is good for this purpose because of use the end points of the integration subintervals, where the integrand is zero, so, more accuracy can be obtained without additional computation (see [6]).

Several authors (see, e.g., Zamfirescu [68], Gautschi [14], Piesens [49], [50], Piesens and Haegemans [53], Davis and Rabinowitz [6]) considered usage of Gaussian formulae for oscillatory weights. Considering the following nonnegative weight functions on $[-1,1]$,

$$c_k(t)=\frac{1}{2}(1+\cos k\pi t),\quad s_k(t)=\frac{1}{2}(1+\sin k\pi t),$$

it is easy to see that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,\mathrm{d}x = 2\int_{-1}^{1} f(\pi t)c_k(t)\,\mathrm{d}t-\int_{-1}^{1} f(\pi t)\,\mathrm{d}t,$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,\mathrm{d}x = 2\int_{-1}^{1} f(\pi t)s_k(t)\,\mathrm{d}t-\int_{-1}^{1} f(\pi t)\,\mathrm{d}t.$$

Gauss-type rules can now be constructed for the first integrals on the right–hand sides of the previous equalities.

Goldberg and Varga [16] (cf. [34], [35]) proposed a method for the computation of Fourier coefficients based on Möbius inversion of Poisson summation formula.

Milovanović [39] proposed complex integration method. Let us for $\delta>0$ denote

$$G=\big\{z\in\mathbb{C}\mid -1\le \operatorname{Re} z\le 1,\, 0\le \operatorname{Im} z\le\delta\big\},\quad \varGamma_\delta=\partial G.$$

Consider the Fourier integral on the finite interval

$$I(f;\omega)=\int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x, \tag{1}$$

where f is an analytic real–valued function in the half–strip of the complex plane, $-1\le \operatorname{Re} z\le 1$, $\operatorname{Im} z\ge 0$, with singularities z_ν, $\nu=1,\dots,m$, in the region $\operatorname{int}\varGamma_\delta$, and

$$2\pi\mathrm{i}\sum_{\nu=1}^{m}\operatorname*{Res}_{z=z_\nu}\Big(f(z)\mathrm{e}^{\mathrm{i}\omega z}\Big)=P+\mathrm{i}Q.$$

If there exist the constants $M>0$ and $\xi<\omega$ such that

$$\int_{-1}^{1}|f(x+\mathrm{i}\delta)|\,\mathrm{d}x\le M\mathrm{e}^{\xi\delta}, \tag{2}$$

then (see [39, Theorem 2.1])

$$\int_{-1}^{1} f(x)\cos\omega x\,\mathrm{d}x = P+\frac{2}{\omega}\int_0^{+\infty}\operatorname{Im}\Big[\mathrm{e}^{\mathrm{i}\omega} f_e\Big(1+\mathrm{i}\frac{t}{\omega}\Big)\Big]\mathrm{e}^{-t}\,\mathrm{d}t,$$

$$\int_{-1}^{1} f(x)\sin\omega x\,\mathrm{d}x = Q-\frac{2}{\omega}\int_0^{+\infty}\operatorname{Re}\Big[\mathrm{e}^{\mathrm{i}\omega} f_o\Big(1+\mathrm{i}\frac{t}{\omega}\Big)\Big]\mathrm{e}^{-t}\,\mathrm{d}t,$$

where $f_o(z)$ and $f_e(z)$ are the odd and even part in $f(z)$, respectively. The obtained integrals can be calculated efficiently by using Gauss–Laguerre rule.

The Fourier integral on $(0,+\infty)$,

$$F[f;\omega]=\int_0^{+\infty} f(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x,$$

can be transformed to

$$F[f;\omega] = \frac{1}{\omega}\int_0^{+\infty} f(x/\omega)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x = F[f(\cdot/\omega);1].$$

Thus, it is enough to consider only the case $\omega = 1$.

In order to calculate $F[f;1]$, for a chosen positive number a, one can write

$$K[f;1] = \int_0^a f(x)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x + \int_a^{+\infty} f(x)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x = L_1[f] + L_2[f],$$

where

$$L_1[f] = a\int_0^1 f(at)\mathrm{e}^{\mathrm{i}at}\,\mathrm{d}t \quad \text{and} \quad L_2[f] = \int_a^{+\infty} f(x)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x.$$

If the function $f(z)$ is defined and holomorphic in the region $D = \{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq a > 0,\, \mathrm{Im}\, z \geq 0\}$, such that $|f(z)| \leq A/|z|$ when $|z| \to +\infty$, for some positive constant A, then (see [39, Theorem 2.2])

$$L_2[f] = \mathrm{i}\mathrm{e}^{\mathrm{i}a}\int_0^{+\infty} f(a+\mathrm{i}y)\mathrm{e}^{-y}\,\mathrm{d}y \quad (a>0).$$

In the numerical implementation Gauss–Legendre rule on $(0,1)$ and Gauss–Laguerre rule can be used for for calculating $L_1[f]$ and $L_2[f]$, respectively.

In this paper we .... The paper is organized as follows................

2 Filon–type quadrature rules for weighted Fourier integral

In this section we describe and analyze Filon–type method for generalized Fourier integral in the sense that a weight function is allowed (see Iserles [21]). Let $\mathscr{P}$ be the space of all algebraic polynomials and $\mathscr{P}_n$ be the linear space of all algebraic polynomials of degree at most n.

For a nonnegative sufficiently smooth nonzero weight function $w \in L[0,1]$ and $h > 0$ we consider the following integral

$$I_h[f] = \int_0^h f(x)\mathrm{e}^{\mathrm{i}\omega x} w(x/h)\,\mathrm{d}x = h\int_0^1 f(hx)\mathrm{e}^{\mathrm{i}h\omega x} w(x)\,\mathrm{d}x, \tag{3}$$

where $f \in L[0,h]$ is sufficiently smooth function. Let us choose n distinct points $0 \leq \tau_1 < \tau_2 < \cdots < \tau_n \leq 1$, and interpolate function f by a polynomial of degree $n-1$,

$$f(x) \approx P_n(x;f) = \sum_{k=1}^{n} \ell_k(x/h) f(h\tau_k),$$

where $\ell_k \in \mathscr{P}_{n-1}$, $k = 1,2\ldots,n$, are fundamental polynomials of Lagrange interpolation,

$$\ell_k(x) = \frac{\prod\limits_{\substack{\nu=1\\ \nu\neq k}}^{n} (x-\tau_\nu)}{\prod\limits_{\substack{\nu=1\\ \nu\neq k}}^{n} (\tau_k-\tau_\nu)}.$$

Replacing f by $P_n(x; f)$ in (3) the Filon–type quadrature rule is obtained,

$$Q_n^h[f] = I_h[P_n(x; f)] = h\sum_{k=1}^{n} \sigma_k(\mathrm{i}h\omega) f(h\tau_k), \tag{4}$$

where the weights are given by

$$\sigma_k(\mathrm{i}h\omega) = \int_0^1 \ell_k(x)\mathrm{e}^{\mathrm{i}h\omega x} w(x)\,\mathrm{d}x, \quad k = 1,2,\ldots,n.$$

Obviously, for the remainder term $R_n^h[f] = Q_n^h[f] - I_h[f]$ we have $R_n^h[f] = 0$ for all $f \in \mathscr{P}_{n-1}$. Hence, for sufficiently smooth function f we have $R_n^h[f] = \mathscr{O}(h^{n+1})$.

Remark 1. Let us notice that the same weights can be obtained by solving the following Vandermonde system

$$\sum_{k=1}^{n} \sigma_k(\mathrm{i}h\omega)\tau_k^m = \mu_m(h\omega), \quad m = 0,1,\ldots,n-1,$$

where μ_m are the corresponding moments,

$$\mu_m(\theta) = \int_0^1 x^m \mathrm{e}^{\mathrm{i}\theta x} w(x)\,\mathrm{d}x, \quad m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}.$$

If we set $\theta = h\omega$ and

$$\delta_m(\theta) = \sum_{k=1}^{n} \sigma_k(\mathrm{i}\theta)\tau_k^m - \mu_m(\theta), \quad m \in \mathbb{N}_0,$$

then $\delta_m = 0$ for $m = 0,1,\ldots,n-1$.

Let p be the order of the corresponding Gauss–Christoffel quadrature rule with nodes $0 \le \tau_1 < \tau_2 < \cdots < \tau_n \le 1$. Thus, $p \in \{n, n+1, \ldots, 2n\}$ (see [15]), which means that quadrature rule is exact for all polynomials of degree less than or equal to $p-1$. The maximal algebraic degree od exactness is $2n-1$, i.e., the maximal order is $2n$, if nodes are zeros of the corresponding orthogonal polynomial of $n-$th degree. It is important to point out that here we talk about Gauss–Christoffel quadrature rule with respect to the weight function w on $[0,1]$. In Section 4 we consider quadrature rules with maximal algebraic degree of exactness with respect to the complex oscillatory weight function of the form $\mathrm{e}^{\mathrm{i}\zeta x}w(x)$.

We present estimates of error term $R_n^h[f]$ for sufficiently smooth function f (see [21]) in three situations: $0 < h\omega \ll 1$ (non–oscillatory); $h\omega = \mathscr{O}(1)$ (mildly–oscillatory); $h\omega \gg 1$ (highly–oscillatory).

Let f be an analytic function in the disc $|z| < r$ for some $r > 0$, and its Taylor series

$$f(z) = \sum_{m=0}^{\infty} \frac{f_m}{m!} z^m.$$

Since $R_n^h[x^m] = 0$, for $m = 0, 1, \ldots, n-1$, one can assume without loss of generality that $f_m = 0$, $m = 0, 1, \ldots, n-1$. The function

$$\widetilde{f} = \sum_{m=n}^{\infty} \frac{f_m}{m!} z^m$$

is the essential part of the function f, and $R_n^h[f] = R_n^h[\widetilde{f}]$ regardless of the size of h and ω. Due to analyticity of f one can easily obtain that

$$R_n^h[f] = \sum_{m=n}^{\infty} \frac{f_m}{m!} \delta_m(\theta). \tag{5}$$

For analytic function f, for fixed $\omega > 0$ and $0 < h \ll 1$ we have (see [21]) $R_n^h[f] = \mathscr{O}(h^{p+1})$, where p is order of the corresponding Gauss–Christoffel quadrature rule, while in the case when $h\omega = \mathscr{O}(1)$ the error term behaves like $\mathscr{O}(h^{n+1})$.

