Quadrature rules with multiple nodes

Gradimir V. Milovanović and Marija P. Stanić

Abstract In this paper a brief historical survey of the development of quadrature rules with multiple nodes and the maximal algebraic degree of exactness is given. The natural generalization of such rules are quadrature rules with multiple nodes and the maximal degree of exactness in some functional spaces that are different from the space of algebraic polynomial. For that purpose we present a generalized quadrature rules considered by A. Ghizzeti and A. Ossicini [Quadrature Formulae, Academie–Verlag, Berlin, 1970] and apply their ideas in order to obtain quadrature rules with multiple nodes and the maximal trigonometric degree of exactness. Such quadrature rules are characterized by so called $s-$ and $\sigma-$orthogonal trigonometric polynomials. Numerical method for the construction of such quadrature rules are given, as well as numerical the example which illustrate obtained theoretical results.

1 Introduction and preliminaries

The most significant discovery in the field of numerical integration in 19th century was made by Carl Friedrich Gauß in 1814. His method [14] dramatically improved the earlier Newton's method. Gauss' method was enriched by significant contributions of Jacobi [21] and Christoffel [7]. More than hundred years after Gauß published his famous method there appeared the idea of numerical integration involving multiple nodes. Taking any system of n distinct points $\{\tau_1,\ldots,\tau_n\}$ and n non-negative integers $m_1,\ldots,m_n$, and starting from the Hermite interpolation formula, L. Chakalov [4] in 1948 obtained the quadrature formula

Gradimir V. Milovanović
The Serbian Academy of Sciences and Arts, Kneza Mihaila 35, 11000 Belgrade, Serbia e-mail: gvm@mi.sanu.ac.rs

Marija P. Stanić
Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia e-mail: stanicm@kg.ac.rs

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = \sum_{\nu=1}^{n} \left[A_{0,\nu} f(\tau_\nu) + A_{1,\nu} f'(\tau_\nu) + \cdots + A_{m_\nu-1,\nu} f^{(m_\nu-1)}(\tau_\nu)\right], \tag{1}$$

which is exact for all polynomials of degree at most $m_1 + \cdots + m_n - 1$. He gave a method for computing the coefficients $A_{i,\nu}$ in (1). Such coefficients are given by $A_{i,\nu} = \int_{-1}^{1} \ell_{i,\nu}(t)\,\mathrm{d}t$, $\nu = 1,\ldots,n$, $i = 0,1,\ldots,m_\nu - 1$, where $\ell_{i,\nu}(t)$ are the fundamental functions of Hermite interpolation.

Taking $m_1 = \cdots = m_n = k$ in (1), P. Turán [48] in 1950 studied numerical quadratures of the form

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = \sum_{i=0}^{k-1} \sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,k}(f), \tag{2}$$

Obviously, for any given nodes $-1 \le \tau_1 \le \cdots \le \tau_n \le 1$ the formula (2) can be made exact for $f \in \mathcal{P}_{kn-1}$ ($\mathcal{P}_m$, $m \in \mathbb{N}_0$, is the set of all algebraic polynomials of degree at most m). However, for $k = 1$ the formula (2), i.e.,

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = \sum_{\nu=1}^{n} A_{0,\nu} f(\tau_\nu) + R_{n,1}(f)$$

is exact for all polynomials of degree at most $2n-1$ if the nodes τ_ν are the zeros of the Legendre polynomial P_n (it is the well–known Gauss–Legendre quadrature rule). Because of that it is quite natural to consider whether nodes τ_ν can be chosen in such a way that the quadrature formula (2) will be exact for all algebraic polynomials of degree less that or equal to $(k+1)n-1$. Turán [48] showed that the answer is negative for $k = 2$, while it is positive for $k = 3$. He proved that the nodes τ_ν, $\nu = 1,2,\ldots,n$, should be chosen as the zeros of the monic polynomial $\pi_n^*(t) = t^n + \cdots$ which minimizes the integral $\int_{-1}^{1} [\pi_n(t)]^4\,\mathrm{d}t$, where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. In the general case, the answer is negative for even, and positive for odd k, when τ_ν, $\nu = 1,2,\ldots,n$, must be the zeros of the monic polynomial π_n^* minimizing $\int_{-1}^{1} [\pi_n(t)]^{k+1}\,\mathrm{d}t$. Specially, for $k = 1$, π_n^* is the monic Legendre polynomial $\widehat{P}_n$.

Let us now assume that $k = 2s+1$, $s \ge 0$. Instead of (2), it is also interesting to consider a more general *Gauss–Turán type* quadrature formula

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\lambda(t) = \sum_{i=0}^{2s} \sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,2s}(f), \tag{3}$$

where $\mathrm{d}\lambda(t)$ is a given nonnegative measure on the real line $\mathbb{R}$, with compact or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k\,\mathrm{d}\lambda(t)$, $k = 0,1,\ldots$, exist and are finite, and $\mu_0 > 0$. It is known that formula (3) is exact for all polynomials of degree at most $2(s+1)n-1$, i.e., $R_{n,2s}(f) = 0$ for $f \in \mathcal{P}_{2(s+1)n-1}$. The nodes τ_ν, $\nu = 1,\ldots,n$, in (3) are the zeros of the monic polynomial $\pi_{n,s}(t)$, which minimizes the integral

$$F(a_0,a_1,\ldots,a_{n-1}) = \int_{\mathbb{R}} [\pi_n(t)]^{2s+2}\,\mathrm{d}\lambda(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. This minimization leads to the conditions

$$\frac{1}{2s+2} \cdot \frac{\partial F}{\partial a_k} = \int_{\mathbb{R}} [\pi_n(t)]^{2s+1} t^k \, \mathrm{d}\lambda(t) = 0, \quad k = 0, 1, \ldots, n-1. \tag{4}$$

These polynomials $\pi_n = \pi_{n,s}$ are known as $s-$*orthogonal* (or $s-$*self associated*) *polynomials* on $\mathbb{R}$ with respect to the measure $\mathrm{d}\lambda(t)$ (for more details see [15, 35, 36, 37]. For $s = 0$ they reduce to the standard orthogonal polynomials and (3) becomes the well–known Gauss–Christoffel formula.

Using some facts about monosplines, Micchelli [22] investigated the sign of the Cotes coefficients $A_{i,\nu}$ in the Turán quadrature formula.

For the numerical methods for the construction of $s-$orthogonal polynomials as well as for the construction of Gauss–Turán-type quadrature formulae we refer readers to the articles [23, 24, 17].

A next natural generalization of the Turán quadrature formula (3) is generalization to rules having nodes with different multiplicities. Such rule for $\mathrm{d}\lambda(t) = \mathrm{d}t$ on (a,b) was derived independently by L. Chakalov [5, 6] and T. Popoviciu [38]. Important theoretical progress on this subject was made by D.D. Stancu [43, 44] (see also [46]).

Let us assume that the nodes $\tau_1 < \tau_2 < \cdots < \tau_n$, have multiplicities $m_1, m_2, \ldots, m_n$, respectively (a permutation of the multiplicities $m_1, m_2, \ldots, m_n$, with the nodes held fixed, in general yields to a new quadrature rule).

It can be shown that the quadrature formula (1) is exact for all polynomials of degree less than $2\sum_{\nu=1}^{n}[(m_\nu+1)/2]$. Thus, the multiplicities m_ν that are even do not contribute toward an increase in the degree of exactness, so that it is reasonable to assume that all m_ν are odd integers, i.e., that $m_\nu = 2s_\nu + 1$, $\nu = 1, 2, \ldots, n$. Therefore, for a given sequence of nonnegative integers $\sigma = (s_1, s_2, \ldots, s_n)$ the corresponding quadrature formula

$$\int_{\mathbb{R}} f(t) \, \mathrm{d}\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f)$$

has the maximal degree of exactness $d_{\max} = 2\sum_{\nu=1}^{n} s_\nu + 2n - 1$ *if and only if*

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+1} t^k \, \mathrm{d}\lambda(t) = 0, \quad k = 0, 1, \ldots, n-1. \tag{5}$$

The previous orthogonality conditions correspond to (4) and they could be obtained by the minimization of the integral

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+2} \, \mathrm{d}\lambda(t).$$

The existence of such quadrature rules was proved by Chakalov [5], Popoviciu [38], Morelli and Verna [33], and existence and uniqueness by Ghizzetti and Ossicini [18].

The conditions (5) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in\mathbb{N}_0}$,

$$\pi_{n,\sigma}(t)=\prod_{\nu=1}^{n}(t-\tau_\nu^{(n,\sigma)}),\quad \tau_1^{(n,\sigma)}<\tau_2^{(n,\sigma)}<\cdots<\tau_n^{(n,\sigma)},$$

such that

$$\int_{\mathbb{R}}\pi_{k,\sigma}(t)\prod_{\nu=1}^{n}(t-\tau_\nu^{(n,\sigma)})^{2s_\nu+1}\,\mathrm{d}\lambda(t)=0,\quad k=0,1,\ldots,n-1.$$

These polynomials $\pi_{k,\sigma}$ are called *σ-orthogonal polynomials*, and they correspond to the sequence $\sigma=(s_1,s_2,\ldots)$. Specially, for $s=s_1=s_2=\cdots$, the $\sigma-$orthogonal polynomials reduce to the $s-$orthogonal polynomials.

At the end of this section we mention a general problem investigated by Stancu [43, 44, 46]. Namely, let $\eta_1,\ldots,\eta_m$ $(\eta_1<\cdots<\eta_m)$ be given *fixed* (or *prescribed*) nodes, with multiplicities $\ell_1,\ldots,\ell_m$, respectively, and $\tau_1,\ldots,\tau_n$ $(\tau_1<\cdots<\tau_n)$ be *free* nodes, with given multiplicities $m_1,\ldots,m_n$, respectively. Stancu considered interpolatory quadrature formulae of a general form

$$I(f)=\int_{\mathbb{R}}f(t)\,\mathrm{d}\lambda(t)\cong\sum_{\nu=1}^{n}\sum_{i=0}^{m_\nu-1}A_{i,\nu}f^{(i)}(\tau_\nu)+\sum_{\nu=1}^{m}\sum_{i=0}^{\ell_\nu-1}B_{i,\nu}f^{(i)}(\eta_\nu),\tag{6}$$

with an algebraic degree of exactness at least $M+L-1$, where $M=\sum_{\nu=1}^{n}m_\nu$ and $L=\sum_{\nu=1}^{m}\ell_\nu$.

Using free and fixed nodes we can introduce two polynomials

$$Q_M(t):=\prod_{\nu=1}^{n}(t-\tau_\nu)^{m_\nu}\quad\text{and}\quad q_L(t):=\prod_{\nu=1}^{m}(t-\eta_\nu)^{\ell_\nu}.$$

Choosing the free nodes to increase the degree of exactness leads to the so–called Gaussian type of quadratures. If the free (or *Gaussian*) nodes $\tau_1,\ldots,\tau_n$ are such that the quadrature rule (6) is exact for each $f\in\mathcal{P}_{M+L+n-1}$, then we call it the *Gauss–Stancu formula* (see [26]). Stancu [45] proved that *$\tau_1,\ldots,\tau_n$ are the Gaussian nodes if and only if*

$$\int_{\mathbb{R}}t^kQ_M(t)q_L(t)\,\mathrm{d}\lambda(t)=0,\quad k=0,1,\ldots,n-1.\tag{7}$$

Under some restrictions of node polynomials $q_L(t)$ and $Q_M(t)$ on the support interval of the measure $\mathrm{d}\lambda(t)$ we can give sufficient conditions for the existence of Gaussian nodes (cf. Stancu [45] and [20]). For example, *if the multiplicities of the Gaussian nodes are odd, e.g., $m_\nu=2s_\nu+1$, $\nu=1,\ldots,n$, and if the polynomial with fixed nodes $q_L(t)$ does not change its sign in the support interval of the measure $\mathrm{d}\lambda(t)$, then, in this interval, there exist real distinct nodes τ_ν, $\nu=1,\ldots,n$.* This condition for the polynomial $q_L(t)$ means that the multiplicities of the internal fixed nodes must be even. Defining a new (nonnegative) measure $\mathrm{d}\hat{\lambda}(t):=|q_L(t)|\,\mathrm{d}\lambda(t)$, the “orthogonality conditions” (7) can be expressed in a simpler form

$$\int_{\mathbb{R}} t^k Q_M(t)\,\mathrm{d}\hat{\lambda}(t) = 0, \quad k = 0,1,\ldots,n-1.$$

This means that the general quadrature problem (6), under these conditions, can be reduced to a problem with only Gaussian nodes, but with respect to another modified measure. Computational methods for this purpose are based on Christoffel’s theorem and described in details in [16] (see also [20]).

