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# ON BIRKHOFF (0,3) AND (0,4) QUADRATURE FORMULAE

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ABSTRACT. Quadrature formulas of Birkhoff type ((0,3) and (0,4) cases) which are exact for all algebraic polynomials of degree at most $2n-1$ is developed. The nodes of such quadrature are taken in the zeros of the polynomial $(1-x^2)P_n''(x)$, where $P_n$ is the Legendre polynomial of degree $n$. The corresponding quadrature formulas based on the zeros of (1.2) still remain open.

**1. Introduction.** Let $Q_n$ be the class of algebraic polynomials of degree at most $n$. The following problem was raised by P. Turán [6]. Let $x_1, x_2 \ldots x_n$ satisfying

$$-1 = x_n < x_{n-1} < \cdots < x_2 < x_1 = 1 \tag{1.1}$$

be the zeros of the polynomial

$$\Pi_n(x) = (1-x^2)P_{n-1}'(x) \tag{1.2}$$

where $P_k$ is the Legendre polynomial of degree $k$. For what choice of $\lambda_k, \mu_k$ $(k = 1, 2, \ldots, n)$ do we have

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} \lambda_k f(x_k) + \sum_{k=1}^{n} \mu_k f''(x_k) \tag{1.3}$$

for every $f \in Q_{2n-1}$?

Using Balázs and Turán [1] fundamental polynomials of (0,2) interpolation based on the nodes of (1.1) one of us [8] solved this problem. Later he [8] gave a simple method for finding the weights $\lambda_k, \mu_k$ $(k = 1, 2, \ldots, n)$. A natural extension of Turáns problem is to obtain quadrature formulas in the case when function values and $m^{th}$ derivative are prescribed at the zeros of $\pi_n(x)$ defined by (1.2). This problem appears to be difficult. Recently, only some modification of $(0,3)$ and $(0,4)$ Birkhoff quadrature formulas were considered by Varma and Saxena [9]. The method used in [9] does not provide the solution of the problem in the case when only function values and third derivative are prescribed at the zeros of (1.1). At the Banach center conference on Approximation theory held in 1992 one of us raised whether it is possible to choose the nodes $1, -1$ and different $x_k$'s, $k = 2, 3 \ldots n - 1$ for which ($m$ being positive integer)

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} f(x_i)\lambda_i + \sum_{i=1}^{n} f^{(m)}(x_i)\mu_i \tag{1.4}$$

is valid for $f \in Q_{2n-1}$.

In this paper an affirmative answer to this problem is obtained for the case $m = 3$, $m = 4$. Theorem 1 will reveal that by choosing a special choice of nodes we are able to construct such a quadrature formula. For other related results we refer to the work of Nevai and Varma [4] and Dimitrov [2]. Concerning Birkhoff interpolation and quadrature we refer to the book [3].

**2. Main Results.** Let $\left\{P_k^{(\alpha,\beta)}(x)\right\}_{k=0}^{\infty}$ be the set of Jacobi polynomials orthogonal on the interval $(-1,1)$ with respect to the weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1$. Let us fix the nodes $x_i$'s as the zeros of the polynomial

$$r_n(x) = (1-x^2)P_n''(x) = \frac{1}{4}(n+1)(n+2)(1-x^2)P_{n-2}^{(2,2)}(x). \tag{2.1}$$

**Theorem 1.** *Let $n \geq 3$ and let $x_k$ be the zeros of the polynomial (2.1). Then there exists a unique quadrature formula*

$$\int_{-1}^{1} f(x)dx = \frac{16}{(n+3)(n^2+n+4)}\left[\frac{n+3}{3}(f(1)+f(-1))\right.$$
$$+\frac{n^2}{2(n-2)}\sum_{k=2}^{n-1}\frac{f(x_k)}{(1-x_k^2)\left(P_{n-1}^{(1,1)}(x_k)\right)^2}-\frac{4(f'''(1)-f'''(-1))}{n(n^2-1)(n^2-4)} \tag{2.2}$$
$$\left.-\frac{2n}{(n^2-1)(n^2-4)}\sum_{k=2}^{n-1}\frac{x_k f'''(x_k)}{\left(P_{n-1}^{(1,1)}(x_k)\right)^2}\right]$$

*which is exact for every polynomial* $f \in Q_{2n-1}$.

**Theorem 2.** *Let* $n \geq 4$ *and let* $x_k$*'s be the zeros of the polynomial* $P_{n-2}^{(2,2)}(x)$. *Then there exists a unique quadrature formula*

$$\int_{-1}^{1} f(x)dx = \frac{1}{(n^4+2n^3-n^2-2n-24)}\left[\frac{16}{3}(n+3)(n-2)(f(1)+(-1))\right.$$
$$+8n^2\sum_{k=2}^{n-1}\frac{f(x_k)}{(1-x_k^2)(P_{n-1}^{(1,1)}(x_k))^2}$$
$$-\frac{8n}{(n+2)(n^2-1)}\sum_{k=2}^{n-1}\frac{(1-x_k^2)f^{(iv)}(x_k)}{[P_{n-1}^{(1,1)}(x_k)]^2}\ ].$$

*valid for* $f \in Q_{2n-1}$.