Now, we pay our attention to the highly–oscillatory situation, when the standard Gauss–Christoffel quadrature rules became useless. Let

$$p(t) = \prod_{k=1}^{n} (t - \tau_k) = \sum_{k=1}^{n} a_k t^k$$

be the nodal polynomial. Let $h > 0$ be small and characteristic frequency $\theta = h\omega$ large. The main idea presented in [21] is to keep $h > 0$ fixed and consider the asymptotic expansion of the error term in negative powers of θ. It is easy to get the following asymptotic expansions for the moments:

$$\begin{aligned}
\mu_0(\theta) &\sim \frac{w(1)\mathrm{e}^{\mathrm{i}\theta} - w(0)}{\mathrm{i}\theta} + \frac{w'(1)\mathrm{e}^{\mathrm{i}\theta} - w'(0)}{\theta^2} + \mathscr{O}(\theta^{-3}),\\
\mu_1(\theta) &\sim \frac{w(1)\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{i}\theta} + \frac{(w(1) + w'(1))\mathrm{e}^{\mathrm{i}\theta} - w(0)}{\theta^2} + \mathscr{O}(\theta^{-3}),\\
\mu_m(\theta) &\sim \frac{w(1)\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{i}\theta} + \frac{(mw(1) + w'(1))\mathrm{e}^{\mathrm{i}\theta}}{\theta^2} + \mathscr{O}(m^2\theta^{-3}), \quad m \geq 2.
\end{aligned}$$

By using the fact that $p(\tau_k) = 0$, $k = 1, 2, \ldots, n$, and the obtained asymptotic expansions for the moments, it was shown (see [21, Proposition 3]) that there exist two sequences of numbers $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ such that

$$\delta_m(\theta) \sim \frac{\alpha_m w(0)p(0) - \beta_m w(1)p(1)\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{i}\theta} + \mathscr{O}(m\theta^{-2}), \quad m \in \mathbb{N}_0. \tag{6}$$

Here, $\alpha_m = \beta_m = 0$ for $m = 0,1,\ldots,n-1$, $\alpha_n = \beta_n = 1$. If $w(0)p(0) \neq 0$ and $w(1)p(1) \neq 0$, then α_m and β_m satisfy recurrence relations

$$\sum_{k=0}^{n} a_k \alpha_{k+m} = 0, \quad \sum_{k=0}^{n} a_k \beta_{k+m} = -1, \quad m \geq 1.$$

The general solutions of these equations are

$$\alpha_m = \sum_{k=1}^{n} c_k \tau_k^m, \quad \beta_m = \sum_{k=1}^{n} d_k \tau_k^m - \frac{1}{p(1)}, \quad m \geq 0,$$

where the constants c_k and d_k, $k = 1,2,\ldots,n$, can be determined from the initial values by solving a Vanredmonde linear system. If $w(0)p(0) = 0$, then $\alpha_m = 0$, while if $w(1)p(1) = 0$, then $\beta_m = 0$.

Finally, from (5), (6) and the fact that $R_n^h[f] = R_n^h[\widetilde{f}]$ the following result can be proved (see [21, Theorem 2]).

Theorem 1. *Let function f be analytic and $\theta = h\omega \gg 1$. If both $\tau_1 w(0) = 0$ and $(1-\tau_n)w(1) = 0$ then $R_n^h[f] \sim \mathscr{O}(h^{n+1}\theta^{-2})$; otherwise $R_n^h[f] \sim \mathscr{O}(h^{n+1}\theta^{-1})$.*

According to the previous theorem, we can conclude that for general weight function the best choice of nodes for the three considered situations is that of Lobatto points (see [6]).

Disadvantages of Filon–type method will be pointed out in Section 5 were Filon–type method for more general integrals will be presented.

3 Exponential fitting quadrature rules

The first results on exponentially fitting quadrature rules for oscillating integrands were given in [25]. Those ideas led to Gauss–type quadrature rules for oscillatory integrands considered in [26]. Namely, we considered the following quadrature formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{k=1}^{n} \sigma_k f(x_k) + R_n[f], \tag{7}$$

where the nodes x_k and the weights σ_k, $k = 1,\ldots,n$, are chosen such that this quadrature formula is exact for all functions from $\mathscr{F}_{2n}(\zeta)$, which is the linear span of the set $\{x^k \cos\zeta x, x^k \sin\zeta x : k = 0,1,\ldots,n-1, \zeta \in \mathbb{R}\}$. Let us notice that for $\zeta \neq 0$ we have $\dim \mathscr{F}_{2n}(\zeta) = 2n$. Obviously, it is enough to consider only the case $\zeta > 0$, because $\mathscr{F}_{2n}(-\zeta) = \mathscr{F}_{2n}(\zeta)$. The case $\zeta = 0$ is trivial, since $\mathscr{F}_{2n}(0)$ reduces to a pure polynomial set, i.e., $\mathscr{F}_{2n}(0) = \mathscr{P}_{n-1}$ (the set of algebraic polynomials of degree at most $n-1$).

Ixary and Paternoster [26] presented numerical method for constructing such quadrature rules with antisymmetric nodes in $(-1,1)$ and symmetric weights, but they have not proved the existence of such quadrature rules. The existence were proved partially in [43] in the case when all nodes are positive (or all negative). In the sequel we briefly explain that proof of existence.

For a given $n \in \mathbb{N}$ and the set of nodes $\{x_1, \ldots, x_n\}$ we denote $\mathbf{x} = (x_1, \ldots, x_n)$ and introduce the nodal polynomial $\omega(x) = \prod_{k=1}^{n}(x - x_k)$. For $\nu, \mu = 1, \ldots, n$, we use the following notation

$$\omega_\nu(x) = \frac{\omega(x)}{x - x_\nu} = \prod_{\substack{k=1 \\ k\neq\nu}}^{n}(x - x_k), \quad \omega_{\nu,\mu}(x) = \frac{\omega(x)}{(x - x_\nu)(x - x_\mu)} = \prod_{\substack{k=1 \\ k\neq\nu,\mu}}^{n}(x - x_k),$$

and $\ell_\nu(x) = \omega_\nu(x)/\omega_\nu(x_\nu)$, as well as

$$\Phi_\nu(\mathbf{x}) = \int_{-1}^{1} \omega_\nu(x) \sin\zeta(x - x_\nu)\,\mathrm{d}x, \quad \nu = 1, \ldots, n. \tag{8}$$

Suppose we are given mutually different nodes x_ν, $\nu = 1, \ldots, n$, of the quadrature rule (7). Then the weights can be expressed as follows (see [43, Theorem 2.1])

$$\sigma_\nu = \int_{-1}^{1} \ell_\nu(x) \cos\zeta(x - x_\nu)\,\mathrm{d}x, \quad \nu = 1, \ldots, n. \tag{9}$$

Therefore, the weights are unique for the given set of nodes, and the weights can be considered as continuous functions of nodes on any closed subset of $\mathbb{R}^n$ which does not contain points with some pair of the same coordinates. The following result is very important for the proof of existence of quadrature rules.

Theorem 2. *The nodes x_ν, $\nu = 1, \ldots, n$, of the quadrature rule* (7) *satisfy the following system of equations*

$$\int_{-1}^{1} \omega_\nu(x) \sin\zeta(x - x_\nu)\,\mathrm{d}x = 0, \quad \nu = 1, \ldots, n. \tag{10}$$

Suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is a solution of the system of equations (10)*, under the assumption $x_k \neq x_j$, $k \neq j$, $k, j = 1, \ldots, n$, we have that x_ν, $\nu = 1, \ldots, n$, are the nodes of the quadrature rule* (7)*.*

Of course, we are interested only on solutions of (10) which are nodes of quadrature rule (7). Let x_ν, $\nu = 1, \ldots, n$, be the nodes of the quadrature rule (7). It was proved in [43] that

$$\left|\partial_{x_k} \Phi_\nu(\mathbf{x})\right|_{\nu,k=1}^{n} = \left(\prod_{k=1}^{n} \sigma_k \omega_k(x_k)\right) \left|\frac{\sin\zeta(x_k - x_\nu)}{x_k - x_\nu}\right|_{\nu,k=1}^{n},$$

as well as that the function $\sin\zeta x/x$ in x is strictly positive definite. Supposing that $\sigma_k \neq 0$, $k = 1, \ldots, n$, it follows that the determinant of the Jacobian at the solution is not zero.

The case when $\sigma_\mu = 0$ for some $\mu = 1, \ldots, n$, is not important since it produces a quadrature rule which does not depend on x_μ at all.

For a fixed ζ, let us consider the following two equations

$$\int_{-1}^{1}\left(\prod_{\nu=1}^{n}(x-x_\nu)\right)\cos\zeta x\,\mathrm{d}x=0,\quad \int_{-1}^{1}\left(\prod_{\nu=1}^{n}(x-x_\nu)\right)\sin\zeta x\,\mathrm{d}x=0,$$

with unknowns x_ν, $\nu=1,\ldots,n$, and let us denote the sets of their solutions by C_n and S_n, respectively. For the proof of existence theorem we need the following properties of the sets C_n and S_n (see Theorem 2.8 and Theorem 2.9 from [43]).

Lemma 1. *The set C_n, $n\geq 2$, is closed, symmetric with respect to the origin and if* $\sin 2\zeta\geq 0$, *we have* $C_n\cap\{\mathbf{x}\in\mathbb{R}^n \mid x_\nu>0,\ \nu=1,\ldots,n\}=\varnothing$.

The set S_n, $n\geq 3$, is closed, symmetric with respect to the origin and if $\sin 2\zeta\leq 0$ *we have* $S_n\cap\{\mathbf{x}\in\mathbb{R}^n \mid x_\nu>0,\ \nu=1,\ldots,n\}=\varnothing$.

We are now ready to present the main result from [43] and give the sketch of the proof.

Theorem 3. *In the case* $\sin 2\zeta\geq 0$ *for* $2\leq n<\zeta/\pi-1/2$, *system of equations* (10) *has at least* $2\binom{[\zeta/\pi-1/2]}{n}$ *solutions which nodes are all positive or all negative.*

In the case $\sin 2\zeta\leq 0$ *for* $3\leq n<\zeta/\pi-1$, *system of equations* (10) *has at least* $2\binom{[\zeta/\pi-1]}{n}$ *solutions which nodes are all positive or all negative.*

Proof. First we consider the case $\sin 2\zeta\geq 0$. For the solutions which satisfy the condition $\int_{-1}^{1}\omega_\nu(x)\cos\zeta x\,\mathrm{d}x\neq 0$, $\nu=1,\ldots,n$, the system of equations (10) can be rewritten in the form

$$x_\nu=\psi_\nu^C(\mathbf{x})=\frac{1}{\zeta}\left(\arctan\frac{\int_{-1}^{1}\omega_\nu(x)\sin\zeta x\,\mathrm{d}x}{\int_{-1}^{1}\omega_\nu(x)\cos\zeta x\,\mathrm{d}x}+k_\nu\pi\right),\quad \nu=1,\ldots,n, \tag{11}$$

where $k_\nu\in\mathbb{Z}$. Defining the functions $p_\nu:\mathbb{R}^n\to\mathbb{R}^n$, $\nu=1,\ldots,n$, by

$$p_\nu(x_1,\ldots,x_\nu,\ldots,x_n)=(x_1,\ldots,x_n,\ldots,x_\nu),$$

the set of solutions of $\int_{-1}^{1}\omega_\nu(x)\cos\zeta x\,\mathrm{d}x=0$, $\nu=1,\ldots,n$, can be described as $p_\nu(C_{n-1}\times\mathbb{R})$, $\nu=1,\ldots,n$. Thus, the transformation holds true for all the solutions which belong to the set $\mathbb{R}^n\backslash(\cup_{\nu=1}^{n}p_\nu(C_{n-1}\times\mathbb{R}))$. Since the set C_n has empty intersection with the set $\{\mathbf{x}\mid x_\nu>0,\ \nu=1,\ldots,n\}$, it follows that $\{\mathbf{x}\mid x_\nu>0,\ \nu=1,\ldots,n\}\subset\mathbb{R}^n\backslash\left(\cup_{\nu=1}^{n}p_\nu(C_{n-1}\times\mathbb{R})\right)$. This means that any solution of the system (10) with all positive nodes will be also the solution of the system (11). Since the set C_{n-1} is symmetric with respect to the origin, the same holds for $\cup_{\nu=1}^{n}p_\nu(C_{n-1}\times\mathbb{R})$, and in the same way one can consider quadrature rule with all negative nodes.