For more details about the concept of power orthogonality and the corresponding quadrature with multiple nodes and the maximal algebraic degree of exactness and about the numerical methods for their construction we refer readers to the article [25] and the references therein, as well as to a nice book by Shi [41].

The next natural generalization represents quadrature rules with multiple nodes and the maximal degree of exactness in some functional spaces which are different from the space of algebraic polynomials.

2 A generalized Gaussian problem

For a finite interval $[a,b]$ and a positive integer N, Ghizzeti and Osicini in [19] considered quadrature rules for the computation of an integral of the type $\int_a^b f(x)\,w(x)\,\mathrm{d}x$, where the weight function w and function f satisfy the following conditions: $w(x) \in L[a,b]$, $f(x) \in AC^{N-1}[a,b]$ ($AC^m[a,b]$, $m \in \mathbb{N}_0$, is the class of functions f whose m-th derivative is absolutely continuous in $[a,b]$).

For a fixed number $m \geq 1$ of points $a \leq x_1 < x_2 < \cdots < x_m \leq b$ and a fixed linear differential operator E of order N of the form

$$E = \sum_{k=0}^{N} a_k(x) \frac{\mathrm{d}^{N-k}}{\mathrm{d}x^{N-k}},$$

where $a_1(x) = 1$, $a_k(x) \in AC^{N-k-1}[a,b]$, $k = 1,2,\ldots,N-1$, and $a_N(x) \in L(a,b)$, they considered (see [19, pp. 41–43]) a generalized Gaussian problem, i.e., they studied the quadrature rule

$$\int_a^b f(x)\,w(x)\,\mathrm{d}x = \sum_{\nu=1}^{m} \sum_{j=0}^{N-1} A_{j,\nu},f^{(j)}(x_\nu) + R(f)$$

such that $E(f) = 0$ implies $R(f) = 0$.

According to [19, p. 28] quadrature rule which is exact for all algebraic polynomials of degree less than or equal $\nu - 1$ are relative to the differential operator $E = \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}$, and quadrature rule which is exact for all trigonometric polynomials of degree less than or equal to ν are relative to the differential operator (of order $2\nu+1$): $E = \frac{\mathrm{d}}{\mathrm{d}x}\prod_{k=1}^{\nu}\bigl(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\bigr)$.

The considered whether there can exist a rule of the form

$$\int_a^b f(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{m} \sum_{j=0}^{N-p_\nu-1} A_{j,\nu} f^{(j)}(x_\nu) + R(f), \tag{8}$$

with fixed integers p_ν, $0 \le p_\nu \le N-1$, $\nu = 1,2,\ldots,m$, such that at least one of the integers p_ν is greater than or equal to 1, satisfying that $E(f) = 0$ implies $R(f) = 0$, too. The answer is given in the following theorem (see [19, p. 45] for the proof).

Theorem 1. *For the given nodes $x_1, x_2, \ldots, x_m$, the linear differential operator E of order N and the nonnegative integers $p_1, p_2, \ldots, p_m$, $0 \le p_\nu \le N-1$, $\nu = 1,2,\ldots,m$, such that there exists $\nu \in \{1,2,\ldots,m\}$ for which $p_\nu \ge 1$, consider the following homogenous boundary differential problem*

$$E(f) = 0, \quad f^{(j)}(x_\nu) = 0, \quad j = 0,1,\ldots,N-p_\nu-1, \quad \nu = 1,2,\ldots,m. \tag{9}$$

If this problem has no non–trivial solutions $\big($whence $N \le mN - \sum_{\nu=1}^m p_\nu\big)$ it is possible to write a quadrature rule of the type (8) *with $mN - \sum_{\nu=1}^m p_\nu - N$ parameters chosen arbitrarily. If, on the other hand, the problem* (9) *has q linearly independent solutions U_r, $r = 1,2,\ldots,q$, with $N - mN + \sum_{\nu=1}^m p_\nu \le q \le p_\nu$ for all $\nu = 1,2,\ldots,m$, then the formula* (8) *may apply only if the following q conditions*

$$\int_a^b U_r(x)\,w(x)\,\mathrm{d}x = 0, \quad r = 1,2,\ldots,q,$$

are satisfied; if so, $mN - \sum_{\nu=1}^m p_\nu - N + q$ parameters in the formula (8) *can be chosen arbitrary.*

Since the quadrature rules with multiple nodes and the maximal algebraic degree of exactness are widespread in literature, in what follows we restrict our attention to the quadrature rules with multiple nodes and the maximal trigonometric degree of exactness.

3 Quadrature rules with multiple nodes and the maximal trigonometric degree of exactness

For a nonnegative integer n and for $\gamma \in \{0, 1/2\}$ by $\mathcal{T}_n^\gamma$ we denote the linear span of the set $\{\cos(k+\gamma)x, \sin(k+\gamma)x : k = 0,1,\ldots,n\}$. It is obvious that $\mathcal{T}_n^0 = \mathcal{T}_n$ is the linear space of all trigonometric polynomials of degree less than or equal to n, $\mathcal{T}_n^{1/2}$ is the linear space of all trigonometric polynomials of semi–integer degree less than or equal to $n+1/2$ and $\dim(\mathcal{T}_n^\gamma) = 2(n+\gamma)+1$. By $\mathcal{T}^\gamma = \bigcup_{n\in\mathbb{N}_0} \mathcal{T}_n^\gamma$ we denote the set of all trigonometric polynomials (for $\gamma = 0$) and the set of all trigonometric polynomials of semi–integer degree (for $\gamma = 1/2$). For $\gamma = 0$ we simple write $\mathcal{T}$ instead of $\mathcal{T}^0$. Finally, by $\widetilde{\mathcal{T}}_n$ we denote the linear space $\mathcal{T}_n \ominus \mathrm{span}\{\sin nx\}$ or $\mathcal{T}_n \ominus \mathrm{span}\{\cos nx\}$.

We use the following notation: $\widehat{\gamma} = 1 - 2\gamma$, $\gamma \in \{0, 1/2\}$.

In what follows we simple say that trigonometric polynomial has degree $k + \gamma$, $k \in \mathbb{N}_0$, $\gamma \in \{0, 1/2\}$, which always means that for $\gamma = 0$ it is trigonometric polynomial of precise degree k, while for $\gamma = 1/2$ it is trigonometric polynomial of precise semi–integer degree $k + 1/2$.

Obviously, every trigonometric polynomial of degree $n + \gamma$ can be represented in the form

$$T_n^{\gamma}(x) = \sum_{k=0}^{n} \left(c_k \cos(k+\gamma)x + d_k \sin(k+\gamma)x\right), \quad c_n^2 + d_n^2 \neq 0, \tag{10}$$

where $c_k, d_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$. For $\gamma = 0$ we always set $d_0 = 0$. Coefficients c_n and d_n are called *the leading coefficients*.

Let us suppose that w is a weight function, integrable and nonnegative on the interval $[-\pi, \pi)$, vanishing there only on a set of a measure zero.

The quadrature rules with simple nodes and the maximal trigonometric degree of exactness were considered by many authors. For a brief historical survey of available approaches for the construction of such quadrature rules we refer readers to the article [28] and the references therein. Following Turetzkii's approach (see [49]), which is a simulation of the development of Gaussian quadrature rules for algebraic polynomials, quadrature rules with the maximal trigonometric degree of exactness with an odd number of nodes ($\gamma = 1/2$) and with an even number of nodes ($\gamma = 0$) were considered in [28] (see also [9]) and in [47], respectively.

It is well known that in a case of quadrature rule with maximal algebraic degree of exactness the nodes are the zeros of the corresponding orthogonal algebraic polynomial. In a case of quadrature with an odd maximal trigonometric degree of exactness the nodes are the zeros of the corresponding orthogonal trigonometric polynomials, but for an even maximal trigonometric degree of exactness the nodes are not zeros of trigonometric polynomial, but zeros of the corresponding orthogonal trigonometric polynomial of semi–integer degree. Similarly, when the quadrature rules with multiple nodes and the maximal algebraic degree of exactness are in question it is necessary to consider power orthogonality in the space of algebraic polynomials. But, if we want to consider quadrature rules with multiple nodes and the maximal trigonometric degree of exactness, we have to consider power orthogonality in the space of trigonometric polynomials or in the space of trigonometric polynomials of semi–integer degree, which depends on number of nodes.

In mentioned papers [28] and [47] the existence and the uniqueness of orthogonal trigonometric polynomials of integer or of semi–integer degree were proved under the assumption that two of the coefficients in expansion (10) are given in advance (it is usual to choose the leading coefficients in advance). Also, if we directly compute the zeros of orthogonal trigonometric polynomial (of integer or semi–integer degree), we can fix one of them in advance since for $2(n+\gamma)$ zeros we have $2(n+\gamma) - 1$ orthogonality conditions.

Let n be positive integer, $\gamma \in \{0, 1/2\}$, s_ν nonnegative integers for $\nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n$, and $\sigma = (s_{\widehat{\gamma}}, s_{\widehat{\gamma}+1}, \ldots, s_{2n})$. For the given weight function w we considered a quadrature rule of the following type

$$\int_{-\pi}^{\pi} f(x)\,w(x)\,\mathrm{d}x=\sum_{\nu=\widehat{\gamma}}^{2n}\sum_{j=0}^{2s_\nu} A_{j,\nu} f^{(j)}(x_\nu)+R_n(f). \tag{11}$$

which have maximal trigonometric degree of exactness, i.e., for which $R_n(f)=0$ for all $f\in\mathfrak{T}_{N_1}$, where $N_1=\sum_{\nu=\widehat{\gamma}}^{2n}(s_\nu+1)-1$. In this case boundary differential problem (9) has the following form

$$E(f)=0,\quad f^{(j)}(x_\nu)=0,\quad j=0,1,\ldots,2s_\nu,\quad \nu=\widehat{\gamma},\widehat{\gamma}+1,\ldots,2n, \tag{12}$$

where E is the following differential operator of order $N=2N_1+1$:

$$E=\frac{\mathrm{d}}{\mathrm{d}x}\prod_{k=1}^{N_1}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+k^2\right).$$

The boundary problem (12) has the following $2n-\widehat{\gamma}$ linear independent nontrivial solutions

$$U_\ell(x)=\prod_{\nu=\widehat{\gamma}}^{2n}\left(\sin\frac{x-x_\nu}{2}\right)^{2s_\nu+1}\cos(\ell+\gamma)x,\quad \ell=0,1,\ldots,n-1,$$
$$V_\ell(x)=\prod_{\nu=\widehat{\gamma}}^{2n}\left(\sin\frac{x-x_\nu}{2}\right)^{2s_\nu+1}\sin(\ell+\gamma)x,\quad \ell=\widehat{\gamma},\widehat{\gamma}+1,\ldots,n-1.$$

According to Theorem 1, the nodes $x_{\widehat{\gamma}},x_{\widehat{\gamma}+1},\ldots,x_{2n}$ of the quadrature rule (11) satisfy conditions

$$\int_{-\pi}^{\pi}\prod_{\nu=\widehat{\gamma}}^{2n}\left(\sin\frac{x-x_\nu}{2}\right)^{2s_\nu+1}\cos(\ell+\gamma)x\,w(x)\,\mathrm{d}x=0,\quad \ell=0,1,\ldots,n-1,$$
$$\int_{-\pi}^{\pi}\prod_{\nu=\widehat{\gamma}}^{2n}\left(\sin\frac{x-x_\nu}{2}\right)^{2s_\nu+1}\sin(\ell+\gamma)x\,w(x)\,\mathrm{d}x=0,\quad \ell=\widehat{\gamma},\widehat{\gamma}+1,\ldots,n-1,$$

i.e.,

$$\int_{-\pi}^{\pi}\prod_{\nu=\widehat{\gamma}}^{2n}\left(\sin\frac{x-x_\nu}{2}\right)^{2s_\nu+1} t_{n-1}^{\gamma}(x)\,w(x)\,\mathrm{d}x=0,\quad \text{for all } t_{n-1}^{\gamma}\in\mathfrak{T}_{n-1}^{\gamma}. \tag{13}$$

Trigonometric polynomial $T_{\sigma,n}^{\gamma}=\prod_{\nu=\widehat{\gamma}}^{2n}\sin(x-x_\nu)/2\in\mathfrak{T}_n^{\gamma}$ which satisfies orthogonality conditions (13) is called $\sigma-$*orthogonal trigonometric polynomial* of degree $n+\gamma$ with respect to weight function w on $[-\pi,\pi)$. Therefore, the nodes of quadrature rule (11) with the maximal trigonometric degree of exactness are the zeros of the corresponding $\sigma-$orthogonal trigonometric polynomial.