*Remark 1.* The quadrature formula of Theorem 2 is exact for $f \in Q_{2n-1}$ but it requires only $2n-2$ information. The coefficients of $f^{(iv)}(\pm 1)$ are indeed zero.

*Remark 2.* Following quadrature formula

$$\int_{-1}^{1} f(x)dx = \frac{1}{(n^4+2n^3-n^2-2n-24)}\left[\frac{16}{3}(n+3)(n-2)(f(1)+f(-1))\right.$$
$$\left.+8n^2\sum_{k=2}^{n-1}\frac{f(x_k)}{(1-x_k^2)(P_{n-1}^{(1,1)}(x_k))^2}\right]$$
$$= R_n(f)$$

is exact only for $f \in \pi_3$, but it is easy to see that if $f \in C[-1,1]$ then

$$\lim_{n\to\infty} R_n(f) = \int_{-1}^{1} f(x)dx.$$

Thus in view of Theorem 15.4 [Szegö [6]] such quadrature formula is interesting and useful.

**3. Preliminary.** In order to prove theorem 1 we shall need two lemmas.

**Lemma 1.** For $n \geq 2$ we have

$$\int_{-1}^{1} (P_{n-2}^{(2,2)}(x))^2 dx = \frac{4n(n-1)(n^2+n+3)}{3(n+1)(n+2)} \tag{3.1}$$

and

$$\int_{-1}^{1} (1-x^2)(P_{n-2}^{(2,2)}(x))^2 dx = \frac{8n(n-1)}{(n+1)(n+2)} \tag{3.2}$$

**Proof of Lemma 1.** Since [see Szegö [6] formula (4.21.7)]

$$P_{n-2}^{(2,2)}(x) = \frac{4P_n''(x)}{(n+1)(n+2)} \tag{3.3}$$

it follows from integration by parts that

$$\int_{-1}^{1} (1-x^2)(P_{n-2}^{(2,2)}(x))^2 dx = \frac{32}{(n+1)^2(n+2)^2}\left[P_n''(1) + P_n(-1)P_n''(-1)\right].$$

Now, using the known results

$$P_n''(1) = \frac{(n+2)(n+1)n(n-1)}{8} = (-1)^n P_n''(-1), P_n(-1) = (-1)^n P_n(1) \tag{3.4}$$

we obtain (3.2). To prove (3.1) we need (3.3), integration by parts, orthogonality of the Legendre polynomials and we have

$$\int_{-1}^{1} (P_{n-2}^{(2,2)}(x))^2 dx = \frac{16}{(n+1)^2(n+2)^2}\left[2P_n'(1)P_n''(1) - 2P_n'''(1)\right]. \tag{3.9}$$

Next, note that

$$P_n'(1) = \frac{n(n+1)}{2}, P_n'''(1) = \left[\frac{n(n+1)-6}{6}\right] P_n''(1). \tag{3.10}$$

Now, using (3.4), (3.9) and (3.10) we obtain (3.1). This proves Lemma 1. Next, we state

**Lemma 2.** Let $n \geq 3$ and let $x_k (k = 2, 3, \ldots, n-1)$ be the zeros of the Jacobi polynomial $P_{n-2}^{(2,2)}(x)$. Then there exist the following quadrature formulae

$$\int_{-1}^{1} (1-x^2)^2 h(x)dx = \sum_{k=2}^{n-1} \lambda_k h(x_k) \tag{3.11}$$

$$\int_{-1}^{1} (1-x^2) g(x)dx = B_0(g(1) + g(-1)) + \sum_{k=2}^{n-1} B_k g(x_k), \tag{3.12}$$

$$\int_{-1}^{1} f(x)dx = A_0(f(1)+f(-1)) + A_1(f'(1)-f'(-1)) + \sum_{k=2}^{n-1} A_k f(x_k) \tag{3.13}$$

which are exact for every $h \in Q_{2n-5}, g \in Q_{2n-3}$ and $f \in Q_{2n-1}$, respectively. Moreover

$$B_0 = \frac{16}{n(n+2)(n^2-1)}, \quad A_0 = \frac{(2n^2+2n-3)B_0}{6}, \quad A_1 = -\frac{1}{2}B_0, \tag{3.14}$$

$$\lambda_k = \frac{8n(1-x_k^2)}{(n+2)(n^2-1)[P_{n-1}^{(1,1)}(x_k)]^2}, \quad A_k = \frac{\lambda_k}{(1-x_k^2)^2}, \quad B_k = (1-x_k^2)A_k. \tag{3.15}$$

*Remark.* It is interesting to remark that the quadrature formulas of Lemma 2 are optimal and will play an important role in proving theorem 1.