Let us choose some fixed vector, with strictly increasing coordinates, of positive integers $\mathbf{k}=(k_1,\ldots,k_n)$, with the property $k_n<\zeta/\pi-1/2$. The functions $\psi_\nu^C(\mathbf{x})$, $\nu=1,\ldots,n$ are continuous in $\mathbf{x}$ for $x_\nu>0$, $\nu=1,\ldots,n$. The mapping $\Psi_{\mathbf{k}}^C:\mathbb{R}^n\to\mathbb{R}^n$ defined by $\Psi_{\mathbf{k}}^C(\mathbf{x})=(\psi_1^C(\mathbf{x}),\ldots,\psi_n^C(\mathbf{x}))$ is continuous in $\mathbf{x}$, for $x_\nu>0$, $\nu=1,\ldots,n$. The mapping $\Psi_{\mathbf{k}}^C$ maps continuously the closed convex set

$A_{\mathbf{k}} = \mathop{\times}\limits_{\nu=1}^{n} \left[\left(k_\nu - \frac{1}{2}\right)\frac{\pi}{\zeta}, \left(k_\nu + \frac{1}{2}\right)\frac{\pi}{\zeta}\right]$ into itself. According to the Brouwer fixed point theorem (see, e.g., [47]), the map $\Psi_{\mathbf{k}}^C$ has a fixed point $\mathbf{x_k} \in A_{\mathbf{k}}$. According to the fact that $\int_{-1}^{1} \omega_\nu(x)\cos\zeta x\,\mathrm{d}x \neq 0$, it follows that we cannot have the solution with ν-th coordinate equal to $(k_\nu \pm 1/2)\pi/\zeta$, which means that all coordinates of the solution $\mathbf{x_k}$ are different, according to the fact that the coordinates of the vector $\mathbf{k}$ are different. Thus, $\mathbf{x_k}$ are the nodes of the quadrature rule (7). At this solution all the weights are different from zero.

For the case $\sin 2\zeta \leq 0$, one can rewrite the system of equations (10), in the form

$$x_\nu = \psi_\nu^S(\mathbf{x}) = \frac{1}{\zeta}\left(\operatorname{arccot}\frac{\int_{-1}^{1}\omega_\nu(x)\cos\zeta x\,\mathrm{d}x}{\int_{-1}^{1}\omega_\nu(x)\sin\zeta x\,\mathrm{d}x} + k_\nu\pi\right), \quad \nu = 1,\ldots,n, \tag{12}$$

where $k_\nu \in \mathbb{Z}$, and using the similar arguments as in the previous case prove that the mapping $\Psi_{\mathbf{k}}^S : \mathbb{R}^n \to \mathbb{R}^n$, defined by $\Psi_{\mathbf{k}}^S(\mathbf{x}) = (\psi_1^S(\mathbf{x}),\ldots,\psi_n^S(\mathbf{x}))$, has a fixed point in the set $B_{\mathbf{k}} = \mathop{\times}\limits_{\nu=1}^{n} [k_\nu\pi/\zeta, (k_\nu+1)\pi/\zeta]$.

The number of the solutions can be easily obtained. □

The nodes x_k, $k = 1,\ldots,n$, of the quadrature formula (7) can be obtained by using Newton–Kantorovich method for the system (10) with appropriately chosen starting values. Once nodes are constructed, weights σ_k, $k = 1,\ldots,n$, can be computed by using formula (9). In Table 1 we give two different quadrature rules with all positive nodes for the case $n = 10$, $\zeta = 10000$ (numbers in parenthesis indicate decimal exponents). All computations are performed by using the *Mathematica* package *OrthogonalPolynomials* [5].

By using transformed systems (11) and (12) of nonlinear equations the existence of quadrature rule (7) which has the both positive and negative nodes was proved in [44] under two conjectures, one for the case $\sin 2\zeta < 0$ and the second one for $\sin 2\zeta > 0$. We present those conjectures here, while for the proof of existence of mentioned quadrature rule we refer readers to [44].

First we consider the case $\sin 2\zeta < 0$. Let us denote

$$b_\nu = (N - \nu + 1)\frac{\pi}{\zeta}, \quad \nu = 1,\ldots,N, \quad N = [\zeta/\pi],$$

and

$$I_n^S = \operatorname{sgn}(\sin\zeta)\int_{-1}^{1} t\prod_{\nu=1}^{n}(t^2 - b_\nu^2)\sin\zeta t\,\mathrm{d}t, \quad n = 0,1,\ldots,N.$$

For $\sin 2\zeta > 0$ we denote

$$a_\nu = \left(N - \nu + \frac{1}{2}\right)\frac{\pi}{\zeta}, \quad \nu = 1,\ldots,N, \quad N = [\zeta/\pi],$$

and

$$I_n^C = \operatorname{sgn}(\sin\zeta)\int_{-1}^{1}\prod_{\nu=1}^{n}(t^2 - a_\nu^2)\cos\zeta t\,\mathrm{d}t, \quad n = 0,1,\ldots,N.$$

Table 1 Nodes x_k and weights σ_k, $k = 1,\dots,10$, $\zeta = 10000$

k	x_k	σ_k
1	0.7363923439571446(−1)	−2.804303173754735
2	0.2225507250414224	2.970983118291112(1)
3	0.3673781455056259	1.583146376664096(2)
4	0.4999533548408196	−5.186895526861798(2)
5	0.6234179456200185	−1.207492403800276(3)
6	0.7330595288408521	−2.103809780509887(3)
7	0.8269931488869863	−2.821892746331952(3)
8	0.8998780982412568	2.717303933673183(3)
9	0.9554842880629808	1.718285983811184(3)
10	0.9887851701205169	−5.392769580168855(2)
1	0.1000286064833346	−1.463766698478331(2)
2	0.1499799293036977	−5.661615769092509(2)
3	0.2492542565212690	3.197802441947143(3)
4	0.3010905350216144	6.614998760647455(3)
5	0.3500993801768735	−4.923021376270480(3)
6	0.4500020261498017	1.443577768367478(3)
7	0.5499046722324926	−3.931520115104385(3)
8	0.7500241241587095	−5.332095232080515
9	0.8499267711774567	2.037155323640304(1)
10	0.9507718999014823	2.664975853402439

It is easy to see that for $\zeta > 0$ and $\sin 2\zeta < 0$ inequality

$$I_0^S = \operatorname{sgn}(\sin\zeta)\int_{-1}^{1} t\sin\zeta t\,\mathrm{d}t = \frac{1}{|\sin\zeta|}\,\frac{-\zeta\sin 2\zeta + 2\sin^2\zeta}{\zeta^2} > 0$$

holds, while for $\zeta > 0$ and $\sin 2\zeta < 0$ holds

$$I_0^C = \operatorname{sgn}(\sin\zeta)\int_{-1}^{1} \cos\zeta t\,\mathrm{d}t = \frac{2|\sin\zeta|}{\zeta} > 0.$$

The mentioned conjectures are the following.

Conjecture 1. If $\zeta > 0$ and $\sin 2\zeta < 0$, then $I_n^S > 0$ for each $n = 1,\dots,N$.

Conjecture 2. If $\zeta > 0$ and $\sin 2\zeta > 0$, then $I_n^C > 0$ for each $n = 1,\dots,N$.

Under condition that Conjecture 1 is true in [44] it was proved that in the case when $\zeta > 0$ and $\sin 2\zeta < 0$ for all

$$\mathbf{x} = (x_1,\dots,x_{2n+1}) \mathop{\times}_{\nu=1}^{n} ([-b_\nu,0]\times[0,b_\nu])\times[-b_{n+1},b_{n+1}], \quad n < N,$$

the following inequality

$$\operatorname{sgn}(\sin\zeta)\int_{-1}^{1}\prod_{\nu=1}^{2n+1}(t - x_\nu)\sin\zeta t\,\mathrm{d}t > 0$$

holds. This inequality implies the existence of the quadrature rule (7) in general, for the case $\sin 2\zeta < 0$.

Analogously as in the case $\sin 2\zeta < 0$, under condition that Conjecture 2 is true it can be proved that in the case when $\zeta > 0$ and $\sin 2\zeta > 0$ for all

$$\mathbf{x} = (x_1, \ldots, x_{2n}) \in \mathop{\times}_{\nu=1}^{n} ([-a_\nu, 0] \times [0, a_\nu]), \quad n = 1, \ldots, N,$$

the following inequality

$$\operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\nu=1}^{2n} (t - x_\nu) \cos \zeta t \, \mathrm{d}t > 0$$

holds. As the consequence of this inequality we have the existence of the quadrature rule (7) in general, for the case $\sin 2\zeta > 0$.

Kim, Cools, and Ixaru in [27] and [28] considered the quadrature rules which include derivatives. Van Daele, Vanden Berghe, and Vande Vyver [62] took into account the both polynomial and exponential aspects. Assuming symmetric weights and antisymmetric nodes they considered quadrature rules suited to integrate functions that can be expressed in the form $f(x) = f_1(x) + f_2(x)\cos \zeta x + f_3(x)\sin \zeta x$, where f_1, f_2, and f_3 are assumed smooth enough to be well approximated by polynomials on the wanted interval. The readers can find more details on this topic in the recent survey paper [48].

4 Gaussian rules with respect to some complex oscillatory weights

In this section we consider quadrature rules of Gaussian type

$$\int_{-1}^{1} f(x) w(x) \mathrm{e}^{\mathrm{i}\zeta x} \, \mathrm{d}x = \sum_{k=1}^{n} w_k^{(n)} f(x_k^{(n)}) + R_n(f), \quad \zeta \in \mathbb{R},$$

where $R_n(f) = 0$ for each $f \in \mathscr{P}_{2n-1}$. Thus, we have to consider the following complex measure

$$\mathrm{d}\mu(x) = w(x) \mathrm{e}^{\mathrm{i}\zeta x} \chi_{[-1,1]}(x) \, \mathrm{d}x, \quad \zeta \in \mathbb{R}, \tag{13}$$

supported on the interval $[-1, 1]$ (χ_A is the characteristic function of the set A). The existence of the corresponding orthogonal polynomials is not guaranteed. In order to check existence of orthogonal polynomials with respect to complex oscillatory measure $\mathrm{d}\mu(x)$ we need the general concept of orthogonal polynomials with respect to a moment functional (see [3], [37]).

Let a linear functional $\mathscr{L}$ be given on the linear space $\mathscr{P}$ of all algebraic polynomials, i.e., let the functional $\mathscr{L}$ satisfy following equality

$$\mathscr{L}[\alpha P+\beta Q]=\alpha\mathscr{L}[P]+\beta\mathscr{L}[Q],\quad \alpha,\beta\in\mathbb{C},\quad P,Q\in\mathscr{P}.$$

Because of linearity, the value of the linear functional $\mathscr{L}$ at every polynomial is known if the values of $\mathscr{L}$ at the set of all monomials are known. The corresponding values of the linear functional $\mathscr{L}$ at the set of monomials are called the moments and we denote them by μ_k, $k\in\mathbb{N}_0$. Thus, $\mathscr{L}[x^k]=\mu_k, \in\mathbb{N}_0$.

A sequence of polynomials $\{P_n(x)\}_{n=0}^{+\infty}$ is called the polynomial sequence orthogonal with respect to a moment functional $\mathscr{L}$, provided for all nonnegative integers m and n,

- $P_n(x)$ is polynomial of degree n,
- $\mathscr{L}[P_n(x)P_m(x)]=0$ for $m\neq n$,
- $\mathscr{L}[P_n^2(x)]\neq 0$.