Specially, for $s_{\widehat{\gamma}} = s_{\widehat{\gamma}+1} = \cdots = s_{2n} = s$, $\sigma-$orthogonality conditions (13) reduce to

$$\int_{-\pi}^{\pi} \left(\prod_{\nu=\widehat{\gamma}}^{2n} \sin \frac{x-x_\nu}{2} \right)^{2s+1} t_{n-1}^{\gamma}(x)\, w(x)\, \mathrm{d}x = 0, \quad t_{n-1}^{\gamma} \in \mathfrak{T}_{n-1}^{\gamma}. \tag{14}$$

Trigonometric polynomial $T_{s,n}^{\gamma} = \prod_{\nu=\widehat{\gamma}}^{2n} \sin(x-x_\nu)/2 \in \mathfrak{T}_n^{\gamma}$ which satisfies (14) is called *s−orthogonal trigonometric polynomial* of degree $n+\gamma$ with respect to w on $[-\pi,\pi)$.

For $\gamma = 1/2$, $s-$orthogonal trigonometric polynomials of semi–integer degree were defined and considered in [27], while $\sigma-$orthogonal trigonometric polynomials of semi–integer degree were defined and their main properties were proved in [29]. For $\gamma = 0$, the both $s-$ and $\sigma-$orthogonal trigonometric polynomials were defined and studied in [47].

It is obvious that $s-$orthogonal trigonometric polynomials can be considered as a special case of $\sigma-$orthogonal trigonometric polynomials. However, we first prove the existence and uniqueness of $s-$orthogonal trigonometric polynomials because it can be done in a quite simple way, by using the well–known facts about the best approximation given in the Remark 1 below (see [10, p. 58–60]). The proofs of the existence and uniqueness of $\sigma-$orthogonal trigonometric polynomials are more complicated. We prove them by using theory of implicitly defined orthogonality (see [13, Sect. 5.3] for implicitly defined orthogonal algebraic polynomials).

Remark 1. Let X be a Banach space and Y be a closed linear subspace of X. For each $f \in X$, $\inf_{g\in Y} \|f-g\|$ is the error of approximation of f by elements from Y. If there exists some $g = g_0 \in Y$ for which that infimum is attained, then g_0 is called the best approximation to f from Y. For each finite dimensional subspace X_n of X and each $f \in X$, there exists the best approximation to f from X_n. In addition, if X is a strictly convex space, then each $f \in X$ has at most one element of the best approximation in each closed linear subspace $Y \subset X$.

Analogously as it was proved in [28] and [47] for orthogonal trigonometric polynomials, one can prove the existence and uniqueness of $s-$ and $\sigma-$orthogonal trigonometric polynomials fixing in advance two of their coefficients or one of their zeros. Before we prove the existence and uniqueness of $s-$ and $\sigma-$orthogonal trigonometric polynomials, we prove that all of their zeros in $[-\pi,\pi)$ are simple. Of course, it is enough to prove that for $\sigma-$orthogonal trigonometric polynomials.

Theorem 2. *The $\sigma-$orthogonal trigonometric polynomial $T_{\sigma,n}^{\gamma} \in \mathfrak{T}_n^{\gamma}$ with respect to the weight function $w(x)$ on $[-\pi,\pi)$ has in $[-\pi,\pi)$ exactly $2(n+\gamma)$ distinct simple zeros.*

Proof. It is easy to see that the trigonometric polynomial $T_{\sigma,n}^{\gamma}(x)$ must have at least one zero of odd multiplicity in $[-\pi,\pi)$, because if we assume the contrary, then for $n \in \mathbb{N}$ we obtain that

$$\int_{-\pi}^{\pi} \prod_{\nu=\widehat{\gamma}}^{2n} \left(\sin \frac{x-x_\nu}{2} \right)^{2s_\nu+1} \cos(0+\gamma)x\, w(x)\, \mathrm{d}x = 0,$$

which is impossible, because the integrand does not change its sign on $[-\pi,\pi)$. We know (see [1] and [2]) that $T^{\gamma}_{\sigma,n}(x)$ must change its sign an even number of times for $\gamma=0$ and an odd number of times for $\gamma=1/2$.

First, we suppose that $\gamma=0$ and that the number of zeros of odd multiplicities of $T_{\sigma,n}(x)$ on $[-\pi,\pi)$ is $2m$, $m<n$. We denote these zeros by $y_1,y_2,\ldots,y_{2m}$, and set $t(x)=\prod_{k=1}^{2m}\sin((x-y_k)/2)$. Since $t\in\mathfrak{T}_m$, $m<n$, we have

$$\int_{-\pi}^{\pi}\prod_{\nu=1}^{2n}\Big(\sin\frac{x-x_\nu}{2}\Big)^{2s_\nu+1}t(x)\,w(x)\,\mathrm{d}x=0,$$

which also gives a contradiction, since the integrand does not change its sign on $[-\pi,\pi)$. Thus, $T_{\sigma,n}$ must have exactly $2n$ distinct simple zeros on $[-\pi,\pi)$.

Similarly, for $\gamma=1/2$ by $y_1,y_2,\ldots,y_{2m+1}$, $m<n$, we denote the zeros of odd multiplicities of $T^{1/2}_{\sigma,n}(x)$ on $[-\pi,\pi)$ and set $\widehat{t}(x)=\prod_{k=1}^{2m+1}\sin((x-y_k)/2)$. Since $\widehat{t}\in\mathfrak{T}^{1/2}_m$, $m<n$, we have

$$\int_{-\pi}^{\pi}\prod_{\nu=0}^{2n}\Big(\sin\frac{x-x_\nu}{2}\Big)^{2s_\nu+1}(x)\widehat{t}(x)\,w(x)\,\mathrm{d}x=0,$$

which again gives a contradiction, since the integrand does not change its sign on $[-\pi,\pi)$, which means that $T^{1/2}_{\sigma,n}(x)$ must have exactly $2n+1$ distinct simple zeros on $[-\pi,\pi)$. $\square$

3.1 $s-$orthogonal trigonometric polynomials

Theorem 3. *Trigonometric polynomial $T^{\gamma}_{s,n}(x)\in\mathfrak{T}^{\gamma}_n$, $\gamma\in\{0,1/2\}$, with given leading coefficients, which is s–orthogonal on $[-\pi,\pi)$ with respect to a given weight function w is determined uniquely.*

Proof. Let us set $X=L^{2s+2}[-\pi,\pi]$,

$$u=w(x)^{1/(2s+2)}(c_{n,\gamma}\cos(n+\gamma)x+d_{n,\gamma}\sin(n+\gamma)x)\in L^{2s+2}[-\pi,\pi],$$

and fix the following $2n-\widehat{\gamma}$ linearly independent elements in $L^{2s+2}[-\pi,\pi]$:

$$u_j=w(x)^{1/(2s+2)}\cos(j+\gamma)x,\quad j=0,1,\ldots,n-1,$$
$$v_k=w(x)^{1/(2s+2)}\sin(k+\gamma)x,\quad k=\widehat{\gamma},\widehat{\gamma}+1,\ldots,n-1.$$

Then $Y=\mathrm{span}\{u_j,v_k : j=0,1,\ldots,n-1, k=\widehat{\gamma},\widehat{\gamma}+1,\ldots,n-1\}$ is a finite dimensional subspace of X and, according to Remark 1, for each element from X there exists the best approximation from Y, i.e., there exist $2n-\widehat{\gamma}$ constants $\alpha_{j,\gamma}$, $j=0,1,\ldots,n-1$, and $\beta_{k,\gamma}$, $k=\widehat{\gamma},\widehat{\gamma}+1,\ldots,n-1$, such that the error

$$\left\| u - \left(\sum_{k=0}^{n-1} \alpha_{k,\gamma} u_k + \sum_{k=\widehat{\gamma}}^{n-1} \beta_{k,\gamma} v_k \right) \right\| = \left(\int_{-\pi}^{\pi} \left(c_{n,\gamma} \cos(n+\gamma)x + d_{n,\gamma} \sin(n+\gamma)x \right.\right.$$
$$\left.\left. - \left(\sum_{k=0}^{n-1} \alpha_{k,\gamma} \cos(k+\gamma)x + \sum_{k=\widehat{\gamma}}^{n-1} \beta_{k,\gamma} \sin(k+\gamma)x \right) \right)^{2s+2} w(x)\,\mathrm{d}x \right)^{1/(2s+2)},$$

is minimal, i.e., for every n and for every choice of the leading coefficients $c_{n,\gamma}, d_{n,\gamma}$, $c_{n,\gamma}^2 + d_{n,\gamma}^2 \neq 0$, there exists a trigonometric polynomial of degree $n+\gamma$

$$T_{s,n}^{\gamma}(x) = c_{n,\gamma} \cos(n+\gamma)x + d_{n,\gamma} \sin(n+\gamma)x$$
$$- \left(\sum_{k=0}^{n-1} \alpha_{k,\gamma} \cos(k+\gamma)x + \sum_{k=\widehat{\gamma}}^{n-1} \beta_{k,\gamma} \sin(k+\gamma)x \right),$$

such that

$$\int_{-\pi}^{\pi} (T_{s,n}^{\gamma}(x))^{2s+2} w(x)\,\mathrm{d}x$$

is minimal. Since the space $L^{2s+2}[-\pi,\pi]$ is strictly convex, according to Remark 1, the problem of the best approximation has the unique solution, i.e., the trigonometric polynomial $T_{s,n}^{\gamma}$ is unique.

Therefore, for each of the following $2n - \widehat{\gamma}$ functions

$$F_k^C(\lambda) = \int_{-\pi}^{\pi} \big(T_{s,n}^{\gamma}(x) + \lambda \cos(k+\gamma)x\big)^{2s+2} w(x)\,\mathrm{d}x, \quad k = 0, 1, \ldots, n-1,$$
$$F_k^S(\lambda) = \int_{-\pi}^{\pi} \big(T_{s,n}^{\gamma}(x) + \lambda \sin(k+\gamma)x\big)^{2s+2} w(x)\,\mathrm{d}x, \quad k = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, n-1,$$

its derivative must be equal to zero for $\lambda = 0$. Hence,

$$\int_{-\pi}^{\pi} \big(T_{s,n}^{\gamma}(x)\big)^{2s+1} \cos(k+\gamma)x\, w(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, n-1,$$
$$\int_{-\pi}^{\pi} \big(T_{s,n}^{\gamma}(x)\big)^{2s+1} \sin(k+\gamma)x\, w(x)\,\mathrm{d}x = 0, \quad k = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, n-1,$$

which means that the polynomial $T_{s,n}^{\gamma}(x)$ satisfies (14). □

3.2 $\sigma-$orthogonal trigonometric polynomials

In order to prove the existence and uniqueness of $\sigma-$orthogonal trigonometric polynomials we fix in advance the zero $x_{\widehat{\gamma}} = -\pi$ and consider the $\sigma-$orthogonality conditions (13) as the system with unknown zeros.