**Proof of Lemma 2.** At first, we note that (3.11) is a Gauss quadrature formula in $n-2$ points with respect to the weight function $(1-x^2)^2$ on $[-1,1]$. Then $x_k$'s are the zeros of the polynomial $P_{n-2}^{(2,2)}(x)$ and the Christoffel numbers $\lambda_k$ are given by [see Szegö [6] 15.31 and (4.5.7)]

$$\lambda_k = \frac{32(n-1)(1-x_k^2)}{n(n+1)(n+2)[P_{n-3}^{(2,2)}(x_k)]^2}, \qquad k = 2,3,\ldots,n-1.$$

On using [Szegö [6] formula (4.5.1)]

$$(n-1)(n+3)P_{n-1}^{(2,2)}(x_k) = -n(n+1)P_{n-3}^{(2,2)}(x_k)$$

it follows that

$$\lambda_k = \frac{32n(n+1)(1-x_k^2)}{(n-1)(n+2)(n+3)^2[P_{n-1}^{(2,2)}(x_k)]^2}.$$

Next, using [see Szegö [6] formula (4.5.1), (4.7.1) and (4.7.29)] we have

$$(n+3)P_{n-1}^{(2,2)}(x_k) = 2(n+1)P_{n-1}^{(1,1)}(x_k).$$

Therefore, we obtain

$$\lambda_k = \frac{8n(1-x_k^2)}{(n+2)(n^2-1)[P_{n-1}^{(1,1)}(x_k)]^2}, \qquad k = 2,3,\ldots,n-1.$$

as stated in (3.15).

The existence and uniqueness of quadrature formula (3.12) is guaranteed by integrating the quasi Hermite interpolation formula based on the end points and the zeros of $P_{n-2}^{(2,2)}(x)$. This formula is certainly exact for $g \in Q_{2n-3}$. As far as the determination of the coefficients is concerned, we set $g(x) = (1-x^2)h(x)$, where $h \in Q_{2n-5}$. Then the formula (3.12) reduces to

$$\int_{-1}^{1} (1-x^2)^2 h(x)dx = \sum_{k=2}^{n-1} \lambda_k h(x_k) = \sum_{k=2}^{n-1} B_k(1-x_k^2)h(x_k).$$

From this, we conclude that $\lambda_k = (1-x_k^2)B_k$, as stated in (3.14). Next, we set in (3.12) $g(x) = [P_{n-2}^{(2,2)}(x)]^2$ and use (3.2), we obtain at once $B_0 = \frac{8n(n-1)}{(n+1)(n+2)}$ as stated in (3.14). The existence and uniqueness of (3.13) valid for $f \in Q_{2n-1}$ is valued by integrating Hermite interpolation formula based on the zeros of $r_n(x)$ [see (2.1)]. The determination of the coefficients $A_k$, $k = 1, 2, \dots, n-1$ can be obtained by putting $f(x) = (1-x^2)g(x)$ in (3.13) and comparing it with (3.12). Similarly, putting $f(x) = [P_{n-2}^{(2,2)}(x)]^2$ in (3.13), using (3.1) we obtain $A_0$ as desired. Thus, we have proved Lemma 2.

Now, we are ready to give a proof of Theorem 1.

**Proof of Theorem 1.** From integration by parts it follows that for any $f \in C^3[-1,1]$ we have

$$\begin{aligned}\int_{-1}^{1} x(1-x^2)f'''(x)dx = 2(f'(1) - f'(-1)) - 6(f(1) + f(-1)) \\ + 6\int_{-1}^{1} f(x)dx.\end{aligned} \tag{4.1}$$

Let $f \in Q_{2n-1}$ then $xf'''(x) \in Q_{2n-3}$. Applying (3.12) to the integral on the left-hand side in (4.1), we obtain

$$\begin{aligned}\int_{-1}^{1} f(x)dx = (f(1) + f(-1)) - \frac{1}{3}(f'(1) - f'(-1)) \\ + \frac{1}{6}\left\{B_0(f'''(1) - f'''(-1)) + \sum_{k=2}^{n-1} B_k x_k f'''(x_k)\right\}.\end{aligned}$$

Now, eliminating $f'(1) - f'(-1)$ from this equality and (3.12) we find

$$\int_{-1}^{1} f(x)dx = \left(\frac{A_0+3A_1}{1+3A_1}\right)(f(1)+f(-1)) + \frac{1}{1+3A_1}\sum_{k=2}^{n-1} A_k f(x_k)$$
$$+\frac{A_1 B_0(f'''(1)-f'''(-1))}{2(1+3A_1)} + \frac{A_1}{2(1+3A_1)}\sum_{k=2}^{n-1} x_k B_k f'''(x_k)$$

On using the known expressions from Lemma 2 of the coefficients $A_0, A_1, B_0, A_k, B_k$ the above formula reduces to (2.2). This proves Theorem 1. Proof of Theorem 2 is similar to the proof of Theorem 1, so we omit the details. Our method fails in the case of $(0, m)$ quadrature for $m \geq 5$, based on these nodes.

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