If the sequence of orthogonal polynomials exists for a given linear functional $\mathscr{L}$, then $\mathscr{L}$ is called quasi–definite or regular linear functional. Under the condition $\mathscr{L}[P_n^2(x)]>0$, the functional $\mathscr{L}$ is called positive definite.

By using only linear algebraic tools the following theorem can be proved (see [3, p. 11]).

Theorem 4. *The necessary and sufficient conditions for the existence of a sequence of orthogonal polynomials with respect to the linear functional $\mathscr{L}$ are that for each $n\in\mathbb{N}$ the Hankel determinants*

$$\Delta_n=\begin{vmatrix}\mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{n-1}\\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n\\ \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{n+1}\\ \vdots & \vdots & \vdots & & \vdots\\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-2}\end{vmatrix}\neq 0.$$

We use previous theorem to prove existence of orthogonal polynomials with respect to some linear functionals defined by complex oscillatory measures (13)

$$\mathscr{L}[f]=\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu(x)=\int_{-1}^{1} f(x)w(x)\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x,\quad f\in\mathscr{P},\quad \zeta\in\mathbb{R}. \tag{14}$$

1° The case $w(x)=x$, $\zeta=m\pi\neq 0$, for an integer m, was considered by Milovanović and Cvetković in [40]. Here, the measure is

$$\mathrm{d}\mu_m(x)=x\mathrm{e}^{\mathrm{i}m\pi x}\chi_{[-1,1]}\,\mathrm{d}x,\quad m\in\mathbb{Z}\setminus\{0\},$$

thus, orthogonal polynomials with respect to the moment functional

$$\mathscr{L}[f]=\int_{-1}^{1} f(x)x\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x,\quad f\in\mathscr{P}, \tag{15}$$

i.e., with respect to the following (quasi) inner product

$$(f,g) = \int f(x)g(x)x\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x, \quad f,g \in \mathscr{P} \tag{16}$$

must be considered.

By using an integration by parts it is easy to obtain the following recurrence relation for the moments

$$\mu_{k+1} = \frac{(-1)^m}{\mathrm{i}\zeta}(1-(-1)^{k+2}) - \frac{k+2}{\mathrm{i}\zeta}\mu_k, \quad \mu_0 = 2\frac{(-1)^m}{\mathrm{i}\zeta}.$$

The moments can be expressed explicitly as follows

$$\mu_k = \frac{(-1)^{m+k}(k+1)!}{(\mathrm{i}\zeta)^{k+1}}\sum_{\nu=0}^{k}\frac{(1+(-1)^\nu)(-\mathrm{i}\zeta)^\nu}{(\nu+1)!}.$$

In [40] the following theorem was proved.

Theorem 5. *For every non–zero integer m, the sequence of orthogonal polynomials with respect to the linear functional* (15)*, i.e., the sequence of orthogonal polynomials with respect to the weight function* $x\mathrm{e}^{\mathrm{i}m\pi x}$*, supported on the interval* $[-1,1]$*, exists uniquely.*

Let us notice that in general, if $m \notin \mathbb{Z}$, the existence of orthogonal polynomials is not assured (e.g., the smallest positive solution of equation $\Delta_3 = 0$ is $\zeta \approx 7.134143996368961\ldots$).

As a consequence of the following property $(xf,g) = (f,xg)$ of the inner product (16), we have that the monic orthogonal polynomials with respect to the weight function $x\mathrm{e}^{\mathrm{i}m\pi x}$ on $[-1,1]$ satisfy the following three–term recurrence relation

$$p_{n+1}(x) = (x-\mathrm{i}\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad n = 0,1,\ldots,$$

with $p_0(x) = 1$ and $p_{-1}(x) = 0$. The recursion coefficients α_n and β_n can be expressed in terms of Hankel determinants,

$$\mathrm{i}\alpha_n = \frac{\Delta'_{n+1}}{\Delta_{n+1}} - \frac{\Delta'_n}{\Delta_n}, \quad n \in \mathbb{N}_0; \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad n \in \mathbb{N},$$

where and Δ'_n is the Hankel determinant Δ_{n+1} with the penultimate column and the last row removed.

In [40] the first four recursion coefficients were given explicitly. Also, the numerical calculation of recursion coefficients was analyzed. Based on extensive numerical computations, which were done by using a combination of the Chebyshev algorithm and the Stieltjes–Gautschi procedure, applying package of routines written in *Mathematica* (see [5]), the following conjecture was stated.

Conjecture 3. For recursion coefficients the following asymptotic relations are true

$$\alpha_k \to 0, \quad \beta_k \to \frac{1}{4}, \quad k \to +\infty.$$

Numerical calculations indicate that all of the nodes of orthogonal polynomials with respect to the weight function $x\mathrm{e}^{\mathrm{i}m\pi x}$ on $[-1,1]$ are simple, but it was not proved. In the case of multiple zeros of orthogonal polynomials, the Gaussian quadrature rule has the following form

$$G_n[f] = \sum_{\nu=1}^{n} \sum_{k=0}^{m_\nu - 1} w_{\nu,k}^{(n)} f^{(k)}(x_\nu^{(n)}).$$

As a matter of fact, it was proved that at most two nodes in the previous rule may have multiplicity greater than one. As it was said, in all numerical experiments the nodes were simple, so, the Gaussian quadrature rule has standard form

$$G_n[f] = \sum_{\nu=1}^{n} w_\nu^{(n)} f(x_\nu^{(n)}). \tag{17}$$

Methods for numerical calculation of nodes and weights of Gaussian rule were also described in [40].

We present here an application of these quadrature rules for the calculation of Fourier coefficients. Namely,

$$F_m[f] = C_m[f] + \mathrm{i}S_m[f] = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x = \int_{-1}^{1} \frac{f(x)-f(0)}{x} x\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x,$$

so, we can compute it by using Gaussian quadrature rules (17) for the function g given by

$$g(x) = \frac{f(x)-f(0)}{x}, \quad g(0) = f'(0).$$

If function f is analytic in some domain $D \supset [-1,1]$, then g is also analytic in D. Therefore, for some analytic function f, the Fourier coefficients can be calculated as follows

$$F_m[f] = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x \approx \sum_{\nu=1}^{n} \frac{w_\nu^{(n)}}{x_\nu^{(n)}} \left(f(x_\nu^{(n)}) - f(0)\right).$$

2° The case $w(x) = x(1-x^2)^{-1/2}$, $\zeta \in \mathbb{R} \setminus \{0\}$, was considered in [41]. In this case the linear functional $\mathscr{L}$ is given by

$$\mathscr{L}[f] = \int_{-1}^{1} f(x)x(1-x^2)^{-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x, \quad \zeta \in \mathbb{R}\setminus\{0\}, \quad f \in \mathscr{P}. \tag{18}$$

Let $\mu_k(\zeta)$, $k \in \mathbb{N}_0$, be the corresponding sequence of moments. It is easy to see that for each $k \in \mathbb{N}_0$ the following equality

$$\mu_k(\zeta) = \int_{-1}^{1} x^{k+1}(1-x^2)^{-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x = \overline{\int_{-1}^{1} x^{k+1}(1-x^2)^{-1/2}\mathrm{e}^{-\mathrm{i}\zeta x}\,\mathrm{d}x} = \overline{\mu_k(-\zeta)}$$

holds, which means that it is enough to consider only the case $\zeta > 0$, since the corresponding results for $\zeta < 0$ can be obtained by a simple conjugation. The case $\zeta = 0$ is excluded, because for that value the linear functional $\mathscr{L}$, given by (18), is not regular ($\mu_0 = \Delta_0 = 0$).

Let J_ν be the Bessel function of the order ν, defined by (cf. [64, p. 40])

$$J_\nu(z) = \sum_{m=0}^{+\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{19}$$

The sequence of moments $\mu_k(\zeta)$, $k \in \mathbb{N}_0$, satisfy the following recurrence relation (see [41, Theorem 2])

$$\mu_{k+2}(\zeta) = -\frac{k+2}{\mathrm{i}\zeta}\mu_{k+1}(\zeta) + \mu_{k+1}(\zeta) + \frac{k+1}{\mathrm{i}\zeta}\mu_{k-1}(\zeta), \quad k \in \mathbb{N},$$

with the initial conditions

$$\begin{aligned}
\mu_0(\zeta) &= \mathrm{i}\pi J_1(\zeta), \\
\mu_1(\zeta) &= \frac{\pi}{\zeta}\big(\zeta J_0(\zeta) - J_1(\zeta)\big), \\
\mu_2(\zeta) &= \frac{\mathrm{i}\pi}{\zeta^2}\big(\zeta J_0(\zeta) + (\zeta^2 - 2)J_1(\zeta)\big).
\end{aligned}$$

Unfortunately, the sequence of orthogonal polynomials does not exist for all positive ζ. It is not hard to check that Hankel determinant Δ_3 in this case is given by

$$\Delta_3 = \frac{\mathrm{i}\pi^3 J_1^3}{\zeta^6}\left(7\zeta^3\frac{J_0^3}{J_1^3} + (2\zeta^2 - 21)\zeta^2\frac{J_0^2}{J_1^2} + \zeta(5\zeta^2 + 12)\frac{J_0}{J_1} + 2\zeta^4 - 15\zeta^2 + 4\right).$$

The smallest positive solution of the equation $\Delta_3 = 0$ is given by

$$\zeta = 6.4590081519947834555317213970325025438057106691208882\ldots,$$

and for this ζ the sequence of orthogonal polynomials does not exist. So, the task is to find ζ for which the existence of orthogonal polynomials with respect to the linear functional (18) is ensured. For that purpose we notice that for the moment sequence we have the following representation

$$\mu_k(\zeta) = \frac{\mathrm{i}\pi}{(\mathrm{i}\zeta)^k}(P_k J_1(\zeta) + \zeta Q_k J_0(\zeta)), \quad k \in \mathbb{N}_0,$$

where P_k and Q_k are polynomials in ζ^2 with integer coefficients of degrees $2[k/2]$ and $2[(k-1)/2]$, respectively (see [41, Theorem 3]). This expression can be easily obtained from the recurrence relation for the moments. Let ζ be any positive zero of the Bessel function $J_0(\zeta)$. Then $J_1(\zeta) \neq 0$, due to the interlacing property of the positive zeros of the Bessel functions (see [64, p. 479]), and sequence of moments becomes

$$\mu_k(\zeta) = \frac{\mathrm{i}\pi}{(\mathrm{i}\zeta)^k} P_k J_1(\zeta).$$

The following theorem was proved in [41].

Theorem 6. *If ζ is a positive zero of the Bessel function J_0, then the sequence of polynomials orthogonal with respect to the functional $\mathscr{L}$, given by* (18)*, exists.*

Remark 2. With a matrix Riemann-Hilbert problem formulation of the orthogonality relations, Aptekarev and Van Assche [1] considered the linear functional of the form $\mathscr{L}[f] = \int_{-1}^{1} f(x)\rho(x)(1-x^2)^{-1/2}\,\mathrm{d}x$, where ρ is a complex valued, non–vanishing on $[-1,1]$, which is holomorphic in some domain containing the interval $[-1,1]$. In the special case $\rho(x) = \mathrm{e}^{\mathrm{i}\zeta x}$ the linear functional (14) with $w(x) = (1-x^2)^{-1/2}$ is obtained.