Theorem 4. *Let w be the weight function on $[-\pi,\pi)$ and let p be a nonnegative continuous function, vanishing only on a set of a measure zero. Then there exists a* *trigonometric polynomial* T_n^γ*,* $\gamma \in \{0,1/2\}$*, of degree* $n+\gamma$*, orthogonal on* $[-\pi,\pi)$ *to every trigonometric polynomial of degree less than or equal to* $n-1+\gamma$ *with respect to the weight function* $p\big(T_n^\gamma(x)\big)w(x)$*.*

Proof. By $\widehat{\mathfrak{T}}_n^\gamma$ we denote the set of all trigonometric polynomials of degree $n+\gamma$ which have $2(n+\gamma)$ real distinct zeros $-\pi = x_{\widehat{\gamma}} < x_{\widehat{\gamma}+1} < \cdots < x_{2n} < \pi$ and denote

$$S_{2n-\widehat{\gamma}} = \big\{\mathbf{x} = (x_{\widehat{\gamma}+1}, x_{\widehat{\gamma}+2}, \ldots, x_{2n}) \in \mathbb{R}^{2n-\widehat{\gamma}} : -\pi < x_{\widehat{\gamma}+1} < x_{\widehat{\gamma}+2} < \cdots < x_{2n} < \pi\big\}.$$

For a given function p and an arbitrary $Q_n^\gamma \in \widehat{\mathfrak{T}}_n^\gamma$, we introduce the inner product as follows

$$\big(f,g\big)_{Q_n^\gamma} = \int_{-\pi}^{\pi} f(x)g(x)\,p(Q_n^\gamma(x))w(x)\,\mathrm{d}x.$$

It is obvious that there is one to one correspondence between the sets $\widehat{\mathfrak{T}}_n^\gamma$ and $S_{2n-\widehat{\gamma}}$. Indeed, for every element $\mathbf{x} = (x_{\widehat{\gamma}+1}, x_{\widehat{\gamma}+2}, \ldots, x_{2n}) \in S_{2n-\widehat{\gamma}}$ and for

$$Q_n^\gamma(x) = \cos\frac{x}{2}\prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-x_\nu}{2} \tag{15}$$

there exists a unique system of orthogonal trigonometric polynomials $U_k^\gamma \in \widehat{\mathfrak{T}}_k^\gamma$, $k=0,1,\ldots,n$, such that $\big(U_n^\gamma, \cos(k+\gamma)x\big)_{Q_n^\gamma} = \big(U_n^\gamma, \sin(j+\gamma)x\big)_{Q_n^\gamma} = 0$, for all $k = 0,1,\ldots,n-1$, $j = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, n-1$, and $\big(U_n^\gamma, U_n^\gamma\big)_{Q_n^\gamma} \neq 0$.

In such a way, we introduce a mapping $F_n : S_{2n-\widehat{\gamma}} \to S_{2n-\widehat{\gamma}}$, defined in the following way: for any $\mathbf{x} = (x_{\widehat{\gamma}+1}, x_{\widehat{\gamma}+2}, \ldots, x_{2n}) \in S_{2n-\widehat{\gamma}}$ we have $F_n(\mathbf{x}) = \mathbf{y}$, where $\mathbf{y} = (y_{\widehat{\gamma}+1}, y_{\widehat{\gamma}+2}, \ldots, y_{2n}) \in S_{2n-\widehat{\gamma}}$ is such that $-\pi, y_{\widehat{\gamma}+1}, y_{\widehat{\gamma}+2}, \ldots, y_{2n}$ are the zeros of the orthogonal trigonometric polynomial of degree $n+\gamma$ with respect to the weight function $p(Q_n^\gamma(x))w(x)$, where $Q_n^\gamma(x)$ is given by (15). It is easy to see that the function $p\big(\cos(x/2)\prod_{\nu=\widehat{\gamma}+1}^{2n}\sin((x-x_\nu)/2)\big)w(x)$ is also an admissible weight function for arbitrary $\mathbf{x} = (x_{\widehat{\gamma}+1}, x_{\widehat{\gamma}+2}, \ldots, x_{2n}) \in \overline{S}_{2n-\widehat{\gamma}} \backslash S_{2n-\widehat{\gamma}}$.

Now, we prove that F_n is continuous mapping on $\overline{S}_{2n-\widehat{\gamma}}$. Let $\mathbf{x} \in \overline{S}_{2n-\widehat{\gamma}}$ be an arbitrary point, $\{\mathbf{x}^{(m)}\}$, $m \in \mathbb{N}$, a convergent sequence of points from $S_{2n-\widehat{\gamma}}$, which converges to $\mathbf{x}$, $\mathbf{y} = F_n(\mathbf{x})$, and $\mathbf{y}^{(m)} = F_n(\mathbf{x}^{(m)})$, $m \in \mathbb{N}$. Let $\mathbf{y}^* \in \overline{S}_{2n-\widehat{\gamma}}$ be an arbitrary limit point of the sequence $\{\mathbf{y}^{(m)}\}$ when $m \to \infty$. Thus,

$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-y_\nu^{(m)}}{2} \cos(k+\gamma)x$$
$$\times\, p\left(\cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-x_\nu^{(m)}}{2}\right) w(x)\,\mathrm{d}x = 0, \quad k=0,1,\dots,n-1,$$
$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-y_\nu^{(m)}}{2} \sin(j+\gamma)x$$
$$\times\, p\left(\cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-x_\nu^{(m)}}{2}\right) w(x)\,\mathrm{d}x = 0, \quad j=\widehat{\gamma},\widehat{\gamma}+1,2,\dots,n-1.$$

Applying Lebesgue Theorem of dominant convergence (see [32, p. 83]), when $m \to +\infty$ we obtain

$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-y_\nu^{*}}{2} \cos(k+\gamma)x$$
$$p\left(\cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-x_\nu}{2}\right) w(x)\,\mathrm{d}x = 0, \quad k=0,1,\dots,n-1,$$
$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-y_\nu^{*}}{2} \sin(j+\gamma)x$$
$$\times\, p\left(\cos\frac{x}{2} \prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-x_\nu}{2}\right) w(x)\,\mathrm{d}x = 0, \quad j=\widehat{\gamma},\widehat{\gamma}+1,\dots,n-1,$$

i.e., $\cos(x/2)\prod_{\nu=\widehat{\gamma}+1}^{2n} \sin((x-y_\nu^*)/2)$ is the trigonometric polynomial of degree $n+\gamma$ which is orthogonal to all trigonometric polynomials from $\mathcal{T}_{n-1}^{\gamma}$ with respect to the weight function $p\big(\cos(x/2)\prod_{\nu=\widehat{\gamma}+1}^{2n} \sin((x-x_\nu)/2)\big)w(x)$ on $[-\pi,\pi)$. Such trigonometric polynomial has $2(n+\gamma)$ distinct simple zeros in $[-\pi,\pi)$ (see [28] and [47]). Therefore, $\mathbf{y}^* \in S_{2n-\widehat{\gamma}}$ and $\mathbf{y}^* = F_n(\mathbf{x})$. Since $\mathbf{y} = F_n(\mathbf{x})$, because of the uniqueness we have $\mathbf{y}^* = \mathbf{y}$, i.e., the mapping F_n is continuous on $\overline{S}_{2n-\widehat{\gamma}}$.

Finally, we prove that the mapping F_n has a fixed point. The mapping F_n is continuous on the bounded, convex and closed set $\overline{S}_{2n-\widehat{\gamma}} \subset \mathbb{R}^{2n-\widehat{\gamma}}$. Applying the Brouwer fixed point theorem (see [34]) we conclude that there exists a fixed point of F_n. Since $F_n(\mathbf{x}) \in S_{2n-\widehat{\gamma}}$ for all $\mathbf{x} \in \overline{S}_{2n-\widehat{\gamma}} \backslash S_{2n-\widehat{\gamma}}$, the fixed point of F_n belongs to $S_{2n-\widehat{\gamma}}$.

If we denote the fixed point of F_n by $\mathbf{x} = (x_{\widehat{\gamma}+1}, x_{\widehat{\gamma}+2}, \dots, x_{2n})$, then for $T_n(x) = \cos(x/2)\prod_{\nu=\widehat{\gamma}+1}^{2n} \sin((x-x_\nu)/2)$ we get

$$\int_{-\pi}^{\pi} T_n^{\gamma}(x) \cos(k+\gamma)x\, p\big(T_n^{\gamma}(x)\big) w(x)\,\mathrm{d}x = 0, \quad k=0,1,\dots,n-1,$$
$$\int_{-\pi}^{\pi} T_n^{\gamma}(x) \sin(j+\gamma)x\, p\big(T_n^{\gamma}(x)\big) w(x)\,\mathrm{d}x = 0, \quad j=\widehat{\gamma},\widehat{\gamma}+1,\dots,n-1. \quad \square$$

3.3 Construction of quadrature rules with multiple nodes and the maximal trigonometric degree of exactness

The construction of quadrature rule with multiple nodes and the maximal trigonometric degree of exactness is based on the application of some properties of the topological degree of mapping (see [39]). For that purposes we use a certain modification of ideas of Bojanov [3], Shi [40], Shi and Xu [42] (for algebraic $\sigma-$orthogonal polynomials) and Dryanov [11]. Using a unique notation for $\gamma\in\{0,1/2\}$ and omitting some details we present here results obtained in [29] and [47] for $\gamma=1/2$ and $\gamma=0$, respectively.

For $a\in[0,1]$, $n\in\mathbb{N}$, and

$$F(\mathbf{x},a)=\int_{-\pi}^{\pi}\Big(\cos\frac{x}{2}\Big)^{2as_{\widehat{\gamma}}+1}\prod_{\nu=\widehat{\gamma}+1}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1}$$
$$\times\operatorname{sgn}\left(\prod_{\nu=\widehat{\gamma}+1}^{2n}\sin\frac{x-x_\nu}{2}\right)t_{n-1}^{\gamma}(x)\,w(x)\,\mathrm{d}x,\quad t_{n-1}^{\gamma}\in\mathcal{T}_{n-1}^{\gamma},$$

we consider the following problem

$$F(\mathbf{x},a)=0\quad\text{for all } t_{n-1}^{\gamma}\in\mathcal{T}_{n-1}^{\gamma},\tag{16}$$

with unknowns $x_{\widehat{\gamma}+1},x_{\widehat{\gamma}+2},\ldots,x_{2n}$.

The σ–orthogonality conditions (13) (with $x_1=-\pi$) are equivalent to the problem (16) for $a=1$, which means that the nodes of the quadrature rule (11) can be obtained as a solution of the problem (16) for $a=1$.

For $a=0$ and $a=1$ the problem (16) has the unique solution in the simplex $S_{2n-\widehat{\gamma}}$. According to Theorem 4 we know that the problem (16) has solutions in the simplex $S_{2n-\widehat{\gamma}}$ for every $a\in[0,1]$. We will prove the uniqueness of the solution $\mathbf{x}\in S_{2n-\widehat{\gamma}}$ of the problem (16) for all $a\in(0,1)$.

Let us denote

$$W(\mathbf{x},a,x)=\prod_{\nu=\widehat{\gamma}+1}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+2}$$

and

$$\phi_k(\mathbf{x},a)=\int_{-\pi}^{\pi}\Big(\cos\frac{x}{2}\Big)^{2as_{\widehat{\gamma}}+1}\frac{W(\mathbf{x},a,x)}{\sin\dfrac{x-x_k}{2}}\,w(x)\,\mathrm{d}x,\tag{17}$$

for $k=\widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n$.

Applying the same arguments as in [11, Lemma 3.2], the following auxiliary result can be easily proved.

Lemma 1. *There exists $\varepsilon>0$ such that for every $a\in[0,1]$ the solutions $\mathbf{x}$ of the problem (16) belong to the simplex*

$$\overline{S}_{\varepsilon,\gamma}=\big\{\mathbf{y}:\varepsilon\le y_{\widehat{\gamma}+1}+\pi,\varepsilon\le y_{\widehat{\gamma}+2}-y_{\widehat{\gamma}+1},\ldots,\varepsilon\le y_{2n}-y_{2n-1},\varepsilon\le\pi-y_{2n}\big\}.$$

Also, the following result holds (see [29, Lemma 2.3] and [47, Lemma 4.4] for the proof).

Lemma 2. *The problem* (16) *and the following problem*

$$\phi_k(\mathbf{x},a)=0,\quad k=\widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n, \tag{18}$$

where $\phi_k(\mathbf{x},a)$ *are given by* (17)*, are equivalent in the simplex* $S_{2n-\widehat{\gamma}}$*.*

It is important to emphasize that the problems (16) and (18) are equivalent in the simplex $\overline{S}_{\varepsilon,\gamma}$, for some $\varepsilon>0$, as well as in the simplex $\overline{S}_{\varepsilon_1,\gamma}$, for all $0<\varepsilon_1<\varepsilon$, but they are not equivalent in $\overline{S}_{2n-\widehat{\gamma}}$.

For the proof of the main result we need the following lemma (for the proof see [29, Lemma 2.4]).

Lemma 3. *Let* $p_{\xi,\eta}(x)$ *be a continuous function on* $[-\pi,\pi]$*, which depends continuously on parameters* $\xi,\eta\in[c,d]$*, i.e., if* (ξ_m,η_m) *approaches* (ξ_0,η_0) *then the sequences* $p_{\xi_m,\eta_m}(x)$ *tends to* $p_{\xi_0,\eta_0}(x)$ *for every fixed x. If the solution* $\mathbf{x}(\xi,\eta)$ *of the problem* (18) *with the weight function* $p_{\xi,\eta}(x)w(x)$ *is always unique for every* $(\xi,\eta)\in[c,d]^2$*, then the solution* $\mathbf{x}(\xi,\eta)$ *depends continuously on* $(\xi,\eta)\in[c,d]^2$*.*

Now, we are ready to prove the main result.

Theorem 5. *The problem* (18) *has a unique solution in the simplex* $S_{2n-\widehat{\gamma}}$ *for all* $a\in[0,1]$*.*

Proof. We call the problem (18) as $(a;s_{\widehat{\gamma}+1},s_{\widehat{\gamma}+2},\ldots,s_{2n};w)$ problem and prove this theorem by mathematical induction on n.

The uniqueness for $n=0$ is trivial.

As an induction hypothesis, we suppose that the $(a;s_{\widehat{\gamma}+1},s_{\widehat{\gamma}+2},\ldots,s_{2n-2};w)$ problem has a unique solution for every $a\in[0,1]$ and for every weight function w, integrable and nonnegative on the interval $[-\pi,\pi)$, vanishing there only on a set of a measure zero.

For $(\xi,\eta)\in[-\pi,\pi]^2$ we define the weight functions

$$p_{\xi,\eta}(x)=\left|\sin\frac{x-\xi}{2}\right|^{2as_{2n-1}+2}\left|\sin\frac{x-\eta}{2}\right|^{2as_{2n}+2}w(x).$$

According to the induction hypothesis, the $(a;s_{\widehat{\gamma}+1},s_{\widehat{\gamma}+2},\ldots,s_{2n-2};p_{\xi,\eta})$ problem has a unique solution $(x_{\widehat{\gamma}+1}(\xi,\eta),x_{\widehat{\gamma}+2}(\xi,\eta),\ldots,x_{2n-2}(\xi,\eta))$ for every $a\in[0,1]$, such that $-\pi<x_{\widehat{\gamma}+1}(\xi,\eta)<\cdots<x_{2n-2}(\xi,\eta)<\pi$, and $x_\nu(\xi,\eta)$ for all $\nu=\widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n-2$ depends continuously on $(\xi,\eta)\in[-\pi,\pi]^2$ (see Lemma 3).

We will prove that the solution of the $(a;s_{\widehat{\gamma}+1},s_{\widehat{\gamma}+2},\ldots,s_{2n};w)$ problem is unique for every $a\in[0,1]$.

Let us denote $x_{2n-1}(\xi,\eta)=\xi$, $x_{2n}(\xi,\eta)=\eta$,

$$W(\mathbf{x}(\xi,\eta),a,x)=\prod_{\nu=\widehat{\gamma}+1}^{2n}\left|\sin\frac{x-x_\nu(\xi,\eta)}{2}\right|^{2as_\nu+2},$$

and

$$\phi_k(\mathbf{x}(\xi,\eta),a) = \int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_{\widehat{\gamma}}+1}\frac{W(\mathbf{x}(\xi,\eta),a,x)}{\sin\dfrac{x-x_k(\xi,\eta)}{2}}\,w(x)\,\mathrm{d}x, \quad k=\widehat{\gamma}+1,\ldots,2n.$$

The induction hypothesis gives us that

$$\phi_k(\mathbf{x}(\xi,\eta),a)=0, \quad k=\widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n-2, \tag{19}$$

for $(\xi,\eta)\in D$, where $D=\{(\xi,\eta): x_{2n-2}(\xi,\eta)<\xi<\eta<\pi\}$.

Let us now consider the following problem in D with unknown $\mathbf{t}=(\xi,\eta)$:

$$\begin{aligned}\varphi_1(\mathbf{t},a) &= \int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_{\widehat{\gamma}}+1}\frac{W(\mathbf{x}(\xi,\eta),a,x)}{\sin\dfrac{x-\xi}{2}}\,w(x)\,\mathrm{d}x=0,\\ \varphi_2(\mathbf{t},a) &= \int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_{\widehat{\gamma}}+1}\frac{W(\mathbf{x}(\xi,\eta),a,x)}{\sin\dfrac{x-\eta}{2}}\,w(x)\,\mathrm{d}x=0.\end{aligned} \tag{20}$$

If $(\xi,\eta)\in D$ is a solution of the problem (20), then, according to (19), $\mathbf{x}(\xi,\eta)$ is a solution of the $(a;s_{\widehat{\gamma}+1},\ldots,s_{2n};w)$ problem in the simplex $S_{2n-\widehat{\gamma}}$. To the contrary, if $\mathbf{x}=(x_{\widehat{\gamma}+1},x_{\widehat{\gamma}+2},\ldots,x_{2n})$ is a solution of the $(a;s_{\widehat{\gamma}+1},\ldots,s_{2n};w)$ problem in the simplex $S_{2n-\widehat{\gamma}}$, then (x_{2n-1},x_{2n}) is a solution of the problem (20). Applying Lemma 1 we conclude that every solution of the problem (20) belongs to $\overline{D}_\varepsilon=\{(\xi,\eta):\varepsilon\le\xi-x_{2n-2}(\xi,\eta),\varepsilon\le\eta-\xi,\varepsilon\le\pi-\eta\}$, for some $\varepsilon>0$.

Let $\mathbf{x}$ be a solution of the problem $(a;s_{\widehat{\gamma}+1},s_{\widehat{\gamma}+2},\ldots,s_{2n};w)$ in $\overline{S}_{\varepsilon,\gamma}$. Then, differentiating φ_1 with respect to the x_k, for all $k=\widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n-2,2n$, we have

$$\begin{aligned}\frac{\partial\varphi_1}{\partial x_k} = -(as_k+1)\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_{\widehat{\gamma}}+1}\prod_{\nu=\widehat{\gamma}+1}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1}\\ \times\operatorname{sgn}\left(\prod_{\nu=\widehat{\gamma}+1}^{2n}\sin\frac{x-x_\nu}{2}\right)\prod_{\substack{\nu=\widehat{\gamma}+1\\ \nu\ne k,\,\nu\ne 2n-1}}^{2n}\sin\frac{x-x_\nu}{2}\cos\frac{x-x_k}{2}\,w(x)\,\mathrm{d}x.\end{aligned}$$

Since

$$\prod_{\substack{\nu=\widehat{\gamma}+1\\ \nu\ne k,\,\nu\ne 2n-1}}^{2n}\sin\frac{x-x_\nu}{2}\cos\frac{x-x_k}{2}\in\mathcal{T}_{n-1}^{\gamma},$$

applying Lemma 2, we conclude that $\frac{\partial\varphi_1}{\partial x_k}=0$, for all $k=\widehat{\gamma}+1,\ldots,2n-2,2n$. Further, applying elementary trigonometric transformations we get the following identity

$$\cos\frac{x-y}{2}=\cos\frac{x}{2}\cos\frac{y}{2}+\sin\frac{x-y}{2}\cos\frac{y}{2}\sin\frac{y}{2}+\cos\frac{x-y}{2}\sin^2\frac{y}{2},$$

and for

$$I := \frac{\partial \varphi_1}{\partial x_{2n-1}} = -\frac{2as_{2n-1}+1}{2}\int_{-\pi}^{\pi}\Big(\cos\frac{x}{2}\Big)^{2as_{\widehat{\gamma}}+1}\prod_{\substack{\nu=\widehat{\gamma}+1\\ \nu\neq 2n-1}}^{2n}\Big|\sin\frac{x-x_\nu}{2}\Big|^{2as_\nu+2}$$
$$\times\Big|\sin\frac{x-x_{2n-1}}{2}\Big|^{2as_{2n-1}}\cos\frac{x-x_{2n-1}}{2}\,w(x)\,\mathrm{d}x,$$

we obtain

$$\cos\frac{x_{2n-1}}{2}I = -\frac{2as_{2n-1}+1}{2}\Big(I_1+\sin\frac{x_{2n-1}}{2}I_2\Big), \tag{21}$$

where

$$I_1 = \int_{-\pi}^{\pi}\Big(\cos\frac{x}{2}\Big)^{2as_{\widehat{\gamma}}+1}\prod_{\substack{\nu=\widehat{\gamma}+1\\ \nu\neq 2n-1}}^{2n}\Big|\sin\frac{x-x_\nu}{2}\Big|^{2as_\nu+2}\Big|\sin\frac{x-x_{2n-1}}{2}\Big|^{2as_{2n-1}}\cos\frac{x}{2}\,w(x)\,\mathrm{d}x,$$

$$I_2 = \int_{-\pi}^{\pi}\Big(\cos\frac{x}{2}\Big)^{2as_{\widehat{\gamma}}+1}\prod_{\substack{\nu=\widehat{\gamma}+1\\ \nu\neq 2n-1}}^{2n}\Big|\sin\frac{x-x_\nu}{2}\Big|^{2as_\nu+2}$$
$$\times\Big|\sin\frac{x-x_{2n-1}}{2}\Big|^{2as_{2n-1}}\sin\frac{x-x_{2n-1}}{2}\,w(x)\,\mathrm{d}x.$$

Obviously, $I_1 > 0$ (the integrand does not change its sign on $[-\pi,\pi)$) and $I_2 = 0$ (because of (20)). Since $-\pi < x_{2n-1} < \pi$, from (21) we get

$$\operatorname{sgn}\Big(\frac{\partial\varphi_1}{\partial x_{2n-1}}\Big) = -1.$$

Analogously,

$$\frac{\partial\varphi_2}{\partial x_k} = 0, \quad k = \widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n-1, \quad \operatorname{sgn}\Big(\frac{\partial\varphi_2}{\partial x_{2n}}\Big) = -1.$$

Thus, at any solution $\mathbf{t} = (\xi,\eta)$ of the problem (20) from $\overline{D}_\varepsilon$ we have

$$\frac{\partial\varphi_1}{\partial\xi} = \sum_{\substack{k=\widehat{\gamma}+1\\ k\neq 2n-1}}^{2n}\frac{\partial\varphi_1}{\partial x_k}\cdot\frac{\partial x_k}{\partial\xi} + \frac{\partial\varphi_1}{\partial x_{2n-1}} = \frac{\partial\varphi_1}{\partial x_{2n-1}}, \quad \frac{\partial\varphi_1}{\partial\eta} = 0,$$
$$\frac{\partial\varphi_2}{\partial\eta} = \sum_{k=\widehat{\gamma}+1}^{2n-1}\frac{\partial\varphi_2}{\partial x_k}\cdot\frac{\partial x_k}{\partial\eta} + \frac{\partial\varphi_2}{\partial x_{2n}} = \frac{\partial\varphi_2}{\partial x_{2n}}, \quad \frac{\partial\varphi_2}{\partial\xi} = 0.$$

Hence, the determinant of the corresponding Jacobian $J(\mathbf{t})$ is positive.