3° The case $w(x) = (1-x^2)^{\lambda-1/2}$, for $\lambda > -1/2$, and $\zeta \in \mathbb{R}\backslash\{0\}$ was considered in [45]. In this case the linear functional (14) becomes

$$\mathscr{L}[f] = \int_{-1}^{1} f(x)(1-x^2)^{\lambda-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x, \quad f \in \mathscr{P}. \tag{20}$$

As in the case of previous weight,it is enough to consider only the $\zeta > 0$, since the case $\zeta < 0$ can be obtain under substitution $x := -x$.

The corresponding moments $\mu_k^\lambda(\zeta)$ can be expressed in the form

$$\mu_k^\lambda(\zeta) = \frac{A}{(\mathrm{i}\zeta)^k}\left(P_k^\lambda(\zeta)J_\lambda(\zeta) + Q_k^\lambda(\zeta)J_{\lambda-1}(\zeta)\right), \quad k \in \mathbb{N}_0,$$

where $A = (2/\zeta)^\lambda\sqrt{\pi}\,\Gamma(\lambda+1/2)$, J_ν is the Bessel function of the order ν, given by (19), and P_k^λ and Q_k^λ are polynomials in ζ, which satisfy the following four–term recurrence relation

$$y_{k+2} = -(k+2\lambda+1)y_{k+1} - \zeta^2 y_k - k\zeta^2 y_{k-1},$$

with the initial conditions $P_0^\lambda(\zeta) = 1$, $P_1^\lambda(\zeta) = -2\lambda$, $P_2^\lambda(\zeta) = 2\lambda(2\lambda+1) - \zeta^2$ and $Q_0^\lambda(\zeta) = 0$, $Q_1^\lambda(\zeta) = \zeta$, $Q_2^\lambda(\zeta) = -(2\lambda+1)\zeta$, respectively (see [45, Theorem 2.1]).

That it obvious that for each $\lambda > -1/2$, if $\zeta > 0$ is an arbitrary zero of the Bessel function J_λ, the polynomials π_n orthogonal with respect to (20) do not exist, because $\Delta_0 = \mu_0 = AJ_\lambda(\zeta) = 0$. In [45] the following result was proved.

Theorem 7. *If λ is a positive rational number and ζ is a positive zero of the Bessel function $J_{\lambda-1}$, then the polynomials π_n orthogonal with respect to* (20) *exist.*

Suppose that parameters λ and ζ are such that provide the existence of orthogonal polynomials with respect to linear functional (20). Due to the property $(zp,q) = (p,zq)$ of the (quasi) inner-product $(p,q) := \mathscr{L}(pq)$, for $\mathscr{L}$ given by (20), the corresponding (monic) orthogonal polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ satisfy the fundamental three–term recurrence relation

$$\pi_{n+1}(x) = (x - \mathrm{i}\alpha_n)\pi_n(x) - \beta_n\pi_{n-1}(x), \quad n \in \mathbb{N},$$

with $\pi_0(x) = 1$, $\pi_{-1}(x) = 0$. The recursion coefficients α_n and β_n can be expressed in terms of Hankel determinants as

$$\mathrm{i}\alpha_n = \frac{\Delta'_{n+1}}{\Delta_{n+1}} - \frac{\Delta'_n}{\Delta_n} = \frac{1}{\mathrm{i}\zeta}\left(\frac{H'_{n+1}}{H_{n+1}} - \frac{H'_n}{H_n}\right), \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = \frac{1}{(\mathrm{i}\zeta)^2}\frac{H_{n+1}H_{n-1}}{H_n^2},$$

where

$$H_n = \begin{vmatrix} P_0^\lambda(\zeta) & \dots & P_{n-1}^\lambda(\zeta) \\ P_1^\lambda(\zeta) & \dots & P_n^\lambda(\zeta) \\ \vdots & \vdots & \vdots \\ P_{n-1}^\lambda(\zeta) & \dots & P_{2n-2}^\lambda(\zeta) \end{vmatrix}, \quad H'_n = \begin{vmatrix} P_0^\lambda(\zeta) & P_1^\lambda(\zeta) & \dots & P_{n-2}^\lambda(\zeta) & P_n^\lambda(\zeta) \\ P_1^\lambda(\zeta) & P_2^\lambda(\zeta) & \dots & P_{n-1}^\lambda(\zeta) & P_{n+1}^\lambda(\zeta) \\ \vdots & \vdots & & \vdots & \vdots \\ P_{n-1}^\lambda(\zeta) & P_n^\lambda(\zeta) & \dots & P_{2n-3}^\lambda(\zeta) & P_{2n-1}^\lambda(\zeta) \end{vmatrix}.$$

The coefficient β_0 can be chosen arbitrary, but it is convenient to take $\beta_0 = \mu_0 = AJ_\lambda(\zeta)$.

Recursion coefficients can be calculated by using the Chebyshev algorithm, implemented in the software package `OrthogonalPolynomials` [5], similarly as in the case $w(x) = x$. According to very extensive numerical calculations the conjecture that the recursion coefficients satisfy the following asymptotic relations

$$\alpha_n \to 0, \quad \beta_n \to \frac{1}{4}, \quad n \to +\infty,$$

was stated in [45]. Let us notice that for $\lambda = 0$, from the result given in [1], it follows that $\alpha_n \to 0$ and $\beta_n \to 1/4$, $n \to +\infty$.

It is easy to see that $\overline{\mu}_k^\lambda(\zeta) = (-1)^k\mu_k^\lambda(\zeta)$, $k \in \mathbb{N}_0$. Using that fact it can be proved that if the sequence of monic orthogonal polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ exists, then $\pi_n(z) = (-1)^n\overline{\pi_n(-\overline{z})}$ and the coefficients α_n and β_n are real. This implies that the zeros $x_k^{(n)}$, $k = 1, \dots, n$, of π_n are distributed symmetrically with respect to the imaginary axis. Some properties of the corresponding orthogonal polynomials were given in [42].

By using functions implemented in package *OrthogonalPolynomials* [5] in extended arithmetics we are able to construct Gaussian rules

$$\int_{-1}^{1} f(x)(1-x^2)^{\lambda-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x = \sum_{k=1}^{n} w_k^{(n)} f(x_k^{(n)}) + R_n[f], \tag{21}$$

where $R_n[f] = 0$ for each $f \in \mathscr{P}_{2n-1}$, which can be successfully applied for numerical calculation of certain type of highly–oscillating integrals. We illustrate this applying Gaussian rule to the integral

$$I(\zeta) = \mathrm{Im}\left\{\int_{-1}^{1} \frac{1}{x-\mathrm{i}}(1-x^2)^{1/4}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x\right\} \approx G_n(\zeta) = \mathrm{Im}\left\{\sum_{k=1}^{n} \frac{w_k^{(n)}}{x_k^{(n)} - \mathrm{i}}\right\},$$

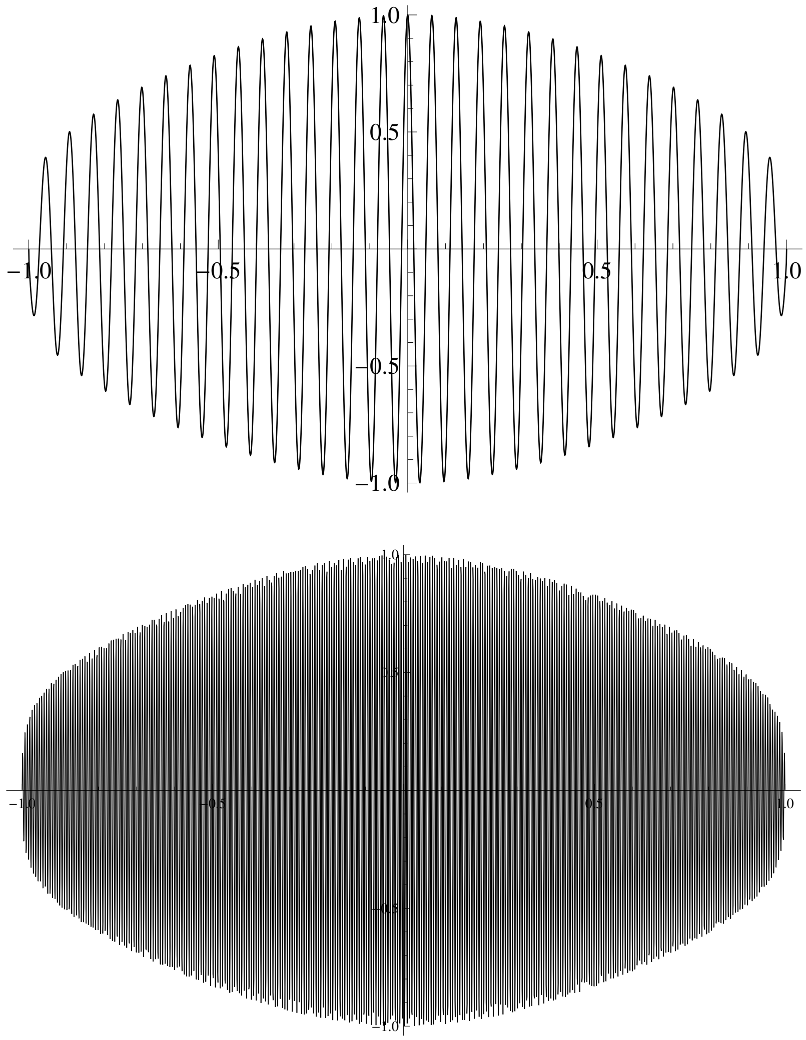

Fig. 1 The graphs of the function $\mathrm{Im}\big((1-x^2)^{1/4}\mathrm{e}^{\mathrm{i}\zeta x}/(x-\mathrm{i})\big)$ for $\zeta=\zeta_1$ (top) and $\zeta=\zeta_2$ (down)

for $\zeta\in\{\zeta_1,\zeta_2\}$, where $\zeta_1=99.35381121792450$ and $\zeta=1000.990052907274$ (here, $\lambda=3/4$ and ζ_1,ζ_2 are zeros of $J_{-1/4}(z)$). The imaginary parts of the corresponding integrands are displayed in Fig. 1.

The exact values of $I(\zeta)$ are

$$I(\zeta_1)=0.0034446765944009118078224282065986452635896 79\ldots,$$
$$I(\zeta_2)=0.00019149147544459801260257921097705042525703 7\ldots.$$

In Table 2, for some selected number of nodes n, the relative errors in Gaussian approximations, $r_n = |(G_n(\zeta_\nu) - I(\zeta_\nu))/I(\zeta_\nu)|$, $\nu = 1,2$, are given, as well as the relative errors r_n^G in Gauss–Gegenbauer approximations with respect to the weight function $x \mapsto (1-x^2)^{1/4}$ (numbers in parenthesis indicate decimal exponents).

Table 2 Relative errors r_n and r_n^G, for $n = 5(5)25$, when $\zeta = \zeta_\nu$, $\nu \in \{1,2\}$

ζ	ζ_1		ζ_2	
n	r_n	r_n^G	r_n	r_n^G
5	7.59(−8)	3.80(2)	5.75(−12)	2.13(2)
10	5.82(−16)	2.24(2)	2.63(−26)	6.65(2)
15	1.16(−19)	2.72(2)	5.58(−35)	6.23(2)
20	4.11(−26)	6.32(1)	3.50(−47)	1.08(3)
25	1.79(−29)	8.46(1)	7.99(−55)	5.14(2)

Numerical experiments indicate that our Gaussian quadrature rule (21) becomes more efficient when ζ increases, while Gauss–Gegenbauer rule becomes unusable.