We are going to prove that the problem (20) has a unique solution in $\overline{D}_{\varepsilon/2}$. We prove it by using the topological degree of a mapping. Let us define the mapping $\varphi(\mathbf{t},a) : \overline{D}_{\varepsilon/2}\times[0,1] \to \mathbb{R}^2$, by $\varphi(\mathbf{t},a) = (\varphi_1(\mathbf{t},a),\varphi_2(\mathbf{t},a))$, $\mathbf{t}\in\mathbb{R}^2$, $a\in[0,1]$. If $\mathbf{x}(\xi,\eta)$ is a solution of the $(a;s_{\widehat{\gamma}+1},\ldots,s_{2n};w)$ problem in $\overline{S}_{\varepsilon,\gamma}$, then the solutions of the problem $\varphi(\mathbf{t},a) = (0,0)$ in $\overline{D}_{\varepsilon/2}$ belong to $\overline{D}_\varepsilon$. The problem $\varphi(\mathbf{t},a) = (0,0)$ has the unique solution in $\overline{D}_{\varepsilon/2}$ for $a=0$. It is obvious that $\varphi(\,\cdot\,,0)$ is differentiable on $D_{\varepsilon/2}$, the mapping $\varphi(\mathbf{t},a)$ is continuous in $\overline{D}_{\varepsilon/2}\times[0,1]$, and $\varphi(\mathbf{t},a)\neq(0,0)$ for all $\mathbf{t}\in\partial D_{\varepsilon/2}$ and $a\in[0,1]$. Then $\deg(\varphi(\,\cdot\,,a),D_{\varepsilon/2},(0,0))=\operatorname{sgn}(\det(J(\mathbf{t})))=1$, for all $a\in[0,1]$, which means that $\deg(\varphi(\,\cdot\,,a),D_{\varepsilon/2},(0,0))$ is a constant independent of a. Therefore, the problem $\varphi(\mathbf{t},a)=(0,0)$ has the unique solution in $D_{\varepsilon/2}$ for all $a\in[0,1]$. Hence, the $(a;s_{\widehat{\gamma}+1},\ldots,s_{2n};w)$ problem has a unique solution in $S_{2n-\widehat{\gamma}}$ which belongs to $\overline{S}_{\varepsilon,\gamma}$. □

Theorem 6. *The solution* $\mathbf{x}=\mathbf{x}(a)$ *of the problem* (18) *depends continuously on* $a\in[0,1]$.

Proof. Let $\{a_m\}$, $a_m\in[0,1]$, $m\in\mathbb{N}$, be a convergent sequence, which converges to $a^*\in[0,1]$. Then for every a_m, $m\in\mathbb{N}$, there exists the unique solution $\mathbf{x}(a_m)=(x_{\widehat{\gamma}+1}(a_m),\ldots,x_{2n}(a_m))$ of the system (18) with $a=a_m$. The unique solution of system (18) for $a=a^*$ is $\mathbf{x}(a^*)=(x_{\widehat{\gamma}+1}(a^*),\ldots,x_{2n}(a^*))$. Let $\mathbf{x}^*=(x^*_{\widehat{\gamma}+1},\ldots,x^*_{2n})$ be an arbitrary limit point of the sequence $\mathbf{x}(a_m)$ when $a_m\to a^*$. According to Theorem 5, for each $m\in\mathbb{N}$ we have

$$\int_{-\pi}^{\pi}\Bigl(\cos\frac{x}{2}\Bigr)^{2a_m s_{\widehat{\gamma}}+1}\frac{W(\mathbf{x}(a_m),a_m,x)}{\sin\dfrac{x-x_k(a_m)}{2}}\,w(x)\,\mathrm{d}x=0,\quad k=\widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n.$$

When $a_m\to a^*$ the above equations lead to

$$\int_{-\pi}^{\pi}\Bigl(\cos\frac{x}{2}\Bigr)^{2a^* s_{\widehat{\gamma}}+1}\frac{W(\mathbf{x}^*,a^*,x)}{\sin\dfrac{x-x_k^*}{2}}\,w(x)\,\mathrm{d}x=0,\quad k=\widehat{\gamma}+1,\widehat{\gamma}+2,\ldots,2n.$$

According to Theorem 5 we get that $\mathbf{x}^*=\mathbf{x}(a^*)$, i.e., $\lim\limits_{a_m\to a^*}\mathbf{x}(a_m)=\mathbf{x}(a^*)$. □

4 Numerical method for construction of quadrature rules with multiple nodes and the maximal trigonometric degree of exactness

Based on the previously given theoretical results we present the numerical method for construction of quadrature rules of the form (11), for which $R_n(f)=0$ for all $f\in\mathcal{T}_{N_1}$, where $N_1=\sum_{\nu=\widehat{\gamma}}^{2n}(s_\nu+1)-1$. The first step is a construction of nodes, and the second one is a construction of weights knowing nodes.

4.1 Construction of nodes

We chose $x_{\widehat{\gamma}} = -\pi$ and obtain the nodes $x_{\widehat{\gamma}+1}, x_{\widehat{\gamma}+2}, \cdots, x_{2n}$ by solving system of nonlinear equations (18) for $a = 1$, applying Newton–Kantorovič method. Obtained theoretical results suggest that for fixed n the system (18) can be solved progressively, for an increasing sequence of values for a, up to the target value $a = 1$. The solution for some $a^{(i)}$ can be used as the initial iteration in Newton–Kantorovič method for calculating the solution for $a^{(i+1)}$, $a^{(i)} < a^{(i+1)} \leq 1$. If for some chosen $a^{(i+1)}$ Newton–Kantorovič method does not converge, we decrease $a^{(i+1)}$ such that it becomes convergent, which is always possible according to Theorem 6. Thus, we can set $a^{(i+1)} = 1$ in each step, and, if that iterative process is not convergent, we set $a^{(i+1)} := (a^{(i+1)} + a^{(i)})/2$ until it becomes convergent. As the initial iteration for the first iterative process, for some $a^{(1)} > 0$, we choose the zeros of the corresponding orthogonal trigonometric polynomial of degree $n+\gamma$, i.e., the solution of system (18) for $a = 0$.

Let us introduce the following matrix notation

$$
\begin{aligned}
\mathbf{x} &= \left[x_{\widehat{\gamma}+1}\ x_{\widehat{\gamma}+2}\ \cdots\ x_{2n}\right]^T, \\
\mathbf{x}^{(m)} &= \left[x^{(m)}_{\widehat{\gamma}+1}\ x^{(m)}_{\widehat{\gamma}+2}\ \cdots\ x^{(m)}_{2n}\right]^T, \quad m = 0, 1, \ldots, \\
\phi(\mathbf{x}) &= \left[\phi_{\widehat{\gamma}+1}(\mathbf{x})\ \phi_{\widehat{\gamma}+2}(\mathbf{x})\ \cdots\ \phi_{2n}(\mathbf{x})\right]^T.
\end{aligned}
$$

Jacobian of $\phi(\mathbf{x})$,

$$
\mathbf{W} = \mathbf{W}(\mathbf{x}) = [w_{i,j}]_{(2n-\widehat{\gamma})\times(2n-\widehat{\gamma})} = \left[\frac{\partial \phi_{i+\widehat{\gamma}}}{\partial x_{j+\widehat{\gamma}}}\right]_{(2n-\widehat{\gamma})\times(2n-\widehat{\gamma})},
$$

has the following entries

$$
\begin{aligned}
\frac{\partial \phi_i}{\partial x_j} = -(1+as_j)\int_{-\pi}^{\pi} &\left(\cos\frac{x}{2}\right)^{2as_{\widehat{\gamma}}+1} \prod_{\nu=\widehat{\gamma}+1}^{2n} \left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1} \operatorname{sgn}\left(\prod_{\nu=\widehat{\gamma}+1}^{2n} \sin\frac{x-x_\nu}{2}\right) \\
&\times \prod_{\substack{\nu=\widehat{\gamma}+1 \\ \nu\neq i,\nu\neq j}}^{2n} \sin\frac{x-x_\nu}{2} \cos\frac{x-x_j}{2}\, w(x)\,\mathrm{d}x, \quad i \neq j,\ i,j = \widehat{\gamma}+1, \widehat{\gamma}+2, \ldots, 2n, \\
\frac{\partial \phi_i}{\partial x_i} = -\frac{1+2as_i}{2}\int_{-\pi}^{\pi} &\left(\cos\frac{x}{2}\right)^{2as_{\widehat{\gamma}}+1} \prod_{\substack{\nu=\widehat{\gamma}+1 \\ \nu\neq i}}^{2n} \left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+2} \\
&\times \left|\sin\frac{x-x_i}{2}\right|^{2as_i} \cos\frac{x-x_i}{2}\, w(x)\,\mathrm{d}x, \quad i = \widehat{\gamma}+1, \widehat{\gamma}+2, \ldots, 2n.
\end{aligned}
$$

All of the above integrals can be computed by using a Gaussian type quadrature rule for trigonometric polynomials (see [28] and [47])

$$\int_{-\pi}^{\pi} f(x)\,w(x)\,\mathrm{d}x = \sum_{\nu=\widehat{\gamma}}^{2N} A_\nu f(x_\nu) + R_N(f),$$

with $2N \geq \sum_{\nu=\widehat{\gamma}}^{2n} s_\nu + 2n$.

The Newton–Kantorovič method for calculating the zeros of the σ–orthogonal trigonometric polynomial $T_{\sigma,n}^{\gamma}$ is given as follows

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} - \mathbf{W}^{-1}(\mathbf{x}^{(m)})\phi(\mathbf{x}^{(m)}), \quad m = 0, 1, \ldots,$$

and for sufficiently good chosen initial approximation $\mathbf{x}^{(0)}$, it has the quadratic convergence.

4.2 Construction of weights

Knowing the nodes of quadrature rule (11), it is possible to calculate the corresponding weights. The weights can be calculating by using the Hermite trigonometric interpolation polynomial (see [11, 12]). Since the construction of the Hermite interpolation trigonometric polynomial is more difficult than in the algebraic case, we use an adaptation on the method which was first given in [17] for construction of Gauss–Turán quadrature rules (for algebraic polynomials) and then generalized for Chakalov–Popoviciu's type quadrature rules (also for algebraic polynomials) in [30, 31]. For $\gamma = 1/2$ and $\gamma = 0$ the corresponding method for construction of weights of quadrature rule (11) is presented in [29] and [47], respectively.

Our method is based on the facts that quadrature rule (11) is of interpolatory type and that it is exact for all trigonometric polynomials of degree less than or equal to $\sum_{\nu=\widehat{\gamma}}^{2n}(s_\nu + 1) - 1 = \sum_{\nu=\widehat{\gamma}}^{2n} s_\nu + 2(n+\gamma) - 1$. Thus, the weights can be calculated requiring that quadrature rule (11) integrates exactly all trigonometric polynomials of degree less than or equal to $\sum_{\nu=\widehat{\gamma}}^{2n} s_\nu + (n+\gamma) - 1$. Additionally, in the case $\gamma = 0$ (i.e., $\widehat{\gamma} = 1$) we require that (11) integrates exactly $\cos\big(\sum_{\nu=1}^{2n} s_\nu + n\big)x$, when $\sum_{\nu=1}^{2n}(2s_\nu + 1)x_\nu = \ell\pi$ for an odd integer ℓ, or $\sin\big(\sum_{\nu=1}^{2n} s_\nu + n\big)x$ when $\sum_{\nu=1}^{2n}(2s_\nu + 1)x_\nu = \ell\pi$ for an even integer ℓ, while if $\sum_{\nu=1}^{2n}(2s_\nu + 1)x_\nu \neq \ell\pi$, $\ell \in \mathbb{Z}$, one can choose to require exactness for $\cos\big(\sum_{\nu=1}^{2n} s_\nu + n\big)x$ or $\sin\big(\sum_{\nu=1}^{2n} s_\nu + n\big)x$ arbitrary (see [1, 2, 47]). In such a way we obtain a system of linear equations for the unknown weights. That system can be solved by decomposing into a set of $2(n+\gamma)$ upper triangular systems. For $\gamma = 0$ we explain that decomposing method in the case when $\sum_{\nu=1}^{2n}(2s_\nu + 1)x_\nu = \ell\pi$, $\ell \in \mathbb{Z}$.

Let us denote

$$\Omega_\nu(x) = \prod_{\substack{i=\widehat{\gamma}\\ i\neq\nu}}^{2n} \left(\sin\frac{x-x_i}{2}\right)^{2s_i+1}, \ \nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n,$$

$$u_{k,\nu}(x) = \left(\sin\frac{x-x_\nu}{2}\right)^{k} \Omega_\nu(x), \quad \nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n, \ k = 0,1,\ldots,2s_\nu,$$

and

$$t_{k,\nu}(x) = \begin{cases} u_{k,\nu}(x)\cos\dfrac{x-x_\nu}{2}, & k\text{−even}, \ \gamma=0, \text{ or } k\text{−odd}, \ \gamma=1/2, \\ u_{k,\nu}(x), & k\text{−odd}, \ \gamma=0, \text{ or } k\text{−even}, \ \gamma=1/2, \end{cases} \tag{22}$$

for all $\nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n$, $k = 0,1,\ldots,2s_\nu$. It is easy to see that $t_{k,\nu}$ is a trigonometric polynomial of degree less than or equal to $\sum_{\nu=\widehat{\gamma}}^{2n} s_\nu + n$, which for $\gamma = 0$ has the leading term $\cos\big(\sum_{\nu=1}^{2n} s_\nu + n\big)x$ or $\sin\big(\sum_{\nu=1}^{2n} s_\nu + n\big)x$, but not the both of them. Quadrature rule (11) is exact for all such trigonometric polynomials, i.e., $R_n(t_{k,\nu}) = 0$ for all $\nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n$, $k = 0,1,\ldots,2s_\nu$.

By using the Leibniz differentiation formula it is easy to check that $t_{k,\nu}^{(j)}(x_i) = 0$, $j = 0,1,\ldots,2s_\nu$, for all $i \neq \nu$. Hence, for all $k = 0,1,\ldots,2s_\nu$ we have

$$\mu_{k,\nu} = \int_{-\pi}^{\pi} t_{k,\nu}(x)\,w(x)\,\mathrm{d}x = \sum_{j=0}^{2s_\nu} A_{j,\nu} t_{k,\nu}^{(j)}(x_\nu), \quad \nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n. \tag{23}$$

In such a way, we obtain $2(n+\gamma)$ independent systems for calculating the weights $A_{j,\nu}$, $j = 0,1,\ldots,2s_\nu$, $\nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n$. Here, we need to calculate the derivatives $t_{k,\nu}^{(j)}(x_\nu)$, $k = 0,1,\ldots,2s_\nu$, $j = 0,1,\ldots,2s_\nu$, for each $\nu = \widehat{\gamma}, \widehat{\gamma}+1, \ldots, 2n$. For that purpose we use the following result (see [29] for details).

Lemma 4. *For the trigonometric polynomials $t_{k,\nu}$, given by* (22) *we have*

$$t_{k,\nu}^{(i)}(x_\nu) = 0, \quad i < k; \quad t_{k,\nu}^{(k)}(x_\nu) = \frac{k!}{2^k}\Omega_\nu(x_\nu),$$

and for $i > k$

$$t_{k,\nu}^{(i)}(x_\nu) = \begin{cases} v_{k,\nu}^{(i)}(x_\nu), \ k\text{−even}, \ \gamma=0, \text{ or } k\text{−odd}, \ \gamma=1/2, \\ u_{k,\nu}^{(i)}(x_\nu), \ k\text{−odd}, \ \gamma=0, \text{ or } k\text{−even}, \ \gamma=1/2, \end{cases}$$

where

$$v_{k,\nu}^{(i)}(x_\nu) = \sum_{m=0}^{[i/2]} \binom{i}{2m} \frac{(-1)^m}{2^{2m}} u_{k,\nu}^{(i-2m)}(x_\nu),$$

and the sequence $u_{k,\nu}^{(i)}(x_\nu)$, $k \in \mathbb{N}_0$, $i \in \mathbb{N}_0$, is the solution of the difference equation

$$f_{k,\nu}^{(i)}=\sum_{m=[(i-k)/2]}^{[(i-1)/2]}\binom{i}{2m+1}\frac{(-1)^m}{2^{2m+1}}f_{k-1,\nu}^{(i-2m-1)}(x_\nu),\quad \nu=\widehat{\gamma},\widehat{\gamma}+1,\ldots,2n,\ k\in\mathbb{N},\ i\in\mathbb{N}_0,$$

with the initial conditions $f_{0,\nu}^{(i)}=\Omega_\nu^{(i)}(x_\nu)$.

Finally, the remaining problem of calculating $\Omega_\nu^{(i)}(x_\nu)$, $\nu=\widehat{\gamma},\widehat{\gamma}+1,\ldots,2n$, is solved in [29, Lemma 3.3].

Lemma 5. *Let us denote*

$$\Omega_{\nu,k}(x)=\prod_{\substack{i\le k\\ i\neq\nu}}\left(\sin\frac{x-x_i}{2}\right)^{2s_i+1}.$$

Then $\Omega_{\nu,\nu-1}(x)=\Omega_{\nu,\nu}(x)$ *and* $\Omega_\nu(x)=\Omega_{\nu,2n}(x)$. *The sequence* $\Omega_{\nu,k,\ell}^{(i)}(x)$, $\ell,i\in\mathbb{N}_0$, *where*

$$\Omega_{\nu,k,\ell}(x)=\left(\sin\frac{x-x_{k+1}}{2}\right)^{\ell}\Omega_{\nu,k}(x),\quad k=\widehat{\gamma},\widehat{\gamma}+1,\ldots,2n-1,\ k\neq\nu-1,$$

is the solution of the following difference equation

$$f_{\nu,k,\ell}^{(i)}(x)=\sum_{m=0}^{i}\binom{i}{m}\frac{1}{2^m}\sin\left(\frac{x-x_{k+1}}{2}+\frac{m\pi}{2}\right)f_{\nu,k,\ell-1}^{(i-m)}(x),\quad \ell\in\mathbb{N}_0,$$

with the initial conditions $f_{\nu,k,0}^{(i)}=\Omega_{\nu,k}^{(i)}$, $i\in\mathbb{N}_0$. *The following equalities*

$$\begin{aligned}\Omega_{\nu,k,2s_{k+1}+1}^{(i)}(x)&=\Omega_{\nu,k+1,0}^{(i)}(x),\quad k\neq\nu-1,\\ \Omega_\nu^{(i)}(x)&=\Omega_{\nu,2n}^{(i)}(x)=\Omega_{\nu,2n-1,2s_{2n}+1}^{(i)}(x),\quad \nu\neq 2n,\\ \Omega_{2n}^{(i)}(x)&=\Omega_{2n,2n-1}^{(i)}(x)=\Omega_{2n,2n-2,2s_{2n-1}+1}^{(i)}(x)\end{aligned}$$

hold.

According to Lemma 4, the systems (23) are the following upper triangular systems

$$\begin{bmatrix} t_{0,\nu}(x_\nu) & t'_{0,\nu}(x_\nu) & \cdots & t_{0,\nu}^{(2s_\nu)}(x_\nu)\\ & t'_{1,\nu}(x_\nu) & \cdots & t_{1,\nu}^{(2s_\nu)}(x_\nu)\\ & & \ddots & \vdots\\ & & & t_{2s_\nu,\nu}^{(2s_\nu)}(x_\nu)\end{bmatrix}\cdot\begin{bmatrix}A_{0,\nu}\\ A_{1,\nu}\\ \vdots\\ A_{2s_\nu,\nu}\end{bmatrix}=\begin{bmatrix}\mu_{0,\nu}\\ \mu_{1,\nu}\\ \vdots\\ \mu_{2s_\nu,\nu}\end{bmatrix},\quad \nu=\widehat{\gamma},\widehat{\gamma}+1,\ldots,2n.$$

Remark 2. For the case $\gamma = 0$, if $\sum_{\nu=1}^{2n}(2s_\nu + 1)x_\nu \neq \ell\pi$, $\ell \in \mathbb{Z}$, then we choose $t_{k,\nu}$, $k = 0,1,\ldots,2s_\nu$, $\nu = 1,2,\ldots,2n$, in one of the following ways

$$t_{k,\nu}(x) = u_{k,\nu}(x)\cos\frac{x + \sum\limits_{\substack{i=1\\ i\neq\nu}}^{2n}(2s_i+1)x_i + 2s_\nu x_\nu}{2}$$

or

$$t_{k,\nu}(x) = u_{k,\nu}(x)\sin\frac{x + \sum\limits_{\substack{i=1\\ i\neq\nu}}^{2n}(2s_i+1)x_i + 2s_\nu x_\nu}{2},$$

which provides that

$$t_{k,\nu} \in \mathcal{T}_{\sum_{\nu=1}^{2n} s_\nu + n} \ominus \operatorname{span}\left\{\cos\left(\sum_{\nu=1}^{2n} s_\nu + n\right)x\right\}$$

or

$$t_{k,\nu} \in \mathcal{T}_{\sum_{\nu=1}^{2n} s_\nu + n} \ominus \operatorname{span}\left\{\sin\left(\sum_{\nu=1}^{2n} s_\nu + n\right)x\right\},$$

$k = 0,1,\ldots,2s_\nu$, $\nu = 1,2,\ldots,2n$, respectively.

4.3 Numerical example

In this section we present one numerical example as illustration of the obtained theoretical results. Numerical results are obtained using our proposed progressive approach. For all computations we use MATHEMATICA and the software package *OrthogonalPolynomials* explained in [8] in double precision arithmetic.