4° The case $w(x) = (1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}$, where $\alpha, \beta > -1/2$ are real numbers such that $\ell = |\beta - \alpha|$ is a positive integer, and $\zeta \in \mathbb{R} \setminus \{0\}$ was considered in [58]. Thus, we are concerned with the following measure

$$\mathrm{d}\mu(x) = (1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\chi_{[-1,1]}(x)\,\mathrm{d}x$$

supported on the interval $[-1,1]$. This measure can be written in the following form

$$\mathrm{d}\mu(x) = \begin{cases} (1+x)^{\ell}(1-x^2)^{\alpha-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\chi_{[-1,1]}(x)\,\mathrm{d}x, & \beta > \alpha, \\ (1-x)^{\ell}(1-x^2)^{\beta-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\chi_{[-1,1]}(x)\,\mathrm{d}x, & \alpha > \beta. \end{cases}$$

Therefore, we consider the measures

$$\mathrm{d}\mu^{\pm}(x) = (1 \pm x)^{\ell}(1-x^2)^{\alpha-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\chi_{[-1,1]}(x)\,\mathrm{d}x,$$

where $\alpha > -1/2$ and ℓ is a positive integer, i.e., we consider orthogonality with respect to the linear functional

$$\mathscr{L}^{\pm}[f] = \int_{-1}^{1} f(x)(1 \pm x)^{\ell}(1-x^2)^{\alpha-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x, \quad f \in \mathscr{P}. \tag{22}$$

Again, we restrict our attention to the case $\zeta > 0$, since the corresponding results for $\zeta < 0$ can be obtained by a simple conjugation.

The moments $\mu_k^{\pm} = \mathscr{L}^{\pm}[x^k]$ can be expressed in the form

$$\mu_k^{\pm} = \frac{A}{(\mathrm{i}\zeta)^{k+\ell}} \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j (\mathrm{i}\zeta)^{\ell-j} \left(P_{k+j}^{\alpha}(\zeta) J_{\alpha}(\zeta) + Q_{k+j}^{\alpha}(\zeta) J_{\alpha-1}(\zeta) \right),$$

where $A=(2/\zeta)^{\alpha}\sqrt{\pi}\,\Gamma(\alpha+1/2)$, J_ν is Bessel function of the order ν, and P_k^{α} and Q_k^{α} are polynomials in ζ, which satisfy the following four-term recurrence relation

$$y_{k+2}=-(k+2\alpha+1)y_{k+1}-\zeta^2 y_k-k\zeta^2 y_{k-1},$$

with the initial conditions $P_0^{\alpha}(\zeta)=1, P_1^{\alpha}(\zeta)=-2\alpha, P_2^{\alpha}(\zeta)=2\alpha(2\alpha+1)-\zeta^2$ and $Q_0^{\alpha}(\zeta)=0, Q_1^{\alpha}(\zeta)=\zeta, Q_2^{\alpha}(\zeta)=-(2\alpha+1)\zeta$, respectively (see [58, Theorem 2.1]). When the existence of orthogonal polynomials with respect to the linear functional (22) is in question, the following result was proved in [58].

Theorem 8. *If $\alpha>-1/2$ is a rational number, ℓ is a positive integer and ζ is a positive zero of the Bessel function $J_{\alpha-1}$, then the polynomials $\pi_n^{\pm}$ orthogonal with respect to the linear functionals $\mathscr{L}^{\pm}$, given by* (22)*, exist.*

The (quasi) inner product $(p,q)=\mathscr{L}^{\pm}[pq]$ has the property $(zp,q)=(p,zq)$, which implies that the corresponding (monic) orthogonal polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ satisfy the fundamental three–term recurrence relation

$$\pi_{n+1}(x)=(x-\alpha_n)\pi_n(x)-\beta_n\pi_{n-1}(x),\quad n\in\mathbb{N},$$

with $\pi_0(x)=1$, $\pi_{-1}(x)=0$. Knowing three-term recurrence coefficients, by using functions implemented in the software package *OrthogonalPolynomials* [5] in extended arithmetics we are able to construct the corresponding quadrature rules of Gaussian type

$$\int_{-1}^{1} f(x)(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x=\sum_{k=1}^{n} w_k^{(n)} f(x_k^{(n)})+R_n[f], \qquad (23)$$

where $R_n[f]=0$ for each polynomial of degree at most $2n-1$. Such rules can be efficiently applied for numerical integration of highly oscillating functions. Analogously as in the case of oscillatory modification of Gegenbauer measure, numerical experiments indicate that Gaussian quadrature rule (23) becomes more efficient when ζ increases, while Gauss–Jacobi rule with respect to weight $x\mapsto(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}$ becomes unusable (see [58] for some examples).

5 Irregular oscillators

In this section we consider the more general highly–oscillating integrand,

$$I[f;g]=\int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x, \qquad (24)$$

where $-\infty<a<b<+\infty$, $|\omega|$ is large, and both f and g are sufficiently smooth functions. The integrand of (24) is often called an irregular oscillators. Such integrals occur in a wide range of practical problems. There are a large number of articles where problems of numerical calculation of such integrals are treated (see [9], [10], [11], [55], [23], [22], [24], [46], [20], [18], [33], [57], etc.). In the case $g(x) = x$, we get so called regular oscillators, which have been already considered through the paper. In this section we briefly describe asymptotic method, Filon–type methods, and Levin–type methods for numerical integration of (24). We consider only the case when $g'(x) \neq 0$ for $a \leq x \leq b$, i.e., the case when g has no stationary points. Notice that from the van der Corput lemma it follows that $I[f;g] = \mathscr{O}(\omega^{-1})$, $|\omega| \to \infty$ (see [59]).

5.1 Asymptotic methods

Asymptotic method was presented by Iserles and Nørsett [24]. Starting by the following simple transformation

$$I[f;g] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x = \frac{1}{\mathrm{i}\omega}\int_a^b \frac{f(x)}{g'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x,$$

and applying an integration by parts we obtain

$$I[f;g] = \frac{1}{\mathrm{i}\omega}\left(\frac{f(x)}{g'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\right)\Bigg|_a^b - \frac{1}{\mathrm{i}\omega}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g'(x)}\right)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x.$$

Denoting

$$Q^A[f;g] = \frac{1}{\mathrm{i}\omega}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\right)\Bigg|_a^b,$$

we have

$$I[f;g] = Q^A[f;g] - \frac{1}{\mathrm{i}\omega}I\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g'(x)}\right);g\right).$$

According to the van der Corput lemma, $I[f;g] - Q^A[f;g] = \mathscr{O}(\omega^{-2})$. Now, we can approximate the error term by the same rule, so, we approximate $I[f;g]$ by

$$Q^A[f;g] - \frac{1}{\mathrm{i}\omega}Q^A\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g'}\right);g\right).$$

In this approximation of $I[f;g]$ the error is $\mathscr{O}(\omega^{-3})$. Continuing in this manner, after s steps we obtain approximation of $I[f;g]$ with error $\mathscr{O}(\omega^{-s-1})$. Thus, we have derived the following asymptotic expansion.

Theorem 9. *Let $f \in C^\infty$ and $g'(x) \neq 0$ for $a \leq x \leq b$. Let*

$$\sigma_1[f](x) = \frac{f(x)}{g'(x)}, \quad \sigma_{k+1}[f](x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sigma_k[f](x)}{g'(x)},\ k = 0,1,\ldots.$$

Then, for $\omega \to \infty$,

$$I[f;g] \sim -\sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \left(\sigma_k[f](b)\mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k[f](a)\mathrm{e}^{\mathrm{i}\omega g(a)} \right).$$

Taking the $s-$th partial sum of the asymptotic expansion we obtain the *asymptotic method*

$$Q_s^A[f;g] = -\sum_{k=1}^{s} \frac{1}{(-\mathrm{i}\omega)^k} \left(\sigma_k[f](b)\mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k[f](a)\mathrm{e}^{\mathrm{i}\omega g(a)} \right).$$

It is easy to see that

$$I[f;g] - Q_s^A[f;g] = \frac{1}{(\mathrm{i}\omega)^s} \int_a^b \sigma_s[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x \sim \mathscr{O}(\omega^{-s-1}).$$

The following result follows from Theorem 9 (see [46]).

Lemma 2. *Suppose* $0 = f^{(k)}(a) = f^{(k)}(b)$, *for all* $k = 0,1,\ldots,s-1$ *for some positive integer s, and that f depend on* ω *as well as that the every function in the set* $\{f, f', \ldots, f^{(s+1)}\}$ *is of asymptotic order* $\mathscr{O}(\omega^{-n})$, $\omega \to \infty$, *for some fixed n. Then,* $I[f;g] \sim \mathscr{O}(\omega^{-n-s-1})$, $\omega \to \infty$.

The drawback of the asymptotic method is that for fixed ω, $Q_s^A[f;g]$, in general, diverges as $s \to \infty$. Also, numerical examples show that asymptotic method may produce very inaccurate approximation for small values of ω in general.

5.2 *Filon–type methods*

Here we describe Filon–type method for numerical computation of (24), presented in [24]. The main idea is to interpolate function f for fixed set of prescribed nodes by using Hermite interpolation and then integrate interpolating polynomial.

Let $\{x_k\}_{k=0}^{\nu}$ be a set of prescribed nodes, such that

$$a = x_0 < x_1 < \cdots < x_\nu = b.$$

Having chosen multiplicities $n_0, n_1, \ldots, n_\nu \in \mathbb{N}$, by $H_n(x) = \sum_{k=0}^{n} a_k x^k$ we denote polynomial of degree n, where $n = \sum_{k=0}^{\nu} n_k - 1$, such that

$$H_n^{(j)}(x_k) = f^{(j)}(x_k), \quad j = 0,1,\ldots,n_k-1, \quad k = 0,1,\ldots,\nu. \tag{25}$$

For $s = \min\{n_0, n_\nu\}$ we define

$$Q_s^F[f;g] = I[H_n;g] = \sum_{k=0}^{n} a_k I(x^k;g).$$

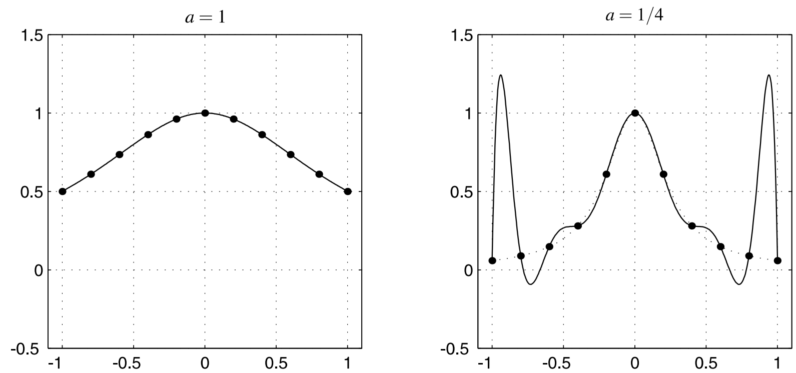

Fig. 2 The Runge's example for $n = 11$ equally spaced nodes, for $a = 1$ (left) and $a = 1/4$ (right)

The function $f - H_n$ satisfies the conditions given in Lemma 2 due to (25). Thus, we have

$$I[f;g] - Q_s^F[f;g] = I[f;g] - I[H_n;g] = I[f - H_n;g] \sim \mathscr{O}(\omega^{-s-1}), \quad \omega \to \infty.$$

Therefore, the asymptotic and the Filon–type methods have the same asymptotic order. In many situations (but it is not always) the accuracy of Filon–type method is significantly higher than that of the asymptotic method (see some examples in [24], [46]).