Example 1. We construct quadrature rule (11) for $\gamma = 0$, $n = 3$, $\sigma = (3,3,3,4,4,4)$, with respect to the weight function

$$w(x) = 1 + \cos 2x, \quad x \in [-\pi,\pi).$$

For calculation of nodes we use proposed progressive approach with only two steps, i.e., two iterative processes: for $a = 1/2$ (with 9 iterations) and for $a = 1$ (with 8 iterations). The nodes x_ν, $\nu = 1,2,\ldots,6$, are the following:

$$\begin{array}{lll} -3.141592653589793, & -2.264556388673865, & -1.179320242581565, \\ -0.1955027724705077, & 0.8612188670819011, & 2.178685249095223. \end{array}$$

The corresponding weight coefficients $A_{j,\nu}$, $j = 0,1,\ldots,2s_\nu$, $\nu = 1,2,\ldots,6$, are given in Table 1 (numbers in parentheses denote decimal exponents).

j	$A_{j,1}$	$A_{j,2}$	$A_{j,3}$
0	1.676594372192496	7.305332409592605(−1)	3.487838013502532(−1)
1	−3.606295932640399(−3)	−8.705060748567067(−2)	6.802235520408829(−2)
2	4.450266475268382(−2)	1.929684165460602(−2)	1.120920249826093(−2)
3	−6.813768854008439(−2)	−9.825070962211830(−4)	8.081866652580276(−4)
4	2.575834718221877(−4)	1.050037121874394(−4)	6.594073161399015(−5)
5	−2.025745530496365(−7)	−2.102329131299247(−6)	1.808851807084914(−6)
6	3.780621226622374(−7)	1.413143818423462(−7)	9.154154335491590(−8)
j	$A_{j,4}$	$A_{j,5}$	$A_{j,6}$
0	1.880646586862865	9.170128528217915(−1)	7.296144529929201(−1)
1	6.328640684710125(−2)	−1.557489854117211(−1)	1.399304206896031(−1)
2	7.316493158593520(−2)	3.683058320865013(−2)	2.916174720939193(−2)
3	1.252054136886738(−3)	−2.974661584717919(−3)	2.588432069654582(−3)
4	7.029326017474412(−4)	3.381006758789474(−4)	2.606855435340879(−4)
5	6.219410600844099(−6)	−1.428080829445919(−5)	1.209621855269665(−5)
6	2.280678345497009(−6)	1.008832884300956(−6)	7.540115736887533(−7)
7	8.397304312672816(−9)	−1.865855244589502(−8)	1.546523556082379(−8)
8	2.276058254314503(−9)	8.981189180630658(−10)	6.501031023479647(−10)

Table 1 Weight coefficients $A_{j,\nu}$, $j = 0,1,\ldots,2s_\nu$, $\nu = 1,2,\ldots,2n$, for $w(x) = 1+\cos 2x$, $x \in [-\pi,\pi)$, $n = 3$ and $\sigma = (3,3,3,4,4,4)$.

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References

1. Cruz–Barroso, R., Darius, L., Gonzáles–Vera, P., Njåstad, O.: Quadrature rules for periodic integrands. Bi–orthogonality and para–orthogonality, Ann. Math. et Informancae. **32** (2005), 5–44.
2. Cruz–Barroso, R., Gonzáles–Vera, P., Njåstad, O.: On bi–orthogonal systems of trigonometric functions and quadrature formulas for periodic integrands, Numer. Algor. **44** (4) (2007), 309–333.
3. Bojanov, B.D.: Oscillating polynomials of least L_1-norm, In: G. Hämmerlin, ed., Numerical Integration, ISNM 57, Birkhäuser, Basel (1982), 25–33.
4. Chakalov, L.: Über eine allgemeine Quadraturformel, C.R. Acad. Bulgar. Sci. **1** (1948), 9–12.
5. Chakalov, L.: General quadrature formulae of Gaussian type, Bulgar. Akad. Nauk Izv. Mat. Inst. **1** (1954), 67–84 (Bulgarian) [English transl. East J. Approx. **1** (1995), 261–276].
6. Chakalov, L.: Formules générales de quadrature mécanique du type de Gauss, Colloq. Math. **5** (1957), 69–73.
7. Christoffel, E.B.: Über die Gaußische Quadratur und eine Verallgemeinerung derselben. J. Reine Angew. Math. **55** (1858), 61–82. [Also in Ges. Math. Abhandlungen I, pp. 42–50.]
8. Cvetković, A.S., Milovanović, G.V.: The Mathematica Package “OrthogonalPolynomials”, Facta Univ. Ser. Math. Inform. **19** (2004), 17–36.
9. Cvetković, A.S., Stanić, M.P.: Trigonometric orthogonal systems, In: Approximation and Computation – In Honor of Gradimir V. Milovanović, Series: Springer Optimization and Its Applications, Vol. 42 (W. Gautschi, G. Mastroianni, Th.M. Rassias eds.), pp. 103–116, Springer–Verlag, Berlin–Heidelberg–New York, 2011.
10. DeVore, R.A., Lorentz, G.G.: Constructive Approximation, Springer–Verlag, Berlin Heildeberg, 1993.

11. Dryanov, D.P.: Quadrature formulae with free nodes for periodic functions, Numer. Math. **67** (1994), 441–464.
12. Du, J., Han, H., Jin, G.: On trigonometric and paratrigonometric Hermite interpolation, J. Approx. Theory **131** (2004), 74–99.
13. Engels, H.: Numerical Quadrature and Qubature, Academic Press, London, 1980.
14. Gauss, C.F.: Methodus nova integralium valores per approximationem inveniendi, Commentationes Societatis Regiae Scientarium Göttingensis Recentiores **3** (1814) [Also in Werke III, pp. 163–196.]
15. Gautschi, W.: A survey of Gauss–Christoffel quadrature formulae, in: P.L. Butzer, F. Fehér (Eds.), E.B. Christoffel – The Influence of his Work on Mathematics and the Physical Sciences, Birkhäuser, Basel, 1981, pp. 72–147.
16. Gautschi, W.: Orthogonal polynomials: applications and computation, Acta Numerica (1996), 45–119.
17. Gautschi, W., Milovanović, G.V.: *S*–orthogonality and construction of Gauss–Turán-type quadrature formulae, J. Comput. Appl. Math. **86** (1997), 205–218.
18. Ghizzetti, A., Ossicini, A.: Sull' esistenza e unicità delle formule di quadratura gaussiane, Rend. Mat. (6) **8** (1975), 1–15.
19. Ghizzeti, A., Ossicini, A.: Quadrature Formulae, Academie–Verlag, Berlin, 1970.
20. Golub, G.H., Kautsky, J.: Calculation of Gauss quadratures with multiple free and fixed knots, Numer. Math. **41** (1983), 147–163.
21. Jacobi, C.G.J.: Ueber Gaußs neue Methode, die Werthe der Integrale näherungsweise zu finden. J. Reine Angew. Math. **1** (1826), 301–308.
22. Micchelli, C.A.: The fundamental theorem of algebra for monosplines with multiplicities, in: P. Butzer, J.P. Kahane, B.Sz. Nagy (Eds.), Linear Operators and Approximation, ISNM vol. 20, Birkhäuser, Basel, 1972, pp. 372–379.
23. Milovanović, G.V.: Construction of *s*–orthogonal polynomials and Turán quadrature formulae, in: G.V. Milovanović (Ed.), Numerical Methods and Approximation Theory III (Niš, 1987), Univ. Niš, Niš, 1988, pp. 311–328.
24. Milovanović, G.V.: *S*–orthogonality and generalized Turán quadratures: construction and applications, in: D.D. Stancu, Ch. Coman, W.W. Breckner, P. Blaga (Eds.), Approximation and Optimization, vol. I (Cluj-Napoca, 1996), Transilvania Press, Cluj-Napoca, Romania, 1997, pp. 91–106.
25. Milovanović, G.V.: Quadrature with multiple nodes, power orthogonality, and moment–preserving spline approximation, Numerical analysis 2000, Vol. V, Quadrature and orthogonal polynomials (W. Gautschi, F. Marcellan, and L. Reichel, eds.), J. Comput. Appl. Math. **127** (2001), 267–286.
26. Milovanović, G.V.: Numerical quadratures and orthogonal polynomials, Stud. Univ. Babeş-Bolyai Math. **56** (2011), 449–464.
27. Milovanović, G.V., Cvetković, A.S., Stanić, M.P.: Trigonometric orthogonal systems and quadrature formulae with maximal trigonometric degree of exactness, NMA 2006, Lecture Notes in Comput. Sci. **4310** (2007), 402–409.
28. Milovanović, G.V., Cvetković, A.S., Stanić, M.P.: Trigonometric orthogonal systems and quadrature formulae, Comput. Math. Appl. **56** (11) (2008), 2915–2931.
29. Milovanović, G.V., Cvetković, A.S., Stanić, M.P.: Quadrature formulae with multiple nodes and a maximal trigonometric degree of axactness, Numer. Math. **112** (2009), 425–448.
30. Milovanović, G.V., Spalević, M.M.: Construction of Chakalov–Popoviciu's type quadrature formulae, Rend. Circ. Mat. Palermo, Serie II, Suppl. **52** (1998), 625–636.
31. Milovanović, G.V., Spalević, M.M., Cvetković, A.S.: Calculation of Gaussian type quadratures with multiple nodes, Math. Comput. Modelling **39** (2004), 325–347.
32. Mirković, B.: Theory of Measures and Integrals, Naučna knjiga, Beograd, 1990 (in Serbian).
33. Morelli, A., Verna, I.: Formula di quadratura in cui compaiono i valori della funzione e delle derivate con ordine massimo variabile da nodo a nodo, Rend. Circ. Mat. Palermo (2) **18** (1969), 91–98.
34. Ortega, J.M., Rheinboldt, W.C.: Iterative solution of nonlinear equations in several variables, In: Classics in Applied Mathematics, vol. 30. SIAM, Philadelphia, 2000.

35. Ossicini, A: Costruzione di formule di quadratura di tipo Gaussiano, Ann. Mat. Pura Appl. (4) **72** (1966), 213–237.
36. Ossicini, A., Rosati, F.: Funzioni caratteristiche nelle formule di quadratura gaussiane con nodi multipli, Boll. Un. Mat. Ital. (4) **11** (1975), 224–237.
37. Ossicini, A., Rosati, F.: Sulla convergenza dei funzionali ipergaussiani, Rend. Mat. (6) **11** (1978), 97–108.
38. Popoviciu, T. : Sur une généralisation de la formule d'intégration numérique de Gauss, Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. **6** (1955), 29–57 (Romanian).
39. Schwartz, J.T.: Nonlinear Functional Analysis, Gordon and Breach, New York, 1969.
40. Shi, Y.G.: A kind of extremal problem of integration on an arbitrary measure, Acta Sci. Math. (Szeged) **65** (1999), 567–575.
41. Shi, Y.G.: Power Orthogonal Polynomials, Nova Science Publishers, Inc., New York, 2006.
42. Shi, Y.G., Xu, G.: Construction of $\sigma-$orthogonal polynomials and Gaussian quadrature formulas, Adv. Comput. Math. **27** (1) (2007), 79–94.
43. Stancu, D.D.: On a class of orthogonal polynomials and on some general quadrature formulas with minimum number of terms, Bull. Math. Soc. Sci. Math. Phys. R. P. Romîne (N.S) **1** (49) (1957), 479–498.
44. Stancu, D.D.: On certain general numerical integration formulas, Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 209–216 (Romanian).
45. Stancu, D.D.: Sur quelques formules générales de quadrature du type Gauss-Christoffel, Mathematica (Cluj) **1 (24)** (1959) 167–182.
46. Stroud, A.H., Stancu, D.D.: Quadrature formulas with multiple Gaussian nodes, J. SIAM Numer. Anal. Ser. B **2** (1965), 129–143.
47. Tomović, T.V., Stanić, M.P.: Quadrature rules with an even number of multiple nodes and a maximal trigonometric degree of exactness, FILOMAT (to appear)
48. Turán, P.: On the theory of the mechanical quadrature, Acta Sci. Math. Szeged **12** (1950), 30–37.
49. Turetzkii, A.H.: On quadrature rule that are exact for trigonometric polynomials, East J. Approx. 11 (2005) 337–359 (translation in English from Uchenye Zapiski, Vypusk 1(149), Seria Math. Theory of Functions, Collection of papers, Izdatel'stvo Belgosuniversiteta imeni V.I. Lenina, Minsk (1959) 31–54).