Unfortunately, there are two problems with Filon–type methods. The first one is obvious from the definition of method. By definition, the Filon–type methods given above require the computation of the moments $I[x^k;g]$ analytically, which is not possible in general. The second problem is connected with the fact that the Filon–type method is based on interpolation, and the accuracy of $Q_s^F[f;g]$ is directly related to accuracy of interpolation. The good example for that is Runge's example from 1901 (see [37, p. 60]). For non–oscillatory functions $f_a(x) = 1/(1+(x/a)^2)$, $x \in [-1,1]$, for sufficiently small a, interpolation polynomials with equally spaced nodes are oscillating (see Figure 2). For such functions the Filon–type methods (especially, when only function values are used) produce less accurate results.

The magnitude of Runge's phenomenon could be reduced by using Chebyshev interpolating points. Another idea is to use cubic spline, but in that case the order is at most $\mathscr{O}(\omega^{-3})$.

Hascelik [18] modified the Filon–type methods such that they can be applied in the cases when f and g have algebraic singularity.

5.3 Levin–type methods

Now, we explain method which does not require the computation of moments, introduced by Levin (see [30], [31], [32]). Levin method can be applied to more general problems, which will be presented in Section 6.

Suppose that $F(x)$ is a function such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\right) = f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}. \tag{26}$$

It is obvious that $I[f;g] = \left(F(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\right)\Big|_a^b$. The idea is to approximate F by some function V, which gives method

$$Q^L[f;g] = \left(V(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\right)\Big|_a^b = V(b)\mathrm{e}^{\mathrm{i}\omega g(b)} - V(a)\mathrm{e}^{\mathrm{i}\omega g(a)}.$$

From (26) we obtain equation $L[F](x) = f(x)$, where L is the operator defined by $L[F] = F' + \mathrm{i}\omega g'F$. If $V(x) = \sum_{k=0}^n a_k x^k$ is the collocation polynomial, satisfying system of equations $L[V](x_k) = f(x_k)$ at points $a = x_0 < x_1 < \cdots < x_\nu = b$, then $I[f;g] - Q^L[f;g] \sim \mathscr{O}(\omega^{-2})$.

There are two natural generalizations of Levin method (see [46], [18]). The first one is to use a polynomial V such that not only the values of f and $L[V]$ are the same at nodes, but also the values of their derivatives up to the given multiplicity. The second generalization is obtained allowing V to be a linear combination of a set of suitable basis functions, not only polynomial.

The following result was proved in [46].

Theorem 10. *Suppose that $g'(x) \neq 0$ for $x \in [a,b]$. Let $\{\psi_k\}_{k=0}^n$ be a basis of functions independent of ω, let $\{x_k\}_{k=0}^\nu$ be a set of nodes such that $a = x_0 < x_1 < \cdots < x_\nu = b$, let $\{n_k\}_{k=0}^\nu$ be a set of multiplicities associated with nodes, and $s = \min\{n_0, n_\nu\}$. Further, suppose that $V = \sum_{k=0}^n a_k\psi_k$, where $n = \sum_{k=0}^\nu n_k - 1$, is the solution of the system of collocation equations*

$$\frac{\mathrm{d}^j L[V]}{\mathrm{d}x^j}(x_k) = f^j(x_k), \quad j = 0,1,\ldots,n_k - 1; \quad k = 0,1,\ldots,\nu,$$

where $L[V] = V' + \mathrm{i}\omega g'V$. Define

$$\mathbf{g}_k = [(g'\psi_k)(x_0)\ \cdots (g'\psi_k)^{(n_0-1)}(x_0)\ \cdots\ (g'\psi_k)(x_\nu)\ \cdots (g'\psi_k)^{(n_\nu-1)}(x_\nu)]^T. \tag{27}$$

If the vectors $\{\mathbf{g}_0, \ldots, \mathbf{g}_n\}$ are linearly indenpendent, then the system has a unique solution and for

$$Q^L_\psi[f;g] = \left(V(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\right)\Big|_a^b = V(b)\mathrm{e}^{\mathrm{i}\omega g(b)} - V(a)\mathrm{e}^{\mathrm{i}\omega g(a)}$$

we have $I[f;g] - Q^L_\psi[f;g] \sim \mathscr{O}(\omega^{-s-1})$, $\omega \to \infty$.

Olver [46] proved that if $\{\psi_k\}_{k=0}^n$ is a Chebyshev set, then the conditions on $\{\mathbf{g}_k\}_{k=0}^n$ of the previous theorem are satisfied for all choices of $\{x_k\}_{k=0}^\nu$ and $\{n_k\}_{k=0}^\nu$. He showed that it is possible to obtain higher asymptotic order of Levin–type method by choosing the basis in the following way

$$\psi_0 = 1, \quad \psi_1 = \frac{f}{g'}, \quad \psi_{k+1} = \frac{\psi_k'}{g'}, \ k = 1, 2, \dots . \tag{28}$$

Suppose that $\{x_k\}_{k=0}^\nu$, $\{n_k\}_{k=0}^\nu$, and $\{\psi_k\}_{k=0}^n$, where $n = \sum_{k=0}^\nu n_k - 1$, satisfy the conditions of Theorem 10. Then, for $s = \min\{n_0, n_\nu\}$ we have (see [46, Theorem 5.1])

$$I[f;g] - Q_\psi^L[f;g] \sim \mathscr{O}(\omega^{-n-s-1}).$$

Levin–type method $Q_\psi^L[f;g]$ with basis $\{\psi_k\}$ given by (28), is significant improvement over $Q^F[f;g]$ and $Q_\psi^L[f;g]$ with standard polynomials basis, when the same nodes and multiplicities are used, and ω is sufficiently large. Also, since $Q_\psi^L[f;g]$ does not require polynomials interpolation, the Runge's phenomenon does not occur.

In general, accuracy of asymptotic, Filon–type and Levin–type methods depends on f and g. Olver [46] presented several examples for comparisons of these three types of methods, including Levin–type method with polynomial basis, and Levin–type method with basis (28).

6 Integrals involving highly oscillating Bessel function

In this section we consider integrals of the form

$$I[f] = \int_a^b f(x) J_\nu(rx)\, \mathrm{d}x, \tag{29}$$

where $J_\nu(rx)$ is Bessel function of the first kind of order ν for some positive real number ν, $r \in \mathbb{R}$ is large, and $0 < a < b \le +\infty$. Such integrals appear in many areas of science and technology and several efficient methods for their numerical calculations are derived (see, e.g., [51], [52], [54], [39], [4], [65], [66], [67]). Here we present Levin–type method [31], [32] for finite b, and Chen's method [2] for the both finite and infinite b.

6.1 Levin–type method

In Section 5 it was explain how Levin–type method [31], [32] can be applied to irregular oscillators, as well as Olver's generalization [46]. Levin's collocation method is applicable to a wide class of oscillating integrals with weight functions satisfying certain differential conditions. It can be efficiently used for computing integral (29) with finite b.

Let $F(x) = [f_1(x) \;\; f_2(x) \;\; \cdots \;\; f_m(x)]^T$ be an $m-$vector of non–oscillating functions, $W(r,x) = [w_1(r,x) \;\; w_2(r,x) \;\; \cdots \;\; w_m(r,x)]^T$ be an $m-$vector of linearly independent highly–oscillating function, depending on r, and let "$\cdot$" denotes the inner product. Let us consider general class of highly oscillatory integrals of the form

$$I[F] = \int_a^b \sum_{k=1}^m f_k(x) w_k(r,x)\,\mathrm{d}x \equiv \int_a^b F(x)\cdot W(r,x)\,\mathrm{d}x. \tag{30}$$

Assume that

$$W'(r,x) = A(r,x)W(r,x),$$

where derivative is with respect to x, and $A(r,x)$ is $m\times m$ matrix of non–oscillating functions. If F were of the form

$$F(x) = Q'(x) + A^T(r,x)Q(x),$$

then the integral (30) could be evaluated as

$$\begin{aligned} I[F] &= \int_a^b \big(Q'(x) + A^T(r,x)Q(x)\big)\cdot W(r,x)\,\mathrm{d}x \\ &= \int_a^b (Q(x)\cdot W(r,x))'\,\mathrm{d}x = Q(b)\cdot W(r,b) - Q(a)\cdot W(r,a). \end{aligned}$$

The main idea of Levin method is to select linearly independent basis function $\{\psi_k\}_{k=1}^n$ and determine

$$P(x) = \left(\sum_{k=1}^n a_k^{(1)}\psi_k(x), \ldots, \sum_{k=1}^n a_k^{(m)}\psi_k(x)\right)$$

such that the following system of equations is satisfied

$$P'(x_j) + A^T(r,x_j)P(x_j) = F(x_j), \quad j = 1,2,\ldots,n,$$

at nodes $x_1, x_2, \ldots, x_n$. The Levin's approximation of $I[F]$ is

$$\begin{aligned} Q_n^L[F] &= \int_a^b \big(P'(x) + A^T(r,x)P(x)\big)\cdot W(r,x)\,\mathrm{d}x \\ &= P(b)\cdot W(r,b) - P(a)\cdot W(r,a). \end{aligned}$$

Levin [32] presented an error analysis for the composite collocation method $Q_{n,h}^L$ with n selected nodes in each subinterval of length h including the endpoints of each subinterval. His error estimate is given in the following theorem.

Theorem 11. *Let $F \in C^{2n+1}[a,b]$, $B(r,x) = (A(x,r)/C(r))^{-1}$ exists, $B \in C^{2n+1}[a,b]$, and its $2n+1$ derivatives are bounded uniformly in r for $C(r) \geq \alpha_0$. Then*

$$\left|I[F]-Q_{n,h}^L[F]\right| < \frac{M(b-a)h^{n-2}}{C(r)^2},$$

for $C(r) \gg 1$, where h is a length of each subinterval and M is a constant independent of r and h.

For integral (29),

$$m=2, \quad W(r,x) = [J_{\nu-1}(rx) \;\; J_\nu(rx)]^T, \quad \text{and} \quad F(x) = [0 \;\; f(x)]^T. \tag{31}$$

Then,

$$A(r,x) = \begin{bmatrix} \dfrac{n-1}{r} & -r \\ r & -\dfrac{n}{r} \end{bmatrix}. \tag{32}$$

Define $C(r) = r$, then by Theorem 11 for $h \to 0$ and $r \to \infty$ we obtain

$$I[F] - Q_{n,h}^L[F] = \mathscr{O}\left(\frac{h^{n-2}}{r^2}\right).$$

Xiang, Gui, and Moa [67] extended Levin's method by using multiple nodes. Let $\{m_k\}_{k=1}^n$ be multiplicities associated with the nodes $a = x_1 < x_2 < \cdots < x_n = b$, $s = \min\{m_1, m_n\}$ and $\{\psi_k\}_{k=0}^N$, where $N = \sum_{k=1}^n m_k - 1$, be a set of linearly independent basis functions such that the matrix $[\mathbf{a}_0 \;\cdots\; \mathbf{a}_N]$ is nonsingular, with $\mathbf{a}_k = [\psi_k(x_1) \;\; \psi_k'(x_1) \;\; \cdots \;\; \psi_k^{(m_1-1)}(x_1) \;\; \cdots \;\; \psi_k(x_n) \;\; \psi_k'(x_n) \;\; \cdots \;\; \psi_k^{(m_n-1)}(x_1)]^T$, $k = 0,1,\ldots,N$. Let $P(x) = \left(\sum_{k=0}^N a_k^{(1)}\psi_k(x), \ldots, \sum_{k=0}^N a_k^{(m)}\psi_k(x)\right)$ satisfy the the following equations

$$\begin{aligned} &P'(x_j) + A^T(r,x_j)P(x_j) = F(x_j), \quad j = 1,2,\ldots,n, \\ &\left[P'(x) + A^T(r,x)P(x)\right]_{x=x_j}^{(k)} = F^{(k)}(x_j), \quad k = 1,2,\ldots,m_j-1; \;\; j = 1,2,\ldots,n. \end{aligned}$$

Levin's approximation of integral $I[F]$, given by (30), is the following

$$Q_s^L = \int_a^b \left(P'(x) + A^T(r,x)P(x)\right) \cdot W(r,x)\,\mathrm{d}x = P(b)\cdot W(r,b) - P(a)\cdot W(r,a).$$

Let $W'(r,x) = A(r,x)W(r,x)$, where $A(r,x)$ is a nonsingular $m \times m$ matrix, and $B(r,x) = (A(r,x)/C(r))^{-1}$ for $r \gg 1$. If

- $W(r,x), B(r,x) \in C^\infty[a,b]$;
- $A(r,x)/C(r)$ and $A^{(k)}(r,x)$, $k = 1,2,\ldots,\max\limits_{1\le j\le n} m_j - 1$, are uniformly bounded for $r \gg 1$ and all $x \in [a,b]$;
- $B(r,x)$ and its $s+1$ derivatives are uniformly bounded for $r \gg 1$ and all $x \in [a,b]$,

then (see [67, Theorem 4.1])

$$I[F]-Q_s^L[F]=\mathscr{O}\left(\frac{\|W(r,x)\|_\infty}{C(r)^{s+1}}\right).$$

Let us now go back to our integral $I[f]$, given by (29), and denote the corresponding Levin's approximation by $Q_s^L[f]$. For the basis $\{\psi_k(x)\}$ we choose the standard polynomial basis. Here $C(r)=r$, and it is easy to see from (31) and (32) that $\frac{1}{r}A(x,r)$, $B(r,x)$, $A^{(k)}(r,x)$, and $B^{(k)}(r,x)$, $k=1,2,\ldots,s+1$, are uniformly bounded for $r\gg 1$ and all $x\in[a,b]$. Therefore, according to previous general estimate and the fact that $\|W(r,x)\|_\infty=\mathscr{O}(r^{-1/2})$ for $f\in C^1[a,b]$ and $r\gg 1$ (see [67, Theorem 2.1]), the following error estimate

$$I[f]-Q_s^L[f]=\mathscr{O}(r^{-s-3/2})$$

holds. Notice that $I[f]=\mathscr{O}(r^{-3/2})$ for $f\in C^1[a,b]$ and $r\gg 1$.

6.2 Chen's method

Chen's [2] presented method for numerical computing of integral $I[f]$ given by (29) following ideas of Milovanović [39] and Huybrechs and Vandewalle [20]. By using integral form of Bessel function and its analytic continuation Chen transformed highly–oscillating integral (29) into non oscillating integral on $[0,+\infty)$, which could be computed efficiently applying Gauss–Laguerre quadrature rule.

Substituting integral representation of Bessel function (see [64])

$$J_\nu(x)=\frac{(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t$$

in (29), we obtain

$$I[f]=\int_a^b f(x)\frac{(rx/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\mathrm{e}^{\mathrm{i}rxt}\,\mathrm{d}t\,\mathrm{d}x. \tag{33}$$

The function $(1-t^2)^{\nu-1/2}\mathrm{e}^{\mathrm{i}rxt}$ is analytic in the half–strip of the complex plane, $-1\le \operatorname{Re} z\le 1$, $\operatorname{Im} z\ge 0$. By using complex integration method (see [2]) it can be proved that

$$\begin{aligned}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\mathrm{e}^{\mathrm{i}rxt}\,\mathrm{d}t=&\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}rx}}{(rx)^{2\nu}}\int_0^{+\infty}\left(u^2+2\mathrm{i}rxu\right)^{\nu-1/2}\mathrm{e}^{-u}\,\mathrm{d}u\\&-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}rx}}{(rx)^{2\nu}}\int_0^{+\infty}\left(u^2-2\mathrm{i}rxu\right)^{\nu-1/2}\mathrm{e}^{-u}\,\mathrm{d}u,\end{aligned}$$

which together with (33) gives

$$I[f] = \frac{\mathrm{i}}{2^\nu r^\nu \sqrt{\pi}\Gamma(\nu+1/2)} \sum_{j=0}^{\nu} (2\mathrm{i}r)^j \binom{\nu}{j} \left(\int_a^b \frac{f(x)\mathrm{e}^{-\mathrm{i}rx}}{x^{\nu-j}} \int_0^{+\infty} \frac{u^{2\nu-j}\mathrm{e}^{-u}}{\sqrt{u^2+2\mathrm{i}rxu}} \,\mathrm{d}u\,\mathrm{d}x \right.$$
$$\left. -(-1)^j \int_a^b \frac{f(x)\mathrm{e}^{\mathrm{i}rx}}{x^{\nu-j}} \int_0^{+\infty} \frac{u^{2\nu-j}\mathrm{e}^{-u}}{\sqrt{u^2-2\mathrm{i}rxu}} \,\mathrm{d}u\,\mathrm{d}x \right).$$

Integrals

$$I_1(\nu,j,rx) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{u^{2\nu-j}\mathrm{e}^{-u}}{\sqrt{u^2+2\mathrm{i}rxu}} \,\mathrm{d}u, \quad I_2(\nu,j,rx) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{u^{2\nu-j}\mathrm{e}^{-u}}{\sqrt{u^2-2\mathrm{i}rxu}} \,\mathrm{d}u$$

can be represented by Whittaker W function (see [17]) as follows

$$I_1(\nu,j,rx) = \prod_{\ell=0}^{2\nu-j-1} (2\ell+1) \frac{(rx\mathrm{i})^{(2\nu-j-1)/2}\mathrm{e}^{rx\mathrm{i}}}{2^{(\nu-j+1)/2}} W_{-(2\nu-j)/2,(2\nu-j)/2}(2rx\mathrm{i}),$$
$$I_2(\nu,j,rx) = \prod_{\ell=0}^{2\nu-j-1} (2\ell+1) \frac{(-rx\mathrm{i})^{(2\nu-j-1)/2}\mathrm{e}^{-rx\mathrm{i}}}{2^{(\nu-j+1)/2}} W_{-(2\nu-j)/2,(2\nu-j)/2}(-2rx\mathrm{i}).$$

It is known that Wittaker W function $W_{\alpha,\beta}(z)$ for $|\arg z| < \pi$ has the following asymptotic expansion (see [17, p. 1016]):

$$W_{\alpha,\beta}(z) \sim z^\alpha \mathrm{e}^{-z/2} \left(1 + \sum_{k=1}^{+\infty} \frac{\prod_{\ell=1}^{k} \left(\beta^2 - (\alpha-\ell+1/2)^2\right)}{k! z^k} \right), \quad |z| \to \infty.$$

Taking a few terms in the corresponding expansions, integrals $I_1(\nu,j,rx)$ and $I_2(\nu,j,rx)$ can be approximated efficiently for large r. Therefore, our integral $I[f]$ is now reduced to the following:

$$I[f] = \frac{\mathrm{i}}{2^\nu r^\nu \Gamma(\nu+1/2)} \sum_{j=0}^{\nu} (2\mathrm{i}r)^j \binom{\nu}{j} \left(\int_a^b \frac{f(x)\mathrm{e}^{-\mathrm{i}rx}}{x^{\nu-j}} I_1(\nu,j,rx)\,\mathrm{d}x \right.$$
$$\left. -(-1)^j \int_a^b \frac{f(x)\mathrm{e}^{\mathrm{i}rx}}{x^{\nu-j}} I_2(\nu,j,rx)\,\mathrm{d}x \right). \quad (34)$$

In the case when $b < +\infty$, by using complex integration method (see [2]), the integrals in (34) can be transformed as follows

$$\int_a^b \frac{f(x)\mathrm{e}^{-\mathrm{i}rx}}{x^{\nu-j}} I_1(\nu,j,rx)\,\mathrm{d}x = \left(\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}rq}}{r}\int_0^{+\infty}\frac{f(q-\mathrm{i}y/r)I_1(\nu,j,r(q-\mathrm{i}y/r))}{(q-\mathrm{i}y/r)^{\nu-j}}\mathrm{e}^{-y}\,\mathrm{d}y\right)\Bigg|_{q=a}^{q=b},$$

$$\int_a^b \frac{f(x)\mathrm{e}^{\mathrm{i}rx}}{x^{\nu-j}} I_2(\nu,j,rx)\,\mathrm{d}x = \left(\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}rq}}{r}\int_0^{+\infty}\frac{f(q+\mathrm{i}y/r)I_2(\nu,j,r(q+\mathrm{i}y/r))}{(q+\mathrm{i}y/r)^{\nu-j}}\mathrm{e}^{-y}\,\mathrm{d}y\right)\Bigg|_{q=b}^{q=a}.$$

Finally, applying a n–point Gauss–Leguerre quadrature rule to the previous integrals, we get the approximation of $I[f]$, which we denote by Q_n^G. If f is an analytic function in the strip of the complex plane $a \le \mathrm{Re}\,z \le b$, then the following error estimate

$$I[f] - Q_n^G[f] = \mathscr{O}\left(\frac{(n!)^2}{(2n)!r^{2n+3/2}}\right), \quad r \gg 1,$$

holds (see [2, Theorem 2.1]).

Suppose now that $b = +\infty$, and that exists constant C such that f satisfies the condition $|f(x)| \le C$ for $x \in [a,+\infty)$. Transforming the integrals on the right hand side on (34) (see [2]), $I[f]$ can be written in the form

$$I[f] = \frac{1}{2^\nu r^\nu \Gamma(\nu+1/2)}\sum_{j=0}^{\nu}(2\mathrm{i}r)^j\binom{\nu}{j}\times \left(\frac{\mathrm{e}^{-\mathrm{i}ra}}{r}\int_0^{+\infty}\frac{f(a-\mathrm{i}y/r)I_1(\nu,j,r(a-\mathrm{i}y/r))}{(a-\mathrm{i}y/r)^{\nu-j}}\mathrm{e}^{-y}\,\mathrm{d}y + (-1)^j\frac{\mathrm{e}^{\mathrm{i}ra}}{r}\int_0^{+\infty}\frac{f(a+\mathrm{i}y/r)I_2(\nu,j,r(a+\mathrm{i}y/r))}{(a+\mathrm{i}y/r)^{\nu-j}}\mathrm{e}^{-y}\,\mathrm{d}y\right).$$

Applying again a n–point Gauss–Leguerre quadrature rule to the integrals on the right hand side of the previous equation, we get approximation Q_n^G of $I[f]$. For and analytic function f in $\{0 \le |\arg z| \le \pi/2\}$, the following error estimate

$$I[f] - Q_n^G[f] = \mathscr{O}\left(\frac{(n!)^2}{(2n)!r^{2n+3/2}}\right), \quad r \gg 1,$$

holds in this case, too (see [2, Theorem 2.2]).

Numerical examples given in [2] show that for $a < b < +\infty$ Chen's method gives better approximation for integral $I[f]$ in comparison with Levin–type method.